PUPILS to whom this textbook is issued must not write on any page or mark any part of it in any way, consumable textbooks excepted.

1. Teachers should see that the pupil's name is clearly written in ink in the spaces above in every book issued.
2. The following terms should be used in recording the condition of the book: New; Good; Fair; Poor; Bad.
Math of bass
Volume
Weight on table
Area
Pressure

Force?
Guts/Personality of the loudspeaker

\[ P = \frac{\mathbf{F}}{A} \]

\[ P = \frac{\mathbf{F}}{A} \]  
Force

calculated

\[ P = \frac{\mathbf{F}}{A} \]

\[ P = \frac{\mathbf{F}}{A} \]

Product

\[ P = \frac{\mathbf{F}}{A} \]

negative reciprocal slope for perpendicular lines

\[ y = mx + b \]

\[ |x| = -x \quad x \leq 0 \]

\[ |x| = -x \quad x \leq 0 \]
Ch. 1: Language of Algebra

- Prime: Assumption
  - Prime number, not divisible by any integer except itself

\[ a + b = a + b \]

Ch. 2: Variation + Graphs

- Direct variation: \( y = kx \)
  - When \( k \) is positive:
    - \( y \) increases as \( x \) increases
  - When \( k \) is negative:
    - \( y \) decreases as \( x \) increases

- Inverse variation: \( \frac{1}{y} = \frac{1}{x} \)

- Variation of \( x \) and \( y \):
  - \( y = \frac{1}{x} + 1 \)
  - \( M = \frac{5}{6} \)
About the Cover

A stroboscopic photograph shows the regularity in the movement of a rotating gyroscope. While the pivot remains stationary, the axis itself revolves, outlining a cone.
Acknowledgments

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A list of the schools that participated in the research and development of this text may be found on page iii.

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We also acknowledge the contribution of the text *Advanced Algebra with Transformations and Applications*, by Zalman Usiskin (Laidlaw, 1975), to some of the conceptualizations and problems used in this book.
The University of Chicago School Mathematics Project (UCSMP) is a long-term project designed to improve school mathematics in grades K–12. UCSMP began in 1983 with a six-year grant from the Amoco Foundation, whose support continued in 1989 with a grant through 1994. Additional funding has come from the Ford Motor Company, the Carnegie Corporation of New York, the National Science Foundation, the General Electric Foundation, GTE, Citibank/Citicorp, and the Exxon Education Foundation.

The project is centered in the Departments of Education and Mathematics of the University of Chicago, and has the following components and directors:

- **Resources**
  Izaak Wirszup, Professor Emeritus of Mathematics

- **Primary Materials**
  Max Bell, Professor of Education

- **Elementary Teacher Development**
  Sheila Sconiers, Research Associate in Education

- **Secondary**
  Sharon L. Senk, Associate Professor of Mathematics, Michigan State University
  Zalman Usiskin, Professor of Education

- **Evaluation**
  Larry Hedges, Professor of Education
  Susan Stodolsky, Professor of Education

From 1983–1987, the director of UCSMP was Paul Sally, Professor of Mathematics. Since 1987, the director has been Zalman Usiskin.

The text *Advanced Algebra* was developed by the Secondary Component (grades 7–12) of the project, and constitutes the fourth year in a six-year mathematics curriculum devised by that component. As texts in this curriculum completed their multi-stage testing cycle, they were published by ScottForesman. A list of the six texts follows:

- *Transition Mathematics*
- *Algebra*
- *Geometry*
- *Advanced Algebra*
- *Functions, Statistics, and Trigonometry*
- *Precalculus and Discrete Mathematics*

A first draft of *Advanced Algebra* was written and piloted during the 1985–86 school year. After a major revision, a field trial edition was used in six schools in 1986–87. A second revision was tested during 1987–88. Results are available by writing UCSMP. This ScottForesman edition is based on improvements suggested by that testing, by the authors and editors, and by some of the many teacher and student users of earlier editions.

Comments about these materials are welcomed. Address queries to Mathematics Product Manager, ScottForesman, 1900 East Lake Avenue, Glenview, Illinois 60025, or to UCSMP, The University of Chicago, 5835 S. Kimbark, Chicago, IL 60637.
UCSMP Advanced Algebra is designed for a second-year course in algebra. It differs from other books for this course in six major ways. First, it has wider scope including substantial amounts of geometry integrated with the algebra. This is to correct the present situation in which many students who finish a second course in algebra forget what they learned in geometry. Also, we want to take advantage of all the mathematics students have had.

Second, reading and problem solving are emphasized throughout. Students can and should be expected to read this book. The explanations were written for students and tested with them. The first set of questions in each lesson is called “Covering the Reading.” The exercises guide students through the reading and check their coverage of critical words, rules, explanations, and examples. The second set of questions is called “Applying the Mathematics.” These questions extend student understanding of the principles and applications of the lesson. To further widen student horizons, “Exploration” questions are provided in every lesson.

Third, there is a reality orientation towards both the selection of content and the methods taught the student in working out problems. Algebra is rich in applications and problem solving. Being able to do algebra is of little ultimate use to individuals unless they can apply that content. Each elementary function is studied in detail for its applications to real-world problems. Real-life situations motivate algebraic ideas and provide the settings for practice of algebra skills. The variety of content of this book permits lessons on problem-solving strategies to be embedded in application settings.

Fourth, fitting the reality orientation, students are expected to use current technology. Calculators are assumed throughout this book because virtually all individuals who use mathematics today find it helpful to have them. Scientific calculators are recommended because they use an order of operations closer to that found in algebra and have numerous keys that are helpful in understanding concepts at this level. Computer activities are used to enhance algebraic concepts, and students are taught how to use a calculator or computer to graph and analyze functions.

Fifth, four dimensions of understanding are emphasized: skill in carrying out various algorithms; developing and using mathematical properties and relationships; applying mathematics in realistic situations; and representing or picturing mathematical concepts. We call this the SPUR approach: Skills, Properties, Uses, Representations.

Sixth, the instructional format is designed to maximize the development of understanding. The book is organized around lessons meant to take one day to cover. Ideas introduced in a lesson are reinforced through “Review” questions in the immediately succeeding lessons; this gives students several nights to learn and practice each idea. The lessons themselves are sequenced into carefully constructed chapters. At the end of each chapter, a carefully focused Progress Self-Test and a Chapter Review, each keyed to objectives in all the dimensions of understanding, are then used to solidify performance of skills and concepts from the chapter so that they may be applied later with confidence. Finally, to increase retention, important ideas are reviewed in questions in later chapters.
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**Models for Operations**

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To the Student

You have studied algebra. Advanced Algebra sounds as if this class will be like algebra was, but more difficult. That is half true. The content of this book is similar to that in first-year algebra. You will study more about variables, equations, and graphs. However, this book is not necessarily more difficult. Some questions are harder, but you know a lot more now than you did then. In particular, you know much more geometry and you have had a year's more practice with algebra.

Advanced Algebra studies a variety of topics, from lines to logarithms, from quadratic equations to conic sections, from systems to statistics, from matrices to trigonometry. It might be best described as "what every high school graduate should know about mathematics." It contains the mathematics that educated people around the world use in conversation and that colleges want or expect you to have studied. But it is not a hodge-podge of topics. The properties of numbers, graphs, expressions, equations, inequalities, and functions are ideas which run throughout the book.

In this course, you will learn the meaning of many of the keys on a scientific calculator that you may never have used before. We recommend a solar-powered calculator so that you do not have to worry about batteries, though some calculators have batteries which can last for many years and work in dim light. You also need to have a ruler and graph paper.

If you plan to buy a calculator, consider buying a graphing calculator. Throughout this book there are questions which are made easier with such a calculator, and you will find such a calculator particularly useful in your future mathematics courses. If you plan to go to college, a graphing calculator will be useful in many courses in business, the sciences, and statistics.

An important goal of this book is to assist you to become able to learn mathematics on your own, so that you will be able to deal with the mathematics that you see in newspapers, magazines, on television, on any job, and in school. The authors, who are all experienced teachers, offer the following advice.

1. You cannot learn much mathematics just by watching other people do it. You must work problems and participate in class. Some teachers have a slogan:
   
   Mathematics is not a spectator sport.

2. You are expected to read each lesson. Read slowly, and keep a pencil with you as you check the mathematics that is done in the book. Use the Glossary or a dictionary to find the meaning of a word you do not understand.

3. You are expected to do homework every day while studying from this book, so put aside time for it. Do not wait until the day before a test if you do not understand something. Try to resolve the difficulty right away and ask questions of your classmates or teacher. You are expected to learn many things by reading, but school is designed so that you do not have to learn everything by yourself.

4. If you cannot answer a question immediately, don't give up! Read the lesson again; read the question again. Look for examples. If you can, go away from the problem and come back to it a little later.

We hope you join the many students who have enjoyed this book. We wish you much success.
CHAPTER 1

The Language of Algebra

Pythagoras (sixth century B.C.)

René Descartes (seventeenth century)

Pierre de Fermat (seventeenth century)

1-1: Describing Situations with Algebra
1-2: Formulas
1-3: Explicit Formulas for Sequences
1-4: Recursive Formulas for Sequences
1-5: Algebra as a Mathematical System
1-6: Reasoning in Algebra
1-7: Solving Equations
1-8: Rewriting Formulas
1-9: Solving Inequalities
Problems that today we solve using algebra were considered by the Babylonians as long ago as 1700 B.C. But neither they nor the Greeks of classical times possessed a language of algebra. To the Greeks, an unknown quantity was imagined as a length. They thought of an unknown quantity multiplied by another as the area of a rectangle. So when today we write

\[ x(x + 4), \]

they would write "the rectangle contained by a line segment and the segment increased by 4."

Slowly over the years the writing was shortened and shortened. By A.D. 850 the Arabs were putting problems in numerical terms. The word "algebra" is derived from the Arabic "al jabr" which means, literally, "the bone setting" or "putting back in place." This described the way the Arabs solved problems. For instance, to find

the number, which when it is multiplied by 15, gives 180,

they would divide by 15 (put the multiplication back in place).

In the years 1200–1600, Arabic manuscripts were discovered by Europeans and, in the last half of that time period, a rebirth (the French word is "renaissance") of learning occurred in Europe. In 1591, the French mathematician François Viète (1540–1603) wrote a book, Introduction to the Analytic Art, in which he established the symbols and principles of what we call algebra.

Over the next 200 years, others refined algebra. By 1607, René Descartes and Pierre Fermat had graphed solutions to equations. In 1770, Leonhard Euler wrote his book Elements of Algebra, which became the standard textbook for many years for Europeans wishing to learn algebra. Many symbols Euler used in his book, like \( \pi \) and \( \sin x \), became the standard symbols of algebra.

Today algebra is truly a universal language. Even in China, Japan, and the Soviet Union, large countries where written alphabets differ from English, algebra is taught using Latin letters like \( a \) and \( b \) and \( x \) and \( y \) and the same operation signs we use. In fact, a Soviet mathematics textbook for 8-year-olds, introducing letters for variables, states, "In mathematics, we use Latin letters, for example \( x \), to denote unknown numbers. Memorize four other letters: \( Aa, Bb, Cc, Dd \)."

This chapter reviews some of the basic ideas of algebra you have studied in previous years. It discusses sequences, one idea that may be new to you, in quite a bit more detail.
LESSON

1-1

Describing Situations with Algebra

The language of algebra is based on numbers and variables. A **variable** is a symbol that can be replaced by any one of a set of numbers or other objects. When numbers and variables are combined, the result is called an **algebraic expression**, or simply an **expression**.

An expression using the variable \( r \) and the numbers \( \pi \) and 2 is \( \pi r^2 \).

In order to apply algebra, a person must be able to write algebraic expressions to describe situations.

One way to describe situations by algebra is by direct translation. For instance, you know that "the sum of \( m \) and \( n \)" is translated as "\( m + n \)" and "twice \( x \)" is translated as "\( 2x \)".

You must be careful when translating directly. For instance, the phrase "less than" sometimes means the operation of subtraction, and sometimes it means the inequality symbol "\(<\)". It can be confused also with the single word "less." Here are examples.

<table>
<thead>
<tr>
<th>Verbal Expression</th>
<th>Mathematical Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 is less than 8</td>
<td>5 &lt; 8</td>
</tr>
<tr>
<td>5 less than 8</td>
<td>8 - 5</td>
</tr>
<tr>
<td>5 less 8</td>
<td>5 - 8</td>
</tr>
</tbody>
</table>

Note how multiplication, subtraction, and an inequality are used to translate the following situation.

**Example 1**  Translate this sentence into algebra.
12 less than twice a number is less than five times the number.

**Solution**  Let \( n \) = the number.
Translate the sentence part by part.

\[
\begin{align*}
12 & \text{ less than } \quad \text{twice a number } \quad \text{is less than } \quad \text{five times the number} \\
2n - 12 & \quad < \quad 5n
\end{align*}
\]

An algebraic sentence (such as \( 2n - 12 < 5n \)) consists of expressions related with a verb. The most common verbs in algebra are = (is equal to), \(<\) (is less than), \(>\) (is greater than), \(\leq\) (is less than or equal to), \(\geq\) (is greater than or equal to), \(\neq\) (is not equal to), and \(\approx\) (is approximately equal to). In the sentence in Example 1, the two expressions \( 2n - 12 \) and \( 5n \) are related by the verb \(<\).

A second way to describe situations with algebra is to find a pattern.
Example 2  Kim has a tape collection now containing 40 tapes. If Kim adds 2 tapes each week, how many tapes will she have after $w$ weeks?

Solution  Make a table. Notice in this table that the arithmetic is not carried out. That would hide the pattern.

<table>
<thead>
<tr>
<th>Weeks from now</th>
<th>Tapes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$40 + 1 \cdot 2$</td>
</tr>
<tr>
<td>2</td>
<td>$40 + 2 \cdot 2$</td>
</tr>
<tr>
<td>3</td>
<td>$40 + 3 \cdot 2$</td>
</tr>
<tr>
<td>4</td>
<td>$40 + 4 \cdot 2$</td>
</tr>
</tbody>
</table>

The number in the left column, which gives the number of weeks, is always in a particular slot in the expressions at right. That shows a pattern:

$$w \quad 40 + w \cdot 2$$

Kim will have $40 + w \cdot 2$, or $40 + 2w$ tapes after $w$ weeks.

Check:  To check the expression $40 + 2w$, pick a value for $w$ not in the table. We pick $w = 10$, indicating 10 weeks from now. Then $40 + 2w = 40 + 2 \cdot 10 = 60$. This is correct. After 10 weeks there would be the 40 Kim started with and 20 more tapes, for a total of 60.

A third way to describe situations by algebra is to recognize common uses of the operations of arithmetic. These uses are summarized by models. A model for an operation is a pattern that describes many of the uses of that operation. For instance, one model for subtraction is "take-away." You learned these models starting in first grade, but you may not have given them names. A list of models is given in Appendix A. You do not need to know the names, but you need to be able to choose the correct operation.
Example 3  Express the cost of \( y \) cans of orange juice at \( x \) cents per can.

Solution 1  Use a special case. 5 cans at 40¢ per can would cost $2.00. That suggests multiplication. So \( y \) cans at \( x \) cents per can will cost \( xy \) cents.

Solution 2  Recognize the model. The unit “cents per can,” which can be written as \( \frac{\text{cents}}{\text{can}} \), signals a rate. The pattern of the rate-factor model for multiplication is \( \frac{\text{cents}}{\text{can}} \) \( \cdot \) \( y \) cans at \( \frac{\text{cents}}{\text{can}} \). The result is \( xy \) cents.

Check  Working with the units gives you a way to check your work. The unit of \( y \) is cans. So the unit of \( xy \) is the “product” of the units.

\[
\frac{\text{cents}}{\text{can}} \cdot y \text{ cans} = xy \text{ cents}
\]

The units work out.

As Example 3 shows, there is often more than one way to translate situations into algebra. You should strive to learn a variety of ways. The expression you get a second way can be used to check the expression you got the first way.

Questions

These questions check your understanding of the reading. If you cannot answer a question, you should go back to the reading to help you find an answer.

1. Who established the principles of what today we call algebra, and when?
2. The word “algebra” comes from the ___ language and means ___.
3. \( 2\pi r \) is an example of an ___.
4. Name any variables in \( 2\pi r \).
5. True or false  Algebraic expressions in Japanese books use letters of the Latin alphabet.

In 6–8, translate into an algebraic expression or sentence.
6. \( p \) less than \( y \)
7. \( p \) is less than \( y \).
8. 7 less than three times a number is less than the number.
9. A person now owns 25 tapes and is buying 3 tapes a week. How many will there be after \( w \) weeks?

10. Answer Question 9 if the person now owns \( N \) tapes.

11. A pattern that describes many of the uses of an operation is called a(n) \(_?\) .

12. Give an example of a rate unit.

13. a. What is the cost of 10 cans of orange juice costing \( c \) cents per can?
   
   b. What is the cost of \( m \) cans of orange juice costing \( c \) cents each?

In 14–16, translate into words.

14. \( \leq \)  

15. \( \neq \)  

16. \( \approx \)

17. Give an example of an algebraic sentence.

These questions extend the content of the lesson. You should take your time, study the examples and explanations, try a variety of methods and check your answers with the ones in the back of the book.

In 18–22, tell whether the answer is \( x + y \), \( x - y \), \( xy \), \( \frac{x}{y} \), or \( \frac{y}{x} \).

18. You give a friend \( y \) dollars. You had \( x \) dollars. How much do you have left?

19. Mrs. Bell is \( y \) years old. A friend is \( x \) years older. How old is the friend?

20. You drove \( x \) miles in \( y \) hours. What was your rate?

21. You buy \( x \) granola bars at \( y \) cents per bar. What is the total cost?

22. A picture of a building is \( x \) times actual size. The height of the building is \( y \). What is the height of the building in the picture?

In 23–26, write an expression to describe each situation.

23. \( T \) liters of a fluid are taken away from a solution that has \( S \) liters in it. How much is left?

24. Barb is \( B \) years old. Her younger sister Yvette is \( Y \) years old. How many times as old as Yvette is Barb?

25. Liz has \( E \) eggs and buys \( F \) eggs. How many eggs does she have altogether?

26. Ben has been in office \( Y \) years. Bess has been in office \( Y + 5 \) years. How much longer than Ben has Bess been in office?
27. Dennis needs to fence the rectangular pasture shown in the figure below. One side borders a river and needs no fencing. He has 1150 ft of fence.

![Diagram of pasture with fence]

a. Let $x$ be the width as labeled. Write an expression for the length, in terms of $x$.
b. Write an expression for the area of the pasture in terms of $x$.
c. Suppose the pasture must enclose at least 60,000 square feet. Write a sentence relating the area expression in part b to the area the pasture must enclose.

Every lesson contains review questions to practice ideas you have studied earlier. If the idea is from an earlier course, the question is designated as being from a previous course.

28. **Multiple choice** Which sentence correctly relates the angle measures in the figure below? *(Previous Course)*

(a) $x + y + z = 180$  
(b) $x + y = 180 + z$
(c) $y + z = x + y$  
(d) none of these

![Diagram of angles]

In 29–32, give the name for the polygon with the indicated number of sides. *(Previous Course)*

29. 4  
30. 5  
31. 6  
32. 8

33. Make up one example of a situation different from those in this lesson that can lead to each expression:

a. $x + y$  
b. $x - y$  
c. $xy$  
d. $\frac{x}{y}$

34. In an encyclopedia, look up one of the mathematicians named on page 3. Find out some of the accomplishments of that person.
Here is a sentence from the language of algebra:

\[ d = \frac{n(n - 3)}{2} \]

This sentence tells you the number of diagonals \( d \) in a polygon having \( n \) sides. Some instances of this general relationship are drawn below.

\[ n = 3, \quad d = 0 \quad n = 4, \quad d = 2 \quad n = 5, \quad d = 5 \quad n = 6, \quad d = 9 \]

Before using a variable, you need to know what values it can have. The set of meaningful numbers or things that can be substituted for a variable is called the replacement set or domain for the variable. The replacement set for \( n \) in the formula above is the set of all possible numbers of sides a polygon can have. It would not make sense to substitute \( n = 2 \) into the formula, because there are no 2-sided polygons! Likewise \( n = 4.7 \) would be ridiculous, since you cannot have part of a side. So, possible values for \( n \) in the sentence

\[ d = \frac{n(n - 3)}{2} \]

are natural numbers greater than 2. That is, \( n = 3, 4, 5, \ldots \). Here are some domains that have their own names:

- the set of natural numbers or counting numbers \( \{1, 2, 3, 4, 5, \ldots \} \);
- the set of whole numbers \( \{0, 1, 2, 3, 4, 5, \ldots \} \);
- the set of integers \( \{0, 1, -1, 2, -2, 3, -3, \ldots \} \);
- the set of rational numbers (those numbers that can be represented by fractions) \( \{\frac{2}{3}, 1\frac{9}{11}, -\frac{34}{10}, 239.6, 0.0004, \text{and so on}\} \);
- the set of real numbers (those numbers that can be represented by decimals) \( \{1, 35 \text{ million}, 2.34, \pi, 0, -7, \sqrt{5}, \text{and so on}\} \);
- the set of positive real numbers (those real numbers greater than zero).

Substituting for the variables and calculating a result is called evaluating an expression. In order to evaluate expressions you must use the rules for grammar and punctuation of the language of algebra. The following rules for order of operations are used to evaluate expressions worldwide.
Rules for Order of Operations:
1. Perform operations within parentheses ( ), brackets [ ], or other grouping symbols like square root symbols or fraction bars, from the inner set of symbols to the outer set. Use the order given in Rules 2, 3, and 4.
2. Take powers.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

An equation is a sentence stating that two expressions are equal. A formula is a sentence stating that a single variable is equal to an expression with one or more different variables on the other side. Thus,

\[ d = \frac{n(n - 3)}{2} \]

is both an equation and a formula. But \( a + b = b + a \) is an equation that is not a formula. Formulas are useful because they express important ideas with very few symbols and because they can be applied easily to many situations.

**Example 1** Below is a dodecagon (12-sided polygon). How many diagonals does it have?

![Dodecagon Diagram]

**Solution** Substitute \( n = 12 \) into the diagonal formula (given at the beginning of the lesson) every place \( n \) appears.

\[ d = \frac{n(n - 3)}{2} \]

Work in parentheses first.

\[ = \frac{12(12 - 3)}{2} \]

\[ = \frac{12(9)}{2} \]

\[ d = 54 \]

A dodecagon has 54 diagonals.
Using a formula is certainly quicker than trying to answer the question in Example 1 by counting the diagonals.

Here is an example using a science formula with more than one variable in the expression. It illustrates how to handle units in formulas.

**Example 2**
The formula \( d = \frac{1}{2}gt^2 \) tells how to find \( d \), the distance an object has fallen during time \( t \), when it is dropped in free fall from near Earth's surface. The variable \( g \) represents the acceleration due to gravity.

Near Earth's surface, \( g = 32 \frac{\text{ft}}{\text{sec}^2} \). About how far will a rock fall in 5 seconds if dropped close to Earth's surface?

**Solution** Substitute \( g = 32 \frac{\text{ft}}{\text{sec}^2} \) and \( t = 5 \text{ sec} \) into the formula.

\[
\begin{align*}
d &= \frac{1}{2}gt^2 \\
&= \frac{1}{2} \left( 32 \frac{\text{ft}}{\text{sec}^2} \right) (5 \text{ sec})^2 \\
&= \frac{1}{2} \left( 32 \frac{\text{ft}}{\text{sec}^2} \right) (25 \text{ sec}^2) \\
&= \left( 16 \frac{\text{ft}}{\text{sec}^2} \right) (25 \text{ sec}^2) \\
d &= 400 \text{ ft}
\end{align*}
\]

In 5 seconds a rock dropped from near Earth's surface will fall about 400 feet.

**Check** The time units cancel out and you are left with feet. This is an appropriate measure for distance, so the unit checks. Does the distance seem reasonable to you?

**Questions**

In 1–3, use the formula \( d = \frac{n(n - 3)}{2} \) for diagonals of polygons.

1. What do the variables \( n \) and \( d \) represent?

2. What is a reasonable replacement set for \( n \)?

3. Find \( d \) when \( n = 8 \). Check your answer with a drawing.

4. Another name for replacement set is ___.

5. a. Name a real number that is not an integer.
   b. Name a real number that is not a rational number.
6. Which integers are not natural numbers?

7. To evaluate $3 \cdot 5^{(4-2)}$, you must do these steps. Put them in order.
   a. Multiply.
   b. Take the power.
   c. Do the subtraction.

In 8 and 9, refer to Example 2. How far will a rock fall in four seconds:

8. if it is dropped from near Earth’s surface?

9. if it is dropped from near the surface of the moon, where
   \[ g = 5.3 \text{ ft/sec}^2 \]

In 10–12, (a) Is the sentence an equation? (b) Is the sentence a formula?
10. $A = \pi r^2$  
11. $6s^2 > 0$  
12. $2(L + W) = 2L + 2W$

13. What is the domain of $n$?

14. Find the total number of degrees in the interior angles of an octagon.

15. The area $A$ of an equilateral triangle with length of side $s$ is given by
    the formula $A = \frac{s^2}{4} \sqrt{3}$.
    a. State the domain for $s$.
    b. Use your calculator to find the area of an equilateral triangle with
       sides of 10 cm. Round to the nearest tenth.
16. Young’s formula, \[ C = \left( \frac{g}{g + 12} \right) A \], has been used to decide how much medicine \( C \) to give to a child under age 13 when the adult dosage \( A \) is known. Here \( g \) is the child’s age measured in years. Suppose an adult dosage for a medicine is 600 milligrams. What is the dosage for a 3-year-old according to this formula? (Caution! Do not apply this formula yourself. Medicines should be taken only under the supervision of a physician or pharmacist.)

In 17–19, evaluate each expression with \( x = 15 \), \( y = -3 \), and \( z = 2 \).

17. \[ \frac{x}{y} - z^3 \]
18. \[ \frac{x}{(y - z)^3} \]
19. \[ \frac{x}{y - z^3} \]

20. Evaluate \( 2x^{y-1} \) when \( x = 3 \) and \( y = 4 \).

Review

A lesson number following a review question indicates a place where the idea of the question is discussed.

In 21–25, tell whether the answer is \( a + b \), \( a - b \), \( b - a \), \( ab \), \( \frac{a}{b} \), or \( \frac{b}{a} \). (Lesson 1-1)

21. A football player gains \( a \) yards on one play and gains \( b \) yards on the next. What is the total gain?
22. A football player gains \( a \) yards on one play and loses \( b \) yards on the next. What is the total gain?
23. How many outfits can be made from \( b \) skirts and \( a \) shirts?
24. Spend \( b \) dollars to buy \( a \) grams of perfume. What is the cost per gram?
25. There are \( a \) juniors and \( b \) sophomores. How many more sophomores than juniors are there?

In 26 and 27, let \( n \) be the number. (Lesson 1-1)

26. A number is doubled. The product is decreased by 7. The difference is divided by 2. What is the final value?
27. Translate into algebra: six greater than a number is less than sixty.

Exploration

28. Using the digits 1, 9, 8, and 7 in the given order, along with the operations of addition, subtraction, multiplication, division and powering, and any parentheses you wish, and taking the opposite of any number or quantity, find a way to calculate every integer from 1 to 10. Two examples are:

\[ 1 = 1^{987} \] and \[ 2 = 1^9 + 8 - 7. \]

[Hint: Taking a square root is raising to the \( \frac{1}{2} \) power.]
In mathematics, a **sequence** is an ordered list. Each item on the list is called a **term**. Many formulas come from observing the pattern of terms in a sequence.

Here are the first five terms of a sequence of figures, staircases made of unit squares

The number of unit squares in each staircase is a sequence of numbers:

\[ 3, 6, 10, 15, 21 \]

There is a pattern, not so easy to see. Notice how the left and right columns are related.

<table>
<thead>
<tr>
<th>Term</th>
<th>Number of squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>( 3 = \frac{3}{2} \cdot 2 = \frac{3}{2} \cdot 3 )</td>
</tr>
<tr>
<td>2nd</td>
<td>( 6 = \frac{3}{2} \cdot 4 = \frac{3}{2} \cdot 4 )</td>
</tr>
<tr>
<td>3rd</td>
<td>( 10 = \frac{4}{2} \cdot 5 = \frac{4}{2} \cdot 5 )</td>
</tr>
<tr>
<td>4th</td>
<td>( 15 = \frac{5}{2} \cdot 6 = \frac{5}{2} \cdot 6 )</td>
</tr>
<tr>
<td>5th</td>
<td>( 21 = \frac{6}{2} \cdot 7 = \frac{6}{2} \cdot 7 )</td>
</tr>
</tbody>
</table>

The general term is called the **nth term**. The number of squares can be given using \( n \).

\[ \text{nth} \quad \frac{n + 1}{2} \cdot (n + 2) = \frac{(n + 1)(n + 2)}{2} \]

For instance, to find the 10th term in the sequence, let \( n = 10 \).

\[ 10 \text{th} \quad \frac{11 \cdot 12}{2} \]

There are 66 squares in the 10th staircase.

To represent the terms of a sequence, variables with subscripts are often used. A **subscript** is a number or variable written below and to the right of a variable. For instance, \( t_1 \), which is read “\( t \) sub 1,” and \( t_n \), which is read “\( t \) sub \( n \),” are **subscripted variables**. The subscript of \( t_1 \) is 1, and the subscript of \( t_n \) is \( n \).
The subscript is often called an index because it indicates the position of the term in the sequence. In general, if \( t_n \) represents the \( n \)th term of the sequence, then the sentence \( t_1 = 3 \) means that the first term of the sequence is 3. In the staircase sequence, \( t_2 = 6 \), \( t_3 = 10 \), \( t_4 = 15 \), \( t_5 = 21 \), and \( t_n = \frac{(n + 1)(n + 2)}{2} \).

The sentence \( t_n = \frac{(n + 1)(n + 2)}{2} \) is called an explicit formula for the \( n \)th term of the sequence 3, 6, 10, 15, 21, ... The domain for an explicit formula for the \( n \)th term of any sequence is the set of counting numbers. Explicit formulas are important because they can be used to calculate any term in the sequence by substituting a particular value of \( n \).

**Example 1** Consider the sequence of cubes of natural numbers:

\[ 1, 8, 27, 64, 125, \ldots \]

Suppose the subscripted variable \( t_n \) represents the \( n \)th term in this sequence.

a. What is the value of \( t_3 \)?

b. Give an explicit formula for \( t_n \).

c. Find the 30th term of this sequence.

**Solution**

a. The third term is \( t_3 \) since the third term is known to be 27, \( t_3 = 27 \).

b. The cube of \( n \) is \( n^3 \), so \( t_n = n^3 \).

c. Substitute 30 for \( n \) on both sides of the explicit formula:

\[ t_{30} = 30^3 = 27,000 \]

The 30th term is 27,000.

When you substitute into a formula, as in part c of Example 1, you are calculating an instance of that formula.

Sequences arise naturally in many situations in science, business, finance, and other areas. The next example looks at a sequence in biology.
Example 2

A microbe reproduces by splitting to make 2 cells. Each of these cells then splits in half to make a total of 4 cells. Each of these splits to make a total of 8, and so on. Each splitting is called a generation. If a colony begins with 500 microbes, the equation

\[ P_n = 500(2)^{n-1} \]

gives the number of microbes in the \( n \)th generation (assuming no microbes die).

Write:

a. the first term;

b. the fifth term of the sequence of populations of microbes given by this formula.

Solution

Notice that the variable is in the exponent.

a. For the population in the first generation, substitute 1 for \( n \) in the formula.

\[ P_1 = 500(2)^{1-1} = 500(2)^0 = 500(1) = 500 \]

This checks with the given information: there are 500 microbes in the first generation.

b. For the population in the fifth generation, use \( n = 5 \) in the formula.

\[ P_5 = 500(2)^{5-1} = 500(2)^4 = 500(16) = 8000 \]

There are 8000 microbes in the fifth generation. You can check this by doubling the first population of 500 four times: \( P_1 = 500, P_2 = 1000, P_3 = 2000, P_4 = 4000, P_5 = 8000. \)

When you know an explicit formula for the \( n \)th term of a sequence, you can use a computer program to generate many terms of the sequence very quickly. Appendix C summarizes important features of the BASIC language. Below is a program using a FOR ... NEXT ... loop that prints the first ten perfect cubes. That is, it prints the terms generated by the formula

\[ t_n = n^3 \]

for integer values of \( n \) from 1 to 10. In line 20, the symbol "^ means "to the power." The semicolon at the end of line 20 (:) causes the terms to be printed in a row with only one space between successive terms.

```
10 FOR N = 1 TO 10
20 PRINT N ^ 3;
30 NEXT N
40 END
```

The computer will print:

\[
1 \ 8 \ 27 \ 64 \ 125 \ 216 \ 343 \ 512 \ 729 \ 1000
\]
The above program can be modified to produce a given number of terms for any sequence when an explicit formula for the $n$th term is known. With a slight change, as shown in the following example, a computer can print out a table with both the index and the term of a sequence. Notice that variables in computer programs need not be single letters.

**Example 3** What will this BASIC program print when run?

```
10 FOR N = 1 TO 6
20 LET TERMN = 500 * 2^(N-1)
30 PRINT N, TERMN
40 NEXT N
50 END
```

**Solution** The FOR ... NEXT ... loop (lines 10–40) tells you that lines 20 and 30 are repeated for each integer from 1 to 6. Lines 20 and 30 tell you that the computer evaluates the formula $t_n = 500(2)^{n-1}$ and prints out the values of $n$ and $t_n$ each time it goes through the loop. The comma between N and TERMN in line 30 tells the computer to align the numbers in a table of two columns. Thus the computer prints the following.

```
1 500
2 1000
3 2000
4 4000
5 8000
6 16000
```
Questions

1. An ordered list of items is called a ___.
2. Each item in a sequence is called a ___.

In 3–5, refer to the sequence of staircases at the beginning of this lesson.
3. Draw the 6th term of the sequence.
4. Use the explicit formula for \( t_n \) to find the number of squares in the 6th staircase.
5. List what will be printed when this program is run.

```
10 FOR N = 1 TO 8
20 PRINT (N+1)*(N+2)/2;
30 NEXT N
40 END
```

6. The sentence \( a_3 = 10 \) is read ___.
7. The general term of a sequence is called the ___ term.
8. a. In the sentence \( a_4 = 15 \), 4 is called a ___ or an ___.
   b. \( a_4 = 15 \) means that 15 is the ___ term of the sequence.

In 9–11, refer to Example 1.
9. Describe the sequence using words.
10. Give the explicit formula for the \( n \)th term.
11. a. Which term of the sequence is 64?
    b. What is the 64th term of the sequence?
12. Refer to Example 2.
    a. Evaluate \( P_n = 500(2)^{n-1} \) when \( n = 7 \).
    b. How many microbes are in the 10th generation?
13. a. Draw the next term in the sequence.

b. Give a formula for \( S_n \), the number of dots in the \( n \)th term.

In 14 and 15, write the first four terms of the sequence with the given explicit formula.
14. \( a_n = 5n - 3 \)
15. \( S_n = \frac{n(3n - 1)}{2} \)
Review

In 16–18, Martin started with a company at an annual salary of $18,000. He gets an increase of 5% at the end of each year. Then the formula \( s_n = 18,000(1.05)^{n-1} \) gives Martin’s salary in his \( n \)th year.

16. Calculate the first three terms of the sequence. (Use your calculator.)

17. At this growth rate, what would Martin’s salary be in his 30th year with this company?

18. Complete this program so that it will print the first 30 terms of the sequence describing Martin’s salary.

```plaintext
10 FOR N = 1 TO ___
20 PRINT ___
30 NEXT N
40 END
```

Multiple choice In 19 and 20, which is a formula for the \( n \)th term of the sequence?

19. 2, 4, 8, 16, 32, ...
   (a) \( t_n = 2n \)                 (b) \( t_n = n^2 \)                 (c) \( t_n = 2^n \)

20. 2, 9, 28, 65, 126, ...
   (a) \( t_n = 7n - 5 \)            (b) \( t_n = 7n^2 - 2 \)            (c) \( t_n = n^3 + 1 \)

21. Write a BASIC program that prints the first 200 perfect squares.

In 22 and 23, evaluate with \( z = 0.5 \), \( x = -2 \), \( y = -6 \). (Lesson 1-2)

22. \( x + 3yz^2 \)                       23. \( x + (9yz)^2 \)

In 24 and 25, use the formula \( S = \frac{20N}{88 - N} \). In this formula, \( N \) is the number of cars that pass an observation point in one minute, and \( S \) represents the average speed for heavy traffic (in mph).

24. Find the average speed of traffic if 30 cars pass the observation point per minute. (Lesson 1-2)

25. Why are numbers greater than or equal to 88 not part of the domain of \( N \)? (Lesson 1-2)

26. What percent of 80 is 56? (Previous Course)
27. Graph the image of $\triangle ABC$ under a size change of magnitude 3. (Previous Course)

![Graph of $\triangle ABC$]

28. Graph the image of $\triangle ABC$ under a size change of magnitude $\frac{2}{3}$. (Previous Course)

29. The array below is part of an infinite pattern called Pascal’s Triangle. The first and last terms of each row are 1. Each other term, from the third row on, is the sum of the two diagonally above it.

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

- - - - - - - - - -
- - - - - - - - - -

a. Write the next two rows in the array.
b. On what diagonals (oblique lines in the array) can you find the sequence of natural numbers?
c. On what diagonals can you find the number sequence that comes from the staircase pattern at the start of this lesson?
Like computers, calculators can also be used to produce terms of a sequence quickly. For instance, to generate the sequence

\[ 2, 7, 12, 17, 22, \ldots \]

you can begin with 2 and keep adding 5. Enter the following key sequence on your calculator and check that your display matches the one below.

Key Sequence

\[ 2 + 5 = + 5 = + 5 = + 5 = \]

Sequence Displayed

\[ 2 \quad 7 \quad 12 \quad 17 \quad 22 \]

When a sequence is generated by adding, subtracting, multiplying, or dividing by a constant, you should be able to write a key sequence for the numerical sequence.

**Example 1**

Write a key sequence that generates the sequence

\[ 40, 20, 10, 5, 2.5, \ldots \]

**Solution**  
Notice that one way to construct this sequence is to start with 40 and divide by 2 to get the next term. One possible key sequence is

\[ 40 \div 2 = \div 2 = \div 2 = \div 2 = \div 2 = \]

**Check**  
Use your calculator. Verify that as the key sequence is entered, the calculator displays the sequence 40, 20, 10, 5, 2.5. Note that because dividing by 2 gives the same result as multiplying by 0.5, there is at least one more key sequence that could be used to generate this sequence.
Notice that no explicit formula is given for the sequence of Example 1. Instead, you are told how to find each term of a sequence in terms of the preceding one. For instance, to find the 12th term of the sequence

\[ 40, 20, 10, 5, 2.5, \ldots, \]

you would divide the preceding term, the 11th term, by 2. This sequence is said to be formed recursively.

A **recursive formula** or **recursive definition** for a sequence is a set of statements that

a. indicates the first term (or first few terms) and

b. gives a rule for how the \( n \)th term is related to one or more of the previous terms.

A recursive definition for the sequence 40, 20, 10, 5, 2.5, \ldots is

\[
\begin{align*}
  t_1 &= 40 \\
  t_n &= \frac{t_{n-1}}{2}, \text{ for integers } n \geq 2.
\end{align*}
\]

The first line tells you that the first term of the sequence is 40. The second line tells you that the new term, \( t_n \), is calculated by dividing the previous term, \( t_{n-1} \), by 2. The brace at the left, \{, indicates that both lines are needed for the recursive definition.

**Example 2** Evaluate the first four terms of the sequence defined by the recursive formula

\[
\begin{align*}
  t_1 &= 25 \\
  t_n &= t_{n-1} - 4 \text{ for } n \geq 2.
\end{align*}
\]

**Solution** The first term is given: \( t_1 = 25 \). The other terms are evaluated by applying the formula for \( t_n \). That formula says that the \( n \)th term is found by subtracting 4 from the previous term. First we calculate \( t_2 \) from \( t_1 \). Let \( n = 2 \) in the formula for \( t_n \).

\[
l_2 = t_{2-1} - 4 = t_1 - 4 = 25 - 4 = 21
\]

Then use \( t_2 \) to get \( t_3 \).

\[
l_3 = t_{3-1} - 4 = t_2 - 4 = 21 - 4 = 17
\]

Finally, use \( t_3 \) to find \( t_4 \).

\[
l_4 = t_{4-1} - 4 = t_3 - 4 = 17 - 4 = 13
\]

Thus the first four terms are 25, 21, 17, 13.
Recursive definitions are often used to generate terms of sequences on computers. However, in BASIC the subscripts for $t_n$ and $t_{n-1}$ are not necessary because of the way the LET statement works. For instance, the program below generates the first ten terms of the sequence defined recursively in Example 2.

```
10 LET T = 25
20 FOR N = 1 TO 10
30 PRINT T;
40 LET T = T - 4
50 NEXT N
60 END
```

Line 40 tells the computer to assign a new value of $T$ that is 4 less than the old value of $T$. The computer will print this sequence:

```
25 21 17 13 9 5 1 -3 -7 -11
```

Recursive and explicit formulas for sequences are useful at different times. An explicit formula lets you calculate a specific term in a sequence without having to know all the previous terms. For instance, when you evaluate the formula

$$t_n = \frac{(n + 1)(n + 2)}{2}$$

for $n = 99$, you find

$$t_{99} = \frac{(99 + 1)(99 + 2)}{2} = \frac{100 \cdot 101}{2} = 5050.$$  

That is, you can calculate that the 99th term in the staircase pattern at the start of Lesson 1-3 is 5050 without having to draw the pattern or write out all the previous terms.

A recursive formula is useful when the explicit formula is not known or is more difficult to use. For instance, an explicit formula for the Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, \ldots,$$

is quite complicated. However, by observing that each term from the third term on is the sum of the two previous terms, you can quickly calculate as many terms of the Fibonacci sequence as needed. A recursive definition for the Fibonacci sequence is

$$\begin{align*}
t_1 &= 1 \\
t_2 &= 1 \\
t_n &= t_{n-1} + t_{n-2}, \text{ for } n \geq 3.
\end{align*}$$

Later in this course you will learn how to convert certain recursive formulas into explicit formulas for $t_n$, and vice versa. But for now you only need to be able to evaluate both types of formulas and find simple recursive formulas.
Questions

In 1 and 2, write the first four terms of the sequence generated by the instructions.

1. Begin with 24; repeatedly press \[ \times \] 1.5 \[ = \].
2. Begin with 4; repeatedly press \[ - \] 9 \[ = \].

In 3–5, use the sequence 40, 20, 10, 5, ....

3. Write calculator instructions as in Questions 1 and 2, using multiplication to generate the first four terms.
4. The 10th term of this sequence is .078125. What is the 11th term?
5. Complete this program in BASIC so it will print the first 11 terms when run.

```
10 LET T = 40
20 FOR N = 1 TO a. ?
30 PRINT T;
40 LET T = b. ?
50 NEXT N
60 END
```

6. Suppose \( t_n \) is a term in a sequence. What symbol is used to represent the previous term?

In 7 and 8, write the first five terms of the sequence defined by the recursive formula.

7. The first term is 7; each term after the first is 10 more than the previous term.
8. \[ \begin{align*}
    t_1 &= 5 \\
    t_n &= 3t_{n-1}, & \text{for } n \geq 2.
\end{align*} \]

9. Write the next two terms in the Fibonacci sequence after 13.

10. **Multiple choice** Which recursive formula states that the \( n \)th term is four less than the previous term?
    (a) \( t_n = 4 - t_n \)  \hspace{1cm}  (b) \( t_n = 4 - t_{n-1} \)
    (c) \( t_n = t_{n-1} - 4 \)  \hspace{1cm}  (d) \( t_n = t_{n+1} - 4 \)

11. Consider the sequence that begins 100, 94, 88, 82, 76, ....
    a. The first term is ___.
    b. From the second term on, each term is ___ the previous term.
    c. Write a recursive formula for the sequence.
    d. Write a BASIC program that will print the first 20 terms.
12. a. Write the first four terms of the sequence defined by \( x_n = 3(4)^{n-1} \).
   b. Write the first four terms of the sequence defined by
      \[
      \begin{align*}
      y_1 &= 3 \\
      y_n &= 4y_{n-1}, \quad n \geq 2.
      \end{align*}
      \]
   c. True or false The sequences defined in parts a and b have the same terms.

13. What will be printed when this program is run?

   ```plaintext
   10 LET T = 1
   20 FOR N = 1 TO 12
   30 PRINT T;
   40 LET T = T + (2 * N + 1)
   50 NEXT N
   60 END
   ```

14. Suppose that a pair of rabbits one month old cannot produce baby rabbits, but every month later they produce a new pair of rabbits. Suppose that each new pair of rabbits behaves in the same way, and that none of the rabbits dies. Let \( r_n \) represent the number of pairs of rabbits at the start of the \( n \)th month. At the beginning of the first month, there is one pair of rabbits, i.e., \( r_1 = 1 \). Similarly, \( r_2 = 1 \). But \( r_3 = 2 \) because the original pair of rabbits produces a new pair of rabbits.

   Find the following:
   a. \( r_4 \quad b. \ r_5 \quad c. \ r_6 \quad d. \) a recursive formula for \( r_n \)

15. What is the domain for \( n \) in a formula that finds the \( n \)th term in a sequence? (Lesson 1-3)

16. True or false When a computer reads a FOR statement it immediately jumps to the NEXT statement, skipping intermediate lines. (Lesson 1-3)

17. Solve: \( 3 = 5p - 11 \). (Previous course)

18. Recall that the distance between two points with coordinates \((x_1, y_1)\) and \((x_2, y_2)\) is \( \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \), and the midpoint of the line segment joining them has coordinates \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \). (Previous course)

   a. Find the distance between \((-4, -7)\) and \((8, -2)\).
   b. Find the coordinates of the midpoint of the line segment with endpoints at \((-4, -7)\) and \((8, -2)\).

19. a. Give the first five terms of the sequence with the formula

   \[ t_n = n^4 - 10n^3 + 35n^2 - 49n + 24. \]

   b. What lesson should you learn from the answer to part a?
Algebra is a language. It has expressions and verbs. The rules for order of operations are rules for writing and speaking; you can think of them as part of the grammar of algebra. Like other languages, algebra has synonyms, expressions that mean the same thing. For instance, for any numbers \(a\) and \(b\), \(5a + b = b + 5a\).

But algebra is different from other languages in that, like geometry, it is part of a mathematical system following strict rules of logic. Statements in a mathematical system are either assumed true or proved true. If they are assumed, they must be either definitions of terms, or postulates, or part of the given of a particular situation or problem. If they are proved, they are called theorems. Postulates, theorems, and definitions are called properties.

You are familiar with the postulates of addition and multiplication of real numbers. You may have seen them many times. We list them here for reference. Notice that each postulate for addition corresponds to one for multiplication, except for the distributive property, which relates the two. These properties are customarily called the field properties.

### Postulate 1: Field Properties of Addition and Multiplication of Real Numbers

For any real numbers \(a\), \(b\), and \(c\):

| Closure: \(a + b\) is a real number. | Additive: \(a + b = b + a\) |
| Commutative: \(a + b = b + a\) | Multiplicative: \(ab = ba\) |
| Associative: \(a + (b + c) = (a + b) + c\) | \(a(bc) = (ab)c\) |
| Identity: \(a + 0 = a\) | \(a \cdot 1 = a\) |
| Inverse: \(a - a = 0\) | \(a \cdot \frac{1}{a} = 1\) |
| Distributive: \(a(b + c) = ab + ac\) |

The most important definitions in algebra are those of subtraction and division. Subtraction of real numbers is defined in terms of addition.

**Definition:** **Subtraction**

For any real numbers \(a\) and \(b\),

\[a - b = a + (-b)\]
You use this definition when you subtract positive and negative numbers:

\[ 3 - (-4) = 3 + 4 = 7 \]

Division of real numbers is defined in terms of multiplication.

**Definition:**

For real numbers \( a \) and \( b \), with \( b \neq 0 \),

\[ a \div b = a \cdot \frac{1}{b}. \]

You use this definition when you divide fractions:

\[ \frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12} \]

As for theorems, you know many of them, though when you learned them it is likely they were not proved. Here are a few of the more often used theorems.

**Some Theorems of Algebra**

Multiplication Property of 0: For all \( a \), \( a \cdot 0 = 0 \).

Multiplication Property of -1: For all \( a \), \( a \cdot -1 = -a \).

Opposite-of-an-Opposite Property: For all \( a \), \(-(-a) = a \).

(Op-Op Property)

Many theorems are based on the distributive property. Here is a list of some of the more important ones. Each can be called the Distributive Property, but we give it a special name.

**Some Theorems Derived from the Distributive Property**

Opposite of a Sum: For all \( b \), and \( c \):

\[ -(b + c) = -b + -c. \]

Distributive Property of Multiplication over Subtraction:

For all \( a \), \( b \), and \( c \):

\[ a(b - c) = ab - ac. \]

Addition of Like Terms: For all \( a \), \( b \), and \( c \):

\[ ac + bc = (a + b)c. \]

Addition of Fractions: For all \( a \), \( b \), and \( c \neq 0 \):

\[ \frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}. \]

A proof is an argument showing that a statement is true. As in geometry, reasons or justifications in a proof must be postulates, definitions, or theorems. It would take too much space and time to prove all the theorems you have already learned in your study of algebra. But you should be able to supply reasons in a proof from among the properties listed above. Supplying reasons is called justifying the steps.
Example 1  Supply justifications in this proof of the Addition-of-Fractions Theorem.

\[
\frac{a}{c} + \frac{b}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} \quad \text{a. } \quad \frac{1}{c} \cdot a + \frac{1}{c} \cdot b \\
= \frac{1}{c}(a + b) \quad \text{b. } \quad \frac{1}{c}(a + b) \\
= (a + b) \cdot \frac{1}{c} \quad \text{c. } \quad \frac{a + b}{c}
\]

Solution

a. The left side has division of \(a\) and \(b\) by \(c\). The right side converts the division to multiplication. This can be done because of the definition of division.

b. The factors in the multiplication are reversed. This is justified by the Commutative Property of Multiplication.

c. The factor \(\frac{1}{c}\) is distributed over the other two. This is justified by the Distributive Property.

d. Again the order of factors is switched, another use of the Commutative Property of Multiplication.

e. Here multiplication by \(\frac{1}{c}\) is converted to division by \(c\), the definition of division.

You see the phrase “for any” or “for all” preceding the properties mentioned above. This phrase means that any number or expression may be substituted for \(a\), \(b\), or \(c\). So even complicated looking expressions may be instances of these simple properties.

\[-(4x^2 - 14) = -4x^2 + 14 \quad \text{instance of Opposite-of-a-Sum-Theorem and definition of subtraction}\]

\[\sqrt{6y} - \sqrt{6y} = 0 \quad \text{instance of Inverse Property of Addition}\]

\[\frac{a}{y - 4} + \frac{7 - b}{y - 4} = \frac{a + (7 - b)}{y - 4} \quad \text{instance of Addition-of-Fractions Theorem}\]

The major use of these properties is to enable you to write expressions in simpler forms.
Example 2  Diane bought a ring for $P. After one year the ring increased in value 15%. Find an expression for the value of the ring after one year.

Solution  The value of the ring increases 15% of the original price, or .15P. Add the increase in value to the original price; the value is then $P + .15P$. This expression can be simplified.

\[
P + .15P = 1P + .15P \\
= (1 + .15)P \\
= 1.15P
\]

The answer is $1.15P$, or 1.15 times the original price. This is a simpler expression than $P + .15P$. It also shows that an increase of 15% is equivalent to multiplying by 1.15.

The purpose of a proof is to give a convincing argument. If you wish to be careful, put in all steps when simplifying. However, we encourage you to do steps in your head. The expert would go from $P + .15P$ to $1.15P$ in one step.

Questions

1. A postulate is a statement that is \(\square\) to be true, and a \(\square\) is a statement which can be proved.

2. Sara N. Dippity now earns $W per year. She will receive a 5% raise next year. Find an expression for the amount of money she will earn next year.

In 3–5, give an instance of the theorem.

3. Distributive property of Multiplication over Subtraction

4. Op-Op Property

5. Multiplication Property of -1

In 6–8, an instance of what property is given?

6. \(3 + -2y = 3 - 2y\)

7. \(4(x + 5) = 4x + 20\)

8. \(463a + 281a = 744a\)
9. Supply justifications in this proof that the formula for the area of a trapezoid.

\[ A = \frac{1}{2} h(b_1 + b_2). \]

This can be rewritten as

\[ A = \frac{h(b_1 + b_2)}{2}. \]

a. \( \frac{1}{2} h(b_1 + b_2) = \frac{1}{2} [h(b_1 + b_2)] \quad ? \\
\]

b. \( = [h(b_1 + b_2)] \cdot \frac{1}{2} \quad ? \\
\]

10. a. What number is the additive identity?
   b. What number is the multiplicative identity?

11. The distributive property can be used to prove the FOIL Theorem: for all \( a, b, c \), and \( d \), \((a + b)(c + d) = ac + ad + bc + bd\).

   Supply all reasons in this proof.

\[
(a + b)(c + d) = (a + b)c + (a + b)d \\
= (ac + bc) + (ad + bd) \\
= ac + (bc + ad) + bd \\
= ac + (ad + bc) + bd \\
= ac + ad + bc + bd
\]

In 12 and 13, use the FOIL Theorem to rewrite the product as a sum.

12. \((x + 3)(y - 2)\) 
13. \((8x - 4)(3x + 9)\)

14. Show that the expression \( \frac{180(n - 2)}{n} \) for the number of degrees in each angle of a regular \( n \)-gon can be written as \( 180 - \frac{360}{n} \).

15. \( P = 2L + 2W \)
16. \( S = 2\pi rh + \pi r^2 \)

In 17 and 18, use the distributive property to rewrite the formulas.

17. \( a_n = -1.2 + 2.4(n - 3) \)
18. \( t_n = 4(3 - n) - 2(n - 8) \)

In 19–21, simplify each expression.

19. \( \frac{3x + x}{2y} \)
20. \( \frac{1}{3}(2y + 4y) \)
21. \( (8x - 4) - (3x + 9) \)
In 22–24, use the definition of division to simplify the complex fraction. 

(Previous Course)

\[
22. \quad \frac{\frac{3}{4}}{\frac{4}{9}} \\
23. \quad \frac{\frac{c}{5}}{\frac{c}{3}} \\
24. \quad \frac{\frac{2a}{b}}{\frac{6a}{b}}
\]

25. What is the reciprocal of \( \frac{1}{a} \)?

26. Solve. \( 5x - 8 = \frac{x}{2} + 7 \) (Previous Course)

27. Simplify. \( \left( \frac{5}{x} \right)^0 \), \( x \neq 0 \) (Previous Course)

28. A full-grown tree is \( y \) meters tall. A newly planted tree is only \( x \) meters tall. How many times as tall as the newly planted tree is the full-grown tree? (Lesson 1-1)

29. Joe put \( d \) dollars into an account at 5.5% interest. How much interest will Joe earn in a year? (Lesson 1-1)

30. Describe the sequences printed by these two computer programs. (Lesson 1-3)

Program A

\[
\begin{align*}
10 & \text{ FOR } N = 1 \text{ TO } 100 \\
20 & \text{ PRINT } N + 100 \\
30 & \text{ NEXT } N 
\end{align*}
\]

Program B

\[
\begin{align*}
10 & \text{ FOR } N = 100 \text{ TO } 200 \\
20 & \text{ PRINT } N \\
30 & \text{ NEXT } N 
\end{align*}
\]

31. The field properties hold with other operations and sets. Here is a set, with two operations, * and @. The results are found in the tables. For instance, \( 1 * 2 = 0 \) and \( 1 @ 2 = 2 \).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>@</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Calculate \( 1 * 0 \) and \( 1 @ 0 \).
b. Which operation seems more like addition; which more like multiplication?
c. What feature of the table indicates that * is commutative?
d. Verify that \( 1@(2 @ 2) = (1@2) @ 2 \). What property have you verified?
e. Verify that \( 1@ (2 * 2) = (1 @ 2) * (1 @ 2) \). What property have you verified?
f. Make new tables for the two operations * and @ and the set \{0, 1, 2, 3, 4\} that would satisfy the field properties.
Consider this number trick.

Step 1: Pick a number.
Step 2: Subtract 5.
Step 3: Multiply the result by 3.
Step 4: Add 9 to the result.
Step 5: Divide the result by 3.
Step 6: Subtract the original number.

If you follow these steps with four arbitrary numbers, such as 3, -2, 1.4, and 10, the computations can be summarized as below.

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-2</td>
<td>-6</td>
<td>3</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>-7</td>
<td>-21</td>
<td>-12</td>
<td>-4</td>
<td>-2</td>
</tr>
<tr>
<td>1.4</td>
<td>-3.6</td>
<td>-10.8</td>
<td>-1.8</td>
<td>-0.6</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>15</td>
<td>24</td>
<td>8</td>
<td>-2</td>
</tr>
</tbody>
</table>

After looking at these instances, you might guess that the result is always -2. In doing this, you are making a conjecture, or an educated guess. In making a conjecture, you invent or are given some information about a situation. You study that information. Then you state something not given that you believe is true whenever the given is true. For instance, in this number trick it is not given that the final result is always -2. However, you have some evidence that the final result will always be -2.

After a conjecture is made, it is natural to ask whether it is always true. To do this, it is first necessary to do the trick in general. So we begin with a variable.

Step 1: Pick a number.
Step 2: Subtract 5.
Step 3: Multiply by 3.
Step 4: Add 9.
Step 5: Divide by 3.
Step 6: Subtract the original number.

\[
\begin{align*}
\text{Step 1: } & \quad \text{Pick a number.} \\
\text{Step 2: } & \quad \text{Subtract 5.} \\
\text{Step 3: } & \quad \text{Multiply by 3.} \\
\text{Step 4: } & \quad \text{Add 9.} \\
\text{Step 5: } & \quad \text{Divide by 3.} \\
\text{Step 6: } & \quad \text{Subtract the original number.}
\end{align*}
\]

Now the question is: Does this last expression always equal -2? For this, properties are needed. You are asked to supply the proof in the questions.

If you can prove a conjecture, then it is true. But not all conjectures are true. If you can find one counterexample to a conjecture, then it is false.
Example 1  Show by counterexample that

\[ t_n = 2^{n-1} \]

is not an explicit formula for the sequence

1, 2, 4, 8, 15,...

Solution  Only one counterexample is needed to show that the formula is wrong. The formula works for \( n = 1, 2, 3, \) or 4. The fifth term gives the counterexample, for when \( n = 5, \) \( t_n = 2^{5-1} = 16. \) In the given sequence, \( t_5 = 15. \)

Recall that an open sentence is a sentence that may be true or false depending on what values are substituted for the variables. For example, the open sentence \( x = 5 \) is false when 7 is substituted for \( x \) and true when 5 is substituted for \( x. \) When you find values of one or more variables that make an open sentence true, you are solving the sentence. The values that make the sentence true are called the solutions of the sentence.

Notice that the following are all true when \( x = 5. \)

\[
\begin{align*}
x &= 5 \\
2x &= 10 \\
2x + 7 &= 17
\end{align*}
\]

The above equations are examples of equivalent sentences. Equivalent sentences are sentences with the same solutions. Usually, when you solve equations you try to find progressively simpler equivalent sentences until you find one, like \( x = 5, \) that has 5 as its obvious solution.

Below are the two postulates that are often applied to find equivalent equations.

<table>
<thead>
<tr>
<th>Postulate 2: Properties of Equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>For all real numbers ( a, b, ) and ( c: )</td>
</tr>
<tr>
<td>Addition Property of Equality: If ( a = b, ) then ( a + c = b + c. )</td>
</tr>
<tr>
<td>Multiplication Property of Equality: If ( a = b, ) then ( ac = bc. )</td>
</tr>
</tbody>
</table>

It may surprise you that when you solve an equation, you are constructing the statements of a proof.
Example 2  Solve \(12 = 20 - 3t\).

Solution  We put in the statements and justifications. Solvers normally omit the justifications.

\[
12 = 20 + -3t \quad \text{definition of subtraction}
\]
\[-8 = -3t \quad \text{Addition Property of Equality (Add -20 to each side.)}
\]
\[-\frac{1}{3} \cdot -8 = -\frac{1}{3} \cdot -3t \quad \text{Multiplication Property of Equality}
\]
\[\frac{8}{3} = t \quad \text{definition of division (left side)}
\]
\[\text{Inverse Property of Multiplication (right side)}
\]

We have proved: If \(12 = 20 - 3t\), then \(t = \frac{8}{3}\).

Check  Substitute \(\frac{8}{3}\) for \(t\) in the original sentence. Use correct order of operations.

Does \(12 = 20 - 3\left(\frac{8}{3}\right)\)? Yes, because \(12 = 20 - 8\).

Recall from geometry that a conditional statement \(\text{if } p \text{ then } q\) (sometimes written \(p \Rightarrow q\)) has a related statement called its converse. The converse of \("if \ p \ then \ q\" \) is \("if \ q \ then \ p\"\).

In Example 2, the converse of \("if \ 12 = 20 - 3t, \ then \ t = \frac{8}{3}\"\) is \("if \ t = \frac{8}{3}, \ then \ 12 = 20 - 3t\"\).

In Example 2, the check of the solution is the proof of the converse. You need to prove both the statement and its converse to be certain you have solved an equation.

Questions

1. Define: conjecture.
2. What must be done to determine whether a conjecture is true?
3. What can be done to determine if a conjecture is false?
4. In Example 1, verify that the formula is true for \(n = 1\) and \(n = 4\).
5. What is an open sentence?
6. What is a solution to an open sentence?
7. Which of the following sentences are equivalent to $3x = 12$?
   a. $3x - 1 = 11$  
   b. $3x + 1 = 11$  
   c. $x = 4$

8. True or false Equivalent sentences have the same solution(s).

9. What two postulates are often used to find equivalent equations?

10. Give the converse of "If I am in Beijing, then I am in China."

11. What two statements must be proved in order to prove $12 = 20 - 3t$ if and only if $t = \frac{8}{3}$?

12. Prove: $7 + 4x = -13$ if and only if $x = -5$.

In 13 and 14, solve.

13. $200 - 3z = 300$  
14. $128 = 12n - 4$

In 15 and 16, which property was applied to derive the second equation from the first?

15. $3a + 17 = 5a$
   \[17 = 2a\]  
16. $16 = .01c$
   \[1600 = c\]

17. Prove that $2.6(3a - 4) = 9.1$ if and only if $a = 2.5$.

18. Show by counterexample that

   \[t_n = n^2 - n + 2\]

   is not a formula that generates the sequence

   $2, 4, 8, 15, 26, \ldots$. 

---

**LESSON 1-6  Reasoning in Algebra  35**
19. Find a counterexample to this statement: For all \( a, b, \) and \( x, (a + b)^x = a^x + b^x. \)

20. Supply reasons in this proof of the number trick conjecture from this lesson.

\[
\frac{3(n - 5) + 9}{3} - n = \frac{3(n - 5)}{3} + \frac{9}{3} - n
\]

\[= 3(n - 5) \cdot \frac{1}{3} + 3 - n\]

\[= (n - 5) \cdot 3 \cdot \frac{1}{3} + 3 - n\]

\[= (n - 5) \cdot 1 + 3 - n\]

\[= n - 5 + 3 - n\]

\[= n + (-5 + 3) + (-n)\]

\[= n + (-n) + (-5 + 3)\]

\[= 0 + (-5 + 3)\]

\[= -2\]

21. Consider the following number trick.

Step 1: Choose a number.

Step 2: Subtract 4.

Step 3: Multiply by 3.

Step 4: Add 9.

Step 5: Divide by 3.

Step 6: Add 1.

a. Try this trick with -3, 2, and 5.7.

b. Make a conjecture about this trick.

c. Show statements that prove your conjecture. (You need not put in reasons.)

22. Simplify. \( 2(3x - 5) - \frac{1}{4} (20 - 8x) \) (Lesson 1-5)

23. Write the first five terms of the sequence generated by this recursive formula: \( \begin{cases} t_1 = 3 \\ t_n = 4 - t_{n-1} \end{cases} \) (Lesson 1-4)

In 24–26, simplify. (Previous course, Lesson 1-5)

24. \( \frac{2}{5} \quad 25. \quad \frac{a}{b} \quad 26. \quad \frac{xy}{x} \)

27. Consider the sequence generated by the formula \( t_n = n^2 - n + 41. \)

a. Write a program in BASIC to print \( t_n \) for \( n = 1 \) to \( n = 50. \)

b. If you print the terms of the sequence, you will see that most of them are prime. Find a counterexample to the conjecture that all the terms are prime.
In this lesson you will examine several situations in which the Distributive Property or related theorems are used to solve equations.

When an equation must be solved for a variable in the denominator of a fraction, as illustrated in Example 1, the Multiplication Property of Equality is applied twice.

Example 1

The grade of a highway measures its steepness as a ratio of the vertical rise over the horizontal run. A highway has an average grade of 2%. Over what horizontal distance does a car travel when it rises 150 meters on this road?

Solution

Write a sentence representing this situation. Use the definition of grade:

\[ \text{grade} = \frac{\text{rise}}{\text{run}}. \]

Let \( x \) represent the run (horizontal distance). Then, since 2% = .02, we have:

\[ .02 = \frac{150}{x} \]

Multiply by \( x \).

\[ .02x = 150 \]

Divide by .02 (This is equivalent to multiplying by \( \frac{1}{.02} \)).

\[ x = \frac{150}{.02} = 7500 \text{ meters} \]

Check

We need to know: If a car travels 7500 meters and rises 150 meters, is the grade 2%?
Is \( \frac{150}{7500} = 2\%? \) Yes, \( \frac{150}{7500} = .02 = 2\% \).
So the car travels 7500 m to rise 150 m.
You can solve equations involving more than one fraction by multiplying each side of the equation by the least common denominator of the fractions. This process eliminates the fractions from the equation.

**Example 2** Stuart Dent works part time to earn spending money and to save for college. He spends $\frac{1}{4}$ of his monthly earnings on clothes, $\frac{1}{5}$ of his earnings on entertainment, and $\frac{1}{6}$ on transportation. He finds he has $46$ left each month for savings. What are his monthly earnings?

**Solution** First write a sentence to represent the situation. Let $E$ represent Stu’s monthly earnings. The total of all Stu’s expenditures and his savings equals his earnings.

$$\frac{1}{4}E + \frac{1}{5}E + \frac{1}{6}E + 46 = E$$

To eliminate or “clear” the fractions, multiply both sides of the equation by $60$, which is the least common denominator of $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$.

$$60(\frac{1}{4}E + \frac{1}{5}E + \frac{1}{6}E + 46) = 60E$$
$$15E + 12E + 10E + 2760 = 60E$$
$$37E + 2760 = 60E$$

Distributive Property
Addition of Like Terms

You want the variable terms on just one side of the equation, so add $-37E$ to both sides of the equation.

$$2760 = 23E$$
$$120 = E$$

Addition Property of Equality
Multiplication Property of Equality (Multiply by $\frac{1}{23}$)

Stu earns $120$ per month.

**Check** $\frac{1}{4}$ ($120$) = $30$ is spent on clothes.
$\frac{1}{5}$ ($120$) = $24$ is for entertainment.
$\frac{1}{6}$ ($120$) = $20$ is for transportation.
And $30 + 24 + 20 + 46 = 120$.

When an equation contains decimals, you can find an equivalent equation with whole-number coefficients by multiplying each side by a sufficiently large power of 10.
Example 3  Suppose $20,000 is to be invested in two accounts, one of which pays 6\% and one of which pays 8\%. If an annual income of $1500 is needed and s dollars are invested at 6\%, then

\[
.06s + .08(20,000 - s) = 1500.
\]

Solve this equation for s.

Solution  Multiplying both sides by 100 will clear decimals.

\[
100[.06s + .08(20,000 - s)] = 100 \cdot 1500
\]

Distribute the 100.

\[
6s + 8(20,000 - s) = 150,000
\]

Distribute the 8.

\[
6s + 160,000 - 8s = 150,000
\]

Add like terms.

\[
160,000 - 2s = 150,000
\]

Add 2s - 150,000 to each side; reverse order.

\[
s = 5,000
\]

Check  First, check the equation.

Does .06 \cdot 5000 + .08(20,000 - 5000) = 1500? Yes.

Now, check the situation.

If $5000 is invested at 6\% and $15,000 at 8\%, is the interest $1500? Yes, the first interest is $300, the second $1200.

In Example 3, it might seem silly to invest some funds at a lower rate. However, often a higher interest rate means a higher risk. So it is wise not to put all the money in one place.

The theorem from Lesson 1-5 about the opposite of a sum is often applied in solving equations.

Example 4  Solve $6m - (5 - 9m) = 12$.

Solution

\[
6m + -(5 + -9m) = 12 \quad \text{definition of subtraction (used twice)}
\]

\[
6m - 5 + 9m = 12 \quad \text{Opposite-of-a-Sum Theorem}
\]

\[
15m - 5 = 12 \quad \text{Addition of Like Terms}
\]

\[
15m = 17 \quad \text{Addition Property of Equality (Add 5.)}
\]

\[
m = \frac{17}{15} \quad \text{Multiplication Property of Equality (Multiply by \(\frac{1}{15}\)).}
\]

Check  Substitute $\frac{17}{15}$ for m and follow order of operations.

Does $6\left(\frac{17}{15}\right) - (5 - 9 \cdot \frac{17}{15}) = 12$?

Does $\frac{10}{15} - (5 - \frac{153}{15}) = 12$?

Does $\frac{10}{15} - (-\frac{78}{15}) = 12$?

Does $\frac{180}{15} = 12$? Yes.
1. Refer to Example 1. Suppose the grade of the highway is 1.5%. Over what distance must a car travel to rise 0.5 m on this road?

In 2-4, refer to Example 2.

2. True or false The equations
   \[ \frac{4}{5}E + \frac{1}{3}E + \frac{5}{6}E + 46 = E \]
   and \[ 60\left(\frac{4}{5}E + \frac{1}{3}E + \frac{5}{6}E + 46\right) = 60E \]
   are equivalent.

3. Why was each side of the original equation multiplied by 60?

4. Suppose that Stu gets a better paying job. Now he spends \(\frac{1}{3}\) of his earnings on clothes. All the other fractions stay the same and he still saves $46 per month. What is his new salary?

5. Solve the equation in Example 3 by following these steps in order, and write the result after each step.
   a. Distribute .08.
   b. Add like terms.
   c. Subtract 1600 from each side.
   d. Divide each side by the coefficient of \(s\).

6. Why is it often wise to invest money at different interest rates?

7. a. According to the Opposite-of-a-Sum Theorem, \(-(-2x + 9) = \_\_\_\_\_\_\_.\)
   b. Solve the equation \(12x - (-2x + 9) = 26\).

In 8–11, solve each equation.

8. \(\frac{12}{y} = 5\)

9. \(\frac{m}{3} + \frac{m}{7} = 1\)

10. \(4x - (x - 1) = 7\)

11. \(0.05x + 0.1(2x) + 0.25(100 - 3x) = 20\)

In 12 and 13, solve.

12. \(3y + 60 = 5y + 42\)

13. \(2z + 2 = 2 - 2z\)

14. A farmer grows three crops: wheat, corn, and alfalfa. He farms all the land he owns. On his farm \(\frac{1}{3}\) of the land is planted with wheat, \(\frac{2}{5}\) is planted with corn, and 60 acres are planted with alfalfa.
   a. How many acres of crops are on this farm?
   b. How many acres of wheat are there?
   c. How many acres of corn are there?

15. Given \(.07x + .05(100,000 - x) = 6500\).
   a. Solve the equation and check your solution.
   b. Make up a question that could be answered by solving the equation.
16. The \( n \)th term of the sequence 1000, 997, 994, 991, \ldots is given by the formula \( t_n = 1000 - 3(n - 1) \).
   a. Find \( n \) when \( t_n = 850 \).
   b. Which term of the sequence is 850?

17. The \( n \)th term of the sequence \( \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \) is given by the formula
   \[ a_n = \frac{1}{n + 2} \]

Which term of the sequence is \( \frac{1}{99} \)?

18. A cylindrical column of a building has a lateral area of 32.2 \( \text{ft}^2 \). If its radius is .5 \( \text{ft} \), what is its height? (Previous course) (Refer to the Appendix of Geometry Formulas if necessary.)

19. Use the sequence of right triangles below. (Previous course, Lessons 1-2, 1-3, 1-4)

   \[ \begin{array}{c}
   \text{T}_1 \\
   3 \hspace{1cm} 4 \\
   \end{array} \hspace{1cm} \begin{array}{c}
   \text{T}_2 \\
   6 \hspace{1cm} 8 \\
   \end{array} \hspace{1cm} \begin{array}{c}
   \text{T}_3 \\
   9 \hspace{1cm} 12 \\
   \end{array} \]

   a. Draw \( T_4 \) and label the lengths of the legs.
   b. Find the length of each of the four hypotenuses. (Use the Pythagorean Theorem if necessary.) Write your answers in a sequence.
   c. Multiple choice Which recursive formula below gives the sequence of the lengths of the four hypotenuses? (In all cases, \( n \geq 2 \).)
      \[ \begin{array}{l}
      (i) \begin{cases}
      t_1 = 1 \\
      t_n = t_{n-1} + 1 \\
      \end{cases} & (ii) \begin{cases}
      t_1 = 25 \\
      t_n = 5(t_{n-1}) \\
      \end{cases} \\
      (iii) \begin{cases}
      t_1 = 5 \\
      t_n = n + 5 \\
      \end{cases} & (iv) \begin{cases}
      t_1 = 5 \\
      t_n = t_{n-1} + 5 \\
      \end{cases}
      \end{array} \]

20. Simplify: \( \frac{x + 2}{3y} + \frac{2x + 7}{3y} \). (Lesson 1-5)

21. Which of the following sentences is (are) equivalent to \( 20(x + 5) = 55 \)? (Lesson 1-6)
   a. \( 4(x + 5) = 11 \) \hspace{1cm} b. \( x + 5 = \frac{11}{4} \) \hspace{1cm} c. \( 4(x + 1) = 11 \)

22. Explain in one or two sentences the difference between a conjecture and a theorem. (Lessons 1-5, 1-6)

23. The Greek mathematician Diophantus, who lived in the second century A.D., was the first person to replace unknowns by single letters. There is a famous problem by which you can calculate how long he lived. The Greek Authority states: "Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh more as a bachelor. Five years after his marriage was born a son who died four years before his father, at half his father's final age." How long did Diophantus live?

LESSON 1-7 Solving Equations 41
LESSON
1-8
Rewriting Formulas

You have studied how to use properties of algebra to simplify expressions and to solve equations. Algebraic properties also allow you to rewrite formulas in equivalent forms. Many times a formula is more useful if written in another form.

Consider the formula

\[ d = rt, \]

where \( d \) is the distance an object travels, \( r \) is the rate at which it travels, and \( t \) is the time the object travels. If you want to calculate the distance traveled on an airplane going at a rate of 650 \( \text{miles/hour} \) for 2.5 hours, this formula is immediately useful. It gives \( d \) in terms of the variables for which you have values.

\[ d = rt = (650 \text{ miles/hour})(2.5 \text{ hr}) = 1625 \text{ mi} \]

But suppose you wanted to know how much time you would need to travel 380 miles if you drive 50 \( \text{miles/hour} \). It might be helpful to have a formula that gives \( t \) in terms of \( d \) and \( r \). The properties allow you to rewrite \( d = rt \) as follows.

\[
\begin{align*}
\text{Divide both sides by } r. & \quad \frac{d}{r} = \frac{rt}{r} \\
\text{Simplify.} & \quad \frac{d}{r} = t
\end{align*}
\]

Because you now have a formula for \( t \), it is easy to answer the question asked earlier. Substitute \( d = 380 \text{ mi} \) and \( r = 50 \text{ miles/hour} \) to get

\[ t = \frac{d}{r} = \frac{380 \text{ mi}}{50 \text{ miles/hour}} = 7.6 \text{ hr}. \]

You would need 7.6 hours to travel 380 miles.
The formulas \( d = rt \) and \( t = \frac{d}{r} \) are equivalent. The first is solved for \( d \); the second is solved for \( t \). Notice that when a formula is solved for a variable, that variable has coefficient and exponent equal to one.

Example 1 illustrates that the most useful version of a formula might depend on your background and location.

Example 1  Pierre grew up in New Orleans, where he learned to tell temperature using the Fahrenheit scale. When he visited his cousin Rae in Montreal, Canada, he found that temperature was reported in degrees Celsius. Because Celsius temperature readings did not mean much to him, Pierre converted Celsius \( C \) to Fahrenheit \( F \) using this formula:

\[
F = 32 + 1.8C
\]

Rae visited Pierre the following summer. Rewrite the formula so she can use it to convert Fahrenheit to Celsius.

Solution  Given \( F = 32 + 1.8C \)

Subtract 32 from both sides. \( F - 32 = 1.8C \)

Divide both sides by 1.8. \( \frac{F - 32}{1.8} = C \)

So \( C = \frac{F - 32}{1.8} \) is an equivalent formula that is suitable for Rae.

Check  Evaluate the formula for a pair of temperatures you know are equivalent. The boiling point of water is 212° F or 100° C.

Does \( 100 = \frac{212 - 32}{1.8} \)? Yes.
Example 2 shows that you need to be careful when working with formulas that contain subscripted variables.

Example 2  Consider the sequence formula \( t_n = 4n + 11 \).

a. Solve the formula for \( n \).
b. Find \( n \) when \( t_n = 103 \).

Solution

a. \( t_n = 4n + 11 \)  Given
   \[ t_n - 11 = 4n \]  Addition Property of Equality (Add -11.)
   \[ \frac{t_n - 11}{4} = n \]  Multiplication Property of Equality (Multiply by \( \frac{1}{4} \)).

   Observe that \( t_n \) is a single variable. You cannot remove the subscript \( n \) from the \( t \).

b. Substitute 103 for \( t_n \) and evaluate.

\[
   n = \frac{t_n - 11}{4} = \frac{103 - 11}{4} = \frac{92}{4} = 23
\]

Thus 103 is the 23rd term.

Notice that to find a value of \( n \) for a given value of \( t_n \), it is much quicker to use \( n = \frac{t_n - 11}{4} \) than to use \( t_n = 4n + 11 \).

Sometimes you may want to rewrite a sentence without solving for a particular variable. Consider the next example.

Example 3  The formula

\[
   A = \frac{1}{2}d_1d_2
\]

gives the area of a rhombus in terms of the lengths of the diagonals \( d_1 \) and \( d_2 \). What is always true about the product of the diagonal lengths?

Solution  The product of the diagonal lengths is \( d_1d_2 \). Multiplying both sides of \( A = \frac{1}{2}d_1d_2 \) by 2 solves the equation for this expression. So

\[
   2A = d_1d_2.
\]

Thus, in a rhombus, the product of its diagonal lengths is twice its area.
Questions

Covering the Reading

In 1–3, refer to the formula \( d = rt \) at the beginning of the lesson.
1. How far can a race car traveling 190 mph go in 1.4 hr?

2. The formula \( t = \frac{d}{r} \) is solved for \( r \).

3. Find an equivalent formula that is solved for \( r \).

4. Refer to Example 1. Find the Celsius temperature equivalent to 86° F.

5. Refer to Example 3. If the area of a rhombus is 24 cm², find the product of its diagonal lengths.

Applying the Mathematics

6. Multiple choice Which formula is easiest to use if you want to find \( N \) and you are given \( L \) and \( h \)?
   (a) \( N = 7Lh \)
   (b) \( L = \frac{N}{7h} \)
   (c) \( h = \frac{N}{7L} \)

7. Multiple choice Which formula is easiest to use if you want to find \( T \) given \( L \)?
   (a) \( L = 100 + .04T \)
   (b) \( T = 25(L - 100) \)
   (c) \( .04T = L - 100 \)

8. Refer to the diagram below. Two angles with measures \( x \) and \( y \) are supplementary. To find \( y \) in terms of \( x \), the following steps are written. Justify each step.
   \[ x + y = 180° \]
   \[ x + y + -x = 180° + -x \]
   \[ x + -x + y = 180° + -x \]
   \[ 0 + y = 180° + -x \]
   \[ y = 180° + -x \]
   \[ y = 180° - x \]
   a. \( \frac{?}{?} \)
   b. \( \frac{?}{?} \)
   c. \( \frac{?}{?} \)
   d. \( \frac{?}{?} \)
   e. \( \frac{?}{?} \)

9. The formula \( C = 2\pi r \) gives the circumference of a circle in terms of its radius \( r \). Find the ratio of \( C \) to \( r \).

10. a. Solve the sequence formula \( a_n = 4n^2 \) for \( n \).
    b. What term of the sequence is 196?

11. The pitch \( P \) of a roof is a measure of the steepness of the slant of the roof. Pitch is defined as the ratio of the vertical rise \( R \) to the span \( S \) as shown in the sketch below.
    That is, \( P = \frac{R}{S} \).
    a. Solve this formula for \( S \).
    b. If a builder wants a roof to have a pitch of \( \frac{1}{6} \) and a rise of 5 feet, what must be the span of the building?
12. In the late 1660s, Isaac Newton formulated the law of universal gravitation described by the formula

\[ F = \frac{km_1m_2}{d^2}, \]

where \( F \) is the force between two bodies with masses \( m_1 \) and \( m_2 \), \( k \) is the gravitational constant, and \( d \) is the distance between the bodies.

a. Solve for \( m_1 \).

b. Solve the formula for the product of the masses.

13. Solve the sequence formula \( a_n = a_1 + (n - 1)d \) for \( n \).

14. **Skill sequence** Solve each equation. *(Lesson 1-7)*
   a. \( \frac{1}{3}n = 60 \)
   b. \( \frac{1}{3}n + 15 = 60 \)
   c. \( \frac{1}{3}n + \frac{1}{2}n + 15 = 60 \)

15. Solve. \( 5 - 2(x - 7) - (3 - x) = 4 \) *(Lessons 1-5, 1-7)*

16. Give the converse of: if \( 3x - 7 = 11 \), then \( x = 6 \). *(Lesson 1-6)*

17. A car is priced at \( d \) dollars. You can get it at 6\% off this price. Write an expression in terms of \( d \) to describe:
   a. the amount saved
   b. the new price *(Lessons 1-1, 1-5)*

18. Find a counterexample. For all \( x \), \( \frac{6 + x}{8 + x} \) is never equal to 2. *(Lesson 1-6)*

19. Let \( d_1 \) and \( d_2 \) be the lengths of the diagonals of rhombus \( RHOM \). Use the diagram below to explain why the area \( A \) of the rhombus equals \( \frac{1}{2}d_1d_2 \). *(Previous course)*

![Diagram of rhombus with diagonals]

20. **Multiple choice** The complex fraction \( \frac{\frac{a}{b}}{\frac{c}{d}} \) does not equal:

   (a) \( \frac{ad}{bc} \)  (b) \( \frac{a}{b} \div \frac{c}{d} \)  (c) \( \frac{ac}{bd} \)  (d) \( \frac{a}{b} \cdot \frac{d}{c} \)

21. In this lesson you have seen temperature measured on Celsius and Fahrenheit scales. In *War and Peace*, Leo Tolstoy describes a calm frost measured in "degrees Réaumur."
   a. Find out when and where the Réaumur scale was used.
   b. Compare the freezing and boiling points of water on each of the three scales.
Suppose that in order to qualify to compete in the long jump, you must jump at least 5 meters. Then, if \( J \) is the length of your jump, to qualify you must have \( J \geq 5 \). This open sentence is called an **inequality** because it contains one of the symbols \(<, \geq, \leq, \text{ or } \geq\). There are infinitely many solutions to the sentence \( J \geq 5 \), so they cannot all be listed. One way to describe all possible solutions is on a number line. The graph of the solutions to \( J \geq 5 \) is shown below.

The shaded circle means that 5 is included in the solution set. It is used in the graphs of inequalities of the form \( \leq \) or \( \geq \). An open circle at the endpoint indicates that the endpoint is not included in the solution set and is used for inequalities involving \(< \text{ or } \geq\). The graph below shows all solutions of \( m < -0.2 \).

In each case the solution is an **interval**. The inequalities \( J \geq 5 \) and \( m < -0.2 \) describe the intervals symbolically. The graphs describe them visually.

The Postulates of Inequality ensure that solving an inequality is very much like solving an equation.

**Postulate 3: Properties of Inequality**

For all real numbers \( a, b, \text{ and } c \):

**Addition Property of Inequality:**
- If \( a < b \), then \( a + c < b + c \).

**Multiplication Properties of Inequality:**
- If \( a < b \) and \( c > 0 \), then \( ac < bc \).
- If \( a < b \) and \( c < 0 \), then \( ac > bc \).
Example 1  Solve and graph all solutions to \(2m + 57 > 113\).

Solution  
\[
2m + 57 > 113 \\
2m > 56 \quad \text{Addition Property of Inequality (Add -57.)} \\
m > 28 \quad \text{Multiplication Property of Inequality (Multiply by \(\frac{1}{2}\))}
\]

The inequality \(m > 28\) is the simplest sentence equivalent to \(2m + 57 > 113\). It shows there is a set of solutions. This set of solutions is written

\[
\{m: m > 28\}
\]

and is read “the set of all \(m\) such that \(m\) is greater than 28.”

Check  First check the endpoint. Substitute 28 for \(m\) in the original sentence. This checks the endpoint. The two sides should be equal. Does \(2 \cdot 28 + 57 = 113\)? Yes.

Now check the direction of the inequality. Pick a value of \(m\) in the solution set. We pick \(m = 40\). Does it work? Is \(2 \cdot 40 + 57 > 113\)? Yes.

Notice that there are two Multiplication Properties of Inequalities in Postulate 3. Solving inequalities is different from solving equations only when you multiply or divide both sides of an inequality by a negative number. Then you must reverse the inequality sign.

Example 2  Solve and graph all solutions to \(-\frac{1}{3}R \geq 2\).

Solution  Be careful! Multiply both sides by -3 and reverse the inequality.

\[
-3(-\frac{1}{3}R) \leq -3(2) \\
R \leq -6
\]

Now simplify.

So the solution set is \(\{R: R \leq -6\}\). Here is a graph of the solution set:

Check  First substitute -6 for \(R\) to check the endpoint. Is \(-\frac{1}{3}(-6) = 2\)? Yes.

Next pick a different value of \(R\) in \(\{R: R \leq -6\}\). We pick -12.

Is \(-\frac{1}{3}(-12) \geq 2\)? Yes, \(4 \geq 2\), so the solution checks.

There are many applications for inequalities. The words “or less” might clue you to set up the inequality of Example 3.
Example 3

A man weighing 275 lb is starting a diet that allows him to lose 2 lb per week. This means after \( w \) weeks, his weight will be \( 275 - 2w \) pounds. If he is able to stick to the diet, when will his weight be 200 lb or less?

**Solution 1**

His weight will be 200 lb or less when \( w \) satisfies

\[
275 - 2w \leq 200.
\]

Subtract 275 from both sides.

\[-2w \leq -75\]

Divide both sides by -2 and reverse the inequality.

\[
\frac{-2w}{-2} \geq \frac{-75}{-2}
\]

\[
w \geq 37.5.
\]

**Solution 2**

To avoid multiplying by a negative number, add \( 2w \) to both sides of the inequality.

\[
275 \leq 200 + 2w
\]

\[
75 \leq 2w
\]

\[
37.5 \leq w
\]

Either way, the solution set to the inequality is \( \{w : w \geq 37.5\} \). Now you must interpret the solution in the context of the problem. The shortest time needed to bring his weight to 200 pounds is 37.5 weeks. If he continues longer on the diet, his weight will drop below 200 pounds.

**Questions**

1. **Multiple choice** Which inequality is a translation of “\( x \) is greater than or equal to 5”?
   (a) \( x \leq 5 \)  
   (b) \( 5 \leq x \)  
   (c) \( 5 \geq x \)  
   (d) \( x \geq 5 \)

2. Write an inequality for the set of numbers graphed below.

![Graph with numbers -60 to 30]

3. When you multiply or divide both sides of an inequality by a negative number, you must _change_ the inequality sign.

4. Graph all solutions to \( m < 42 \).

5. Translate into words: \( \{x : x < -3\} \).

6. To check that \( b \geq 9 \) is the simplest sentence equivalent to \( 5 - \frac{b}{3} \leq 2 \), what two values would you pick for \( b \)? Why must you pick two values?

7. Solve and graph all solutions to \( 0.2m < 1.4 \).

8. A woman weighing 190 lb plans to lose 2 lb per week. How long will it take her to bring her weight down to 135 lb or less?
Applying the Mathematics

In 9 and 10, (a) translate the English into an algebraic inequality; (b) graph all solutions to the inequality.

9. A soccer rulebook says the circumference of the ball shall not be more than 28 inches.

10. Gusts of over 80 kilometers per hour are expected.

11. Justify each step of the solution to \(7(w + 1) \leq 4(w - 2) + 3\).
   \[
   \begin{align*}
   7w + 7 & \leq 4w - 8 + 3 \\
   7w + 7 & \leq 4w - 5 \\
   3w + 7 & \leq -5 \\
   3w & \leq -12 \\
   w & \leq -4
   \end{align*}
   \]

   In 12 and 13, solve and graph all solutions to each inequality.

12. \(-4n - 5 > 1\) \hspace{2cm} 13. \(2(x + 1) \geq 3(1 - 4x)\)

14. Given \(a_n = 3 - 5n\). Find the first \(n\) for which \(a_n \leq -102\).

15. A truck weighs 5000 kg when empty. It is used for carrying 50 kg sacks of pistachio nuts.
   a. Write a sentence for the total weight \(T\) of the truck loaded with \(s\) sacks of pistachios.
   b. How many sacks of pistachios can the truck carry over a bridge with a weight limit of 8000 kg? [Hint: \(T\) must be less than or equal to the weight limit.]

16. Cheap Rentals rents cars at $10 per day plus $0.12 per mile. Ruby needs a car for four days. How many miles can she drive if the total cost of renting the car is not to exceed $100?

17. A formula for the area of a triangle is \(A = \frac{1}{2}bh\). Solve for \(b\).
   (Lesson 1-8)

18. Solve for \(y\). \(4x + y = 8 - y\) (Lesson 1-8)

In 19 and 20, solve and check. (Lesson 1-7)

19. \(18 = 15 - \frac{7}{a}\) \hspace{2cm} 20. \(4 - 3(x - 2) = 5(6 - x)\)

21. Find the degree measure of each angle in the figure at the right.
   (Previous course, Lesson 1-7)

22. a. Find a counterexample to this statement:
   If \(r\) and \(s\) are real numbers and \(r < s\), then \(r^2 < s^2\).
   b. Under what conditions is \(r^2 \geq s^2\)?
The language of algebra is based on numbers and variables. These are put together in expressions, and two expressions connected with a verb make up a sentence.

The order of operations and postulates are like a grammar; they tell what is allowed and what is not. Algebra also has its synonyms, in the form of equal expressions that take on the same values and equivalent equations that have the same solutions.

To use the language of algebra, you must be able to translate situations into it. This can be done by direct translation of words, by looking for patterns, or by using models for operations.

Algebra is a mathematical system, like geometry. It has definitions, postulates, and theorems. It also has conjectures and proofs.

Three kinds of postulates are discussed in this chapter. The postulates of addition and multiplication of real numbers help in obtaining equal expressions. The postulates of equality help in solving equations. Since formulas are equations, these postulates also help to solve formulas. The postulates of inequality help in solving inequalities. Subtraction is defined in terms of addition and division in terms of multiplication, so other postulates are not needed for these operations.

The formulas you have learned in previous years have been explicit formulas; one variable is given in terms of other variables. For sequences, there are also recursive formulas, in which terms of the sequence are determined by using the immediately preceding terms instead of being calculated directly. Computers can be programmed to work with either explicit or recursive formulas.

Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a general description and a specific example of each and a precise definition for those marked with an asterisk (*).

Lesson 1-1
*variable, algebraic expression, expression, model for an operation

Lesson 1-2
formula, equation, *domain, replacement set for a variable, evaluating an expression, *natural numbers, *counting numbers, *whole numbers, *integers, rational numbers, real numbers, order of operations

Lesson 1-3
sequence, term of sequence, subscript, index subscripted variable, explicit formula, nth term

Lesson 1-4
calculator key sequence, recursive formula, Fibonacci sequence

Lesson 1-5
definition, postulate, theorem, property
field properties of addition and multiplication of real numbers, proof

Lesson 1-6
conjecture, counterexample, open sentence solution to a sentence, equivalent sentences conditional statement; if \( p \), then \( q \), *converse Addition Property of Equality, Multiplication Property of Equality

Lesson 1-8
solving a formula for a specific variable

Lesson 1-9
inequality, interval, Addition Property of Inequality, Multiplication Properties of Inequality
Take this test as you would take a test in class. Use a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

In 1–5, use the two formulas below.

Sequence A | Sequence B
---|---
\[ t_n = 5 + 7n \] | \[ S_1 = 5 \]
\[ S_n = S_{n-1} + 7, \ n > 1 \]

1. Write the first four terms of sequence A.
2. Write the first four terms of sequence B.
3. Find \( t_8 \).
4. Calculate \( S_8 \).
5. Solve the explicit formula for \( n \).

6. What sequence will this program print when run?

```
10 FOR N = 1 TO 4
20   LET NTERM = 3*N + 40
30   PRINT NTERM
40 NEXT N
50 END
```

7. Multiply and simplify. \((2x - 3)(6x - 5)\)
8. Simplify: \(3(4 + a) - (25 - a)\).
9. If \( d = \frac{1}{2}gr^2 \), find \( d \) when \( g = 32 \) and \( r = 3 \).

In 10–12, solve.
10. \( 1.7y = 0.9 + 0.5y \)
11. \( \frac{7}{x} = 3 \)
12. \( .12x + .08(15,000 - x) = 1480 \)
13. Solve and graph the solution set. \( \frac{1}{2}P \geq 1 + p \)
14. Multiple choice Which of the following are not formulas?
   (a) \( x = 3a + y \)
   (b) \( 3E = E + 1 \)
   (c) \( \frac{a + 5}{2} \)
   (d) \( w = \frac{r + 5}{n} \)
15. Multiple choice Which formula below is solved for \( d \)?
   (a) \( 2A = d_1d_2 \)
   (b) \( d = rt \)
   (c) \( d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \)
16. Prove that this statement is not true by finding a counterexample. If \( t^2 = 9 \), then \( t = 3 \).

In 17–19, give a justification for each statement.
17. \( 3 + \sqrt{2} = \sqrt{2} + 3 \)
18. If \( 4(3x - 8) = 11 \), then \( 12x - 32 = 11 \).
19. \( \frac{a}{b} = a \cdot \frac{1}{b} \)

In 20 and 21, use the formula \( V = \frac{1}{3}\pi r^2h \) for the volume of a cone.

```
20. Find the volume of the cone above to the nearest cubic centimeter.
21. What is a reasonable domain for \( r \)?
22. Express your rate if you travel 12 miles in \( t \) hours.
23. John is 3 years older than Jane. If John is \( J \) years old, how old is Jane?
24. Write a recursive formula for the following situation. Cherlyn’s grandparents gave her $20 on her first birthday, $30 on her second, $40 on her third, and so on.
25. If the length of a rectangle is 8.3 m, what values of the width \( w \) will make the perimeter less than 21.7 m?
26. Name a real number that is not an integer.
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations.
The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

**Objective A Evaluate formulas.** *(Lessons 1-2, 1-3, 1-4)*

1. In the formula \( d = \frac{n(n - 3)}{2} \), find \( d \) when \( n = 17 \).
2. If \( d = \frac{1}{2}gt^2 \), find \( d \) when \( g = 32 \) and \( t = 2.5 \).
3. In the formula \( a_n = 20,000(.9)^n \), find \( a_{10} \) to the nearest hundredth.
4. If \( b_n = 2^n - 1 \), find \( b_4 \).
5. Write the first five terms in the sequence
   \[
   \begin{align*}
   t_1 &= 10 \\
   t_n &= t_{n-1} - 7, \quad n > 1.
   \end{align*}
   \]
6. If \( S_1 = 3 \) and \( S_n = 2S_{n-1} \) for \( n > 1 \), find \( S_5 \).

**Objective B Use computer programs to obtain terms of sequences.** *(Lessons 1-3, 1-4)*

7. What sequence will this program print when run?

8. Complete this BASIC program to find and print the first ten perfect squares.

9. Complete this BASIC program to find and print the first 100 terms of the sequence:
   \[20, 50, 80, 110 \ldots\]

10. **Objective C Simplify expressions by using field properties, definitions, or theorems derived from the Distributive Property.** *(Lesson 1-5)*
    
    In 10–16, simplify.
    10. \(-7 + 3(x - 4)\)
    11. \(2y + x - (y - 11)\)
    12. \(\frac{2}{3}(c + 4c) - \frac{c}{3}\)
    13. \(20 - 5(a + b)\)
    14. \((2m + 3)(m - 3)\)
    15. \((a + c)(b + d)\)
    16. \(\frac{3x + 6}{3}\)

**Objective D Solve and check linear equations and linear inequalities.** *(Lessons 1-7, 1-9)*

In 17–23, solve and check.

17. \(\frac{3}{2}x = 9\)
18. \(\frac{3}{10}(t - 20) = \frac{6}{5}\)
19. \(\frac{6}{U} = 8\)
20. \(2V + 17 \leq 23\)
21. \(8 - (w + 7) \leq 0.5\)
22. \(\frac{2}{3}(6r - 3) > -r\)
23. \(.05(4500 - x) + .08x = 1200\)
Objective E  Rewrite formulas. (Lesson 1-8)

24. Solve for \( n \) in the formula \( t_n = 4 - 5n \).

25. The measure of an exterior angle of a regular polygon, \( \theta \), is given by

\[
\theta = \frac{360}{n}
\]

where \( n \) is the number of sides. Solve for \( n \).

26. Recall the formula \( d = \frac{1}{2}gr^2 \). What is the ratio of \( d \) to \( r^2 \)?

27. If \( x = 3y \), then \( \frac{x}{y} = \) ?

28. Which equation(s) is (are) solved for \( t \)?

(a) \( t = \frac{D}{R} \)

(b) \( 10t - 5r^2 = h \)

(c) \( A_t = \frac{1}{2}h(b_1 + b_2) \)

(d) \( 180(n - 2) = t \)

Objective F  Use counterexamples to show errors in reasoning. (Lesson 1-6)

In 29–32, find a counterexample to disprove the conjecture.

29. Division is associative, that is, for all \( a \), \( b \), and \( c \): \( (a + b) + c = a + (b + c) \).

30. For all \( x \), \( x^2 \geq 1 \).

31. If \( m^4 = 16 \), then \( m = 2 \).

32. The terms of the sequence with formula \( t_n = n^2 - n + 11 \) are all prime.

Objective G  Identify justifications in mathematical arguments. (Lessons 1-5, 1-6, 1-9)

In 33–36, identify the property used to get from the first step to the second.

33. \( 5x + 17 = 85 \)

\( 5x = 68 \)

34. \( S = 2\pi r^2 + 2\pi rh \)

\( S = 2\pi r(r + h) \)

35. \( \frac{1}{2}d < 9 \)

\( d < 18 \)

36. \( h^2 - 3h + 4h + 5 = 0 \)

\( h^2 + h + 5 = 0 \)

37. Justify each step in this rewriting of \( -(y - x) \).

a. \( -(y - x) = -(y + -x) \)

b. \( = -y + (-x) \)

c. \( = -y + x \)

d. \( = x + -y \)

e. \( = x - y \)

Objective H  State the domain for a variable in a given situation. (Lessons 1-2, 1-3)

In 38–40, what is an appropriate domain for \( n \) in the following?

38. the formula for the number of diagonals \( d = \frac{n(n - 3)}{2} \), where \( n \) is the number of sides in the polygon

39. an explicit formula for \( t_n \) in a sequence

40. \( C = 1.30n \), the cost of a piece of beef weighing \( n \) pounds
**Objective I** Use sequences in real-world situations. (Lessons 1-3, 1-4)

41. Sandra’s annual salary is $26,000. She gets an increase of 6% at the end of each year. The sentence \(a_n = 26,000(1.06)^{n-1}\) gives Sandra’s salary at the end of \(n\) years.
   a. Write Sandra’s salary for the first five years.
   b. At this growth rate, what would Sandra’s salary be after 20 years with this company?

42. Devin opened a savings account with $50. Each month he adds $15 to his account.
   a. Write the amounts in this account for the first six months.
   b. Write a recursive formula that generates the sequence giving his savings account balance.

**Objective J** Use models for the four fundamental operations to describe situations. (Lessons 1-1, 1-7, 1-9)

43. The number of irrigated acres is \(I\) and the total number of acres is \(T\). How many acres are not irrigated?

44. The dimensions of a building are 100 times as large as the dimensions of its model. If a floor on the model is \(x\) cm long, how long is the floor on the building?

45. There are \(s\) students per bus and \(b\) buses. How many students are there in all?

46. Carol takes \(M\) minutes to walk \(B\) blocks. What is her walking speed?

47. A baby blue whale weighs 4000 lb at birth and gains 200 lb a day while nursing. Then a formula that gives its weight \(W\) after \(d\) days of nursing is \(W = 4000 + 200d\).
   a. Write an inequality that can be used to find the number of days a young blue whale has been nursing if it weighs at least 14,000 lb. (Baby blue whales nurse for 5 to 7 months.)
   b. Solve this inequality.

48. At Central High School all students are in grades 10, 11, or 12. This year \(\frac{2}{3}\) of the students are in grade 10, \(\frac{1}{3}\) are in grade 11, and 320 are in grade 12. How many students are at Central High this year?

**REPRESENTATIONS** deal with pictures, graphs, or objects that illustrate concepts.

**Objective K** Graph solutions to inequalities on a number line. (Lesson 1-9)

49. Graph all solutions to \(y \leq -6\).

50. Solve and graph the solutions: \(4x + 12 > 22\).

51. **Multiple choice** Which inequality is graphed below?

   ![Inequality Graph](image)

   (a) \(-7 > x\)  
   (b) \(-7 < x\)  
   (c) \(-7 \leq x\)  
   (d) \(-7 \geq x\)

52. Write an inequality that describes the graph below.

   ![Inequality Graph](image)
A construction worker walking on a board of a scaffold knows that too much weight will break the board. The largest weight that can be safely supported by a board depends on its width \( w \), thickness \( t \), and on the distance \( d \) between the board's supports.

Using wider or thicker boards makes the scaffold stronger. Increasing the distance between the supports weakens the scaffold. Thus the strength varies as the distance is increased or as the dimensions of the board change. But how does it vary—a lot or a little? In this chapter, you will study variation, which examines how one quantity changes as others are changed.

The chapter begins with a simple relationship called direct variation.
In many places, you can get refunds for returning aluminum cans. For example, in New York, starting in 1987, you would get 5¢ per can returned. Thus, if \( r \) is the refund in cents and \( a \) the number of cans you returned, then

\[ r = 5a. \]

Doubling the number of cans returned doubles your refund. Tripling the number of cans returned triples your refund. We say that \( r \) varies directly as \( a \).

In Michigan you would get 10¢ per can returned, so \( r = 10a \). Again, doubling the number of cans would double the refund; again \( r \) varies directly as \( a \).

With the formula \( A = \pi r^2 \) for the area of a circle, as the radius \( r \) increases, the area \( A \) also increases. But if the radius is doubled, the area is quadrupled. Tripling the radius multiplies the area by 9.

In the formula \( A = \pi r^2 \), \( A \) varies directly as \( r^2 \). Often this wording is used: The area \( A \) varies directly as the square of \( r \).

\[ A = \pi r^2 \quad A = 9\pi r^2 \]

The formulas \( r = 5a \), \( r = 10a \), and \( A = \pi r^2 \) are all of the form \( y = kx^n \), where \( k \) is a nonzero constant, called the constant of variation, and \( n \) is a positive number. They are all direct-variation formulas.
**Definition:**

A direct-variation formula is of the form \( y = kx^n \), with \( k \neq 0 \)
and \( n > 0 \)

When \( y \) varies directly as \( x^n \) we also say \( y \) is **directly proportional**
to \( x^n \). For instance, the formula \( A = \pi r^2 \) can be read “the area of a circle is directly proportional to the square of its radius.” Here \( n = 2 \) and \( k = \pi \), so \( \pi \) is the constant of variation. In the formulas \( r = 5a \) and \( r = 10a \), \( n = 1 \) and the constant of variation is 5 for New York and 10 for Michigan.

Direct variation formulas occur often. For instance, after applying the brakes, the braking distance \( d \) needed to stop a car is directly proportional to the square of its speed \( s \).

\[ d = k s^2 \]

The value of \( k \) depends on the type of car, the condition of the brakes, and the condition of the road.

**Example 1**

A certain car needs 25 ft to come to a stop if the brakes are applied at 20 mph. Assume that braking distance \( d \) and speed \( s \) satisfy the equation \( d = k s^2 \).

**a.** Find \( k \), and write the specific direct variation formula relating \( s \) and \( d \).

**b.** Find the distance needed to stop this car after the brakes are applied at 60 mph.

**Solution**

**a.** You are given that \( d = 25 \) ft when \( s = 20 \) mph. To find \( k \), substitute these values into the given equation \( d = k s^2 \).

\[ 25 = k \cdot 20^2 \]
\[ 25 = 400k \]
\[ k = \frac{1}{16} \]

Substituting \( k = \frac{1}{16} \) into \( d = k s^2 \) gives

\[ d = \frac{1}{16} s^2 \]

as a formula relating speed and braking distance for this situation.
b. Evaluate the formula when \( s = 60 \text{ mph} \).
\[
\begin{align*}
d & = \frac{1}{16}(60)^2 \\
& = \frac{1}{16}(3600) \\
& = 225
\end{align*}
\]

Note that according to this formula, this car will need 225 ft to come to a stop after the brakes are applied at 60 mph.

**Check** The speed 60 mph is 3 times faster than 20 mph. The braking distance is \( 3^2 \) or 9 times farther. That is what you would expect for a direct variation in which \( d \) varies directly as the square of \( s \).

Notice that to use variation to predict values, you carry out three steps.

1. Find the constant of variation.
2. Rewrite the variation formula using the constant.
3. Evaluate the formula.

In the variation formula \( y = kx^n \), the value of \( y \) always depends on the value of \( x \). For this reason, \( y \) is called the **dependent variable** and \( x \) the **independent variable**. In Example 1, the braking distance \( d \) depends on the speed \( s \). In that situation, \( d \) is the dependent variable and \( s \) is the independent variable.

**Example 2** The weight \( w \) of an adult animal of a given species is known to vary directly with the cube of its height \( h \).

a. Write an equation relating \( w \) and \( h \).
b. Which is the dependent variable and which is the independent variable?

**Solution**
a. An equation for the direct variation is \( w = kh^3 \).
b. The dependent variable is \( w \) and the independent variable is \( h \).

**Questions**

1. State an example from geometry of a direct variation formula.

2. In the formula \( y = 3x^5 \), \( ? \) varies directly as \( ? \) and \( ? \) is the constant of variation.

In 3 and 4, assume that \( y \) is directly proportional to the square of \( x \).

3. **Multiple choice** Which equation represents this situation?
   \[
   \begin{align*}
   (a) \quad & y = 2x \\
   (b) \quad & y = kx^2 \\
   (c) \quad & x = ky^2 \\
   (d) \quad & y = 2x^k
   \end{align*}
   \]
4. Which is the dependent variable?

5. Describe a situation in which the variation formula \( r = .05n \) could be used.

In 6 and 7, suppose \( y = -10x \).
6. Is this an example of direct variation?

7. Find \( y \) when \( x = 12 \).

8. What three steps can be followed to predict values using variation?

In 9 and 10, refer to Example 1.
9. Find the distance needed to stop the car if its brakes are applied at 40 mph.

10. Suppose that some other car needs 30 ft to stop if its brakes are applied at 20 mph. What distance would it need to stop if its brakes are applied at 60 mph?

11. Suppose \( W \) varies directly as the fifth power of \( z \) and \( W = 96 \) when \( z = 2 \).
   a. Find the constant of variation.
   b. Find \( W \) when \( z = 10 \).

12. The power \( P \) generated by a windmill is directly proportional to the cube of the wind speed \( w \).
   a. Write an equation relating \( P \) and \( w \).
   Which of these is the dependent and which the independent variable?
   b. If a 10 mph wind generates 150 watts of power, how many watts will a 6 mph wind generate?

In 13 and 14, recall that when lightning strikes in the distance you do not see the flash and hear the thunder at the same time. You first see the lightning. Then you hear the thunder.

13. Write an equation to express this situation: "The distance \( d \) from the observer to the flash varies directly as the time \( t \) between seeing the lightning and hearing the thunder."

14. Suppose that lightning strikes a known point 4 miles away, and that you hear the thunder 20 seconds later. Then, how far away has lightning struck if 30 seconds pass between seeing the flash and hearing the thunder?

15. Refer to the formula \( d = \frac{1}{16}s^2 \) from Example 1.
   a. Pick a value for \( s \) and calculate the braking distance for that speed.
   b. Calculate the braking distance for twice that speed.
   c. According to your answers to parts a and b, when a car doubles its speed, it needs \( \_\_ \_ \) times the braking distance to stop.
In 16–18, refer to the graph below of the profits of the Acme Gadget Company. (Previous courses)

Profits (in $100,000)

16. About how much profit did the company make in 1981?
17. In what year(s) did the company lose money?
18. During what time periods did the profits decline?

In 19 and 20, copy the graph below. Recall that a size change of magnitude \( k \) means that coordinates are multiplied by \( k \). Graph the image of the given figure under a size change of the given magnitude. (Previous courses)

19. magnitude 2 to the circle
20. magnitude \( \frac{1}{3} \) to the triangle

In 21–23, write as a power of 3. (Previous course)

21. \( 3^2 \cdot 3^4 \)  
22. \( \frac{3^9}{3^2} \)  
23. \( (3^3)^5 \)

24. The speed of sound in air is about 1088 ft per second and the speed of light is about 186,000 miles per second.
   a. Convert the speed of sound to miles per second.
   b. Use your answer from part a to find the time it takes sound to travel four miles. Compare this answer to the values in Question 14.
   c. What environmental conditions affect the speed of sound?

25. Find a manual for learning how to drive. Often these manuals contain charts of stopping and braking distances. Do the braking distances given there vary directly as the square of the speed?
In direct variation, as one variable increases in absolute value, so does the other. In another type of variation, as one variable increases in absolute value, the other decreases in a specific way. Here is an example.

Metro Car Sales hires students to wash cars in their display lots. The manager knows from experience that 36 students can wash all the cars in one hour. When fewer students work, each student needs to work more hours. If $s$ equals the number of students who work and $t$ equals the time (in hours) each student needs to work, then by the rate-factor model of multiplication,

$$st = 36 \text{ or } t = \frac{36}{s}.$$

Some combinations of $s$ and $t$ that might be used to finish the job are given in the chart below.

<table>
<thead>
<tr>
<th>$s$</th>
<th>4</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>9</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Notice that as $s$ is doubled (say from 4 to 8), $t$ is halved (from 9 to 4). This formula is of the form $y = \frac{k}{x^n}$ and is one instance of inverse variation. We say $t$ varies inversely as $s$. In this example, the constant of variation $k$ is 36.

**Definition:**

An inverse-variation formula is of the form $y = \frac{k}{x^n}$, with $k \neq 0$ and $n > 0$.

When $y$ varies inversely as $x^n$, we also say $y$ is inversely proportional to $x^n$. As with direct variation, inverse variation occurs in many kinds of situations.

**Law of the lever:** To balance a seesaw, the distance $d$ a person is from the pivot is inversely proportional to his or her weight $w$.

$$d = \frac{k}{w}$$

**Newton's Law of Universal Gravitation:** The weight $W$ of a body varies inversely with the square of its distance $r$ from the center of Earth.

$$W = \frac{k}{r^2}$$
The number \( n \) of oranges you can pack in a box varies inversely with the cube of an orange's diameter \( d \).

\[
n = \frac{k}{d^3}
\]

In the picture below Nancy and Sam are trying to balance on a seesaw.

Example 1
Sam, who weighs 40 pounds, is sitting 6 feet from the pivot.

a. Use the law of the lever to find the constant of variation for this situation.

b. Nancy weighs 45 pounds. How far away from the pivot must she sit to balance Sam?

Solution

a. Let \( d \) = a person's distance in feet from the pivot. Let \( w \) = the person's weight in pounds.

The law of the lever states that on a seesaw, the distance varies inversely as the weight, or

\[
d = \frac{k}{w}
\]

To find \( k \), substitute Sam's weight and distance into this equation.

\[
6 = \frac{k}{40}
\]

\[
6 \cdot 40 = k
\]

\[
240 = k \quad \text{(The unit for \( k \) is "foot-pounds." )}
\]

b. You must find \( d \) when \( w \) equals 45 pounds. The work in part a tells you that the variation formula for this situation is

\[
d = \frac{240}{w}
\]

Evaluate this formula when \( w = 45 \) lb.

\[
d = \frac{240}{45}
\]

\[
d = 5\frac{1}{3}
\]

Nancy must sit \( 5\frac{1}{3} \) feet away from the pivot to balance Sam.

Check Does 6 ft \( \cdot \) 40 lb = \( 5\frac{1}{3} \) ft \( \cdot \) 45 lb? Yes.
You have probably seen pictures of astronauts floating almost weightless in space. Newton’s Law of Universal Gravitation, which is an example of an inverse-square variation, can be used to calculate an astronaut’s weight.

**Example 2** If an astronaut weighs 135 pounds on Earth’s surface, what will the astronaut weigh 18,000 miles above Earth’s surface? (The radius of Earth is approximately 4000 miles.)

**Solution** Let \( W = \) the weight of the astronaut in pounds. Let \( r = \) the distance from the center of Earth to the astronaut. Since \( W \) varies inversely as the square of the distance,

\[
W = \frac{k}{r^2}.
\]

An astronaut 18,000 miles above Earth’s surface is 22,000 miles from the center of the earth. You need to find \( W \) when \( r = 22,000 \).

First find \( k \), the constant of variation, by substituting the values \( W = 135 \) lb and \( r = 4000 \) mi.

\[
135 = \frac{k}{(4000)^2}
\]

\[
135 \cdot 4000^2 = k
\]

\[
135 \cdot 16,000,000 = k
\]

For \( k \) so large, a calculator may use scientific notation. We can write

\[
k = 2.16 \cdot 10^9.
\]

Then

\[
W = \frac{2.16 \cdot 10^9}{r^2}.
\]

Substitute \( r = 22,000 \) into the inverse-square formula and solve for \( W \).

\[
W = \frac{2.16 \cdot 10^9}{(22,000)^2}
\]

\[
\approx 4.4628 \text{ lb}
\]

At 18,000 miles above Earth’s surface, the astronaut weighs only about 4.5 pounds.
Questions

1. The time to finish the job varies inversely as the ___.

2. Only 12 students are found to work. How long will it take them to complete the job?

3. Which equation does not represent an inverse variation? (k is a constant.)
   a. \( y = kx \)  
   b. \( y = \frac{k}{x} \)  
   c. \( xy = k \)  
   d. \( y = \frac{k}{x^2} \)

4. The equation \( y = \frac{k}{x^3} \) means \( y \) varies inversely as ___.

At 70 kg, 2.5 m, 90 kg, find the distance needed to balance the seesaw.

5. If Sam sits 5 feet from the pivot, how far away from the pivot must Nancy sit to balance him?

6. Find the distance needed to balance the seesaw below.


8. Find the weight of an astronaut in a space lab 300 miles above Earth if the astronaut weighs 150 lb on Earth.

9. Suppose that \( y \) varies inversely as \( x^3 \) and that \( y = 10 \) when \( x = 4 \). Find the value of \( y \) when \( x = 2 \).

10. The time \( t \) an appliance can be run on 1 kilowatt hour of electricity is inversely proportional to the wattage rating \( w \) of the appliance.

11. The intensity \( I \) of light varies inversely as the square of the observer’s distance \( D \) from the light source.

12. Suppose in Question 11 that the light intensity is 30 lumens when the observer is 6.7 meters from the light.
   a. Find the constant of variation.
   b. Find the light intensity when the distance between the observer and the light is 20 meters.
In 13–15, complete the sentence with the word “directly” or “inversely.”

13. The volume of a sphere varies \_\_\_ as the cube of its radius.

14. At a given time of day, the height of a tree varies \_\_\_ as the length of its shadow.

15. The number of tiles needed to tile a floor varies \_\_\_ as the square of the length of a side of the tile.

16. At some restaurants, the price of a pizza varies directly with the square of its diameter. If you pay $5.95 for a cheese pizza with a 10-inch diameter, how much should a 14-inch-diameter cheese pizza cost? (Lesson 2-1)

17. In 1985, the basketball player Manute Bol was 7'6" tall, and he weighed only about 200 pounds. Recall that a person’s weight varies directly as the cube of his or her height. How much would you expect a man 5'10" tall with Manute’s shape to weigh? (Lesson 2-1)

18. Line \textit{l} is parallel to line \textit{m} in the figure below. The expressions represent angle measures. Find \(y\). (Previous courses, Lesson 1-6)

\[ \text{(2x-2)}^\circ \text{ } \text{(4x-104)}^\circ \]

19. Solve: \(7y + 2(4y + 1) \leq 10y\) (Lesson 1-9)

In 20–22, simplify (Previous course)

20. \(x^{10} \cdot x^3\) \hspace{1cm} 21. \(\frac{x^{12}}{x^4}\) \hspace{1cm} 22. (2x)^3

23. Besides the Law of Gravitation, Isaac Newton discovered several other laws of classical physics. He was also very influential in the world of mathematics. Write a report about at least one other contribution of this remarkable person.
The Fundamental Theorem of Variation

In Lesson 2-1 you learned that after applying brakes, the distance \( d \) needed to stop a car is directly proportional to the square of its speed \( s \). The braking distance \( d \) (in feet) and the speed \( s \) (in miles per hour) for a certain car were related by the equation \( d = \frac{1}{16} s^2 \). Some values of \( d \) and \( s \) are given in the following table.

<table>
<thead>
<tr>
<th>( s )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>6.25</td>
<td>25</td>
<td>56.25</td>
<td>100</td>
<td>156.25</td>
<td>225</td>
<td>400</td>
</tr>
</tbody>
</table>

What happens to the braking distance when the speed is tripled? One way to answer this question is to compare values from the table when the speed is tripled. For example,

- if you go 20 mph, you need 25 feet to stop;
- and
- if you go 60 mph, you need 225 feet to stop.

Notice that when \( s \) is tripled, \( d \) is multiplied by nine. This pattern also holds if you compare the ordered pairs \((10, 6.25)\) and \((30, 56.25)\) from the above table:

\[
30 = 3 \cdot 10
\]

and

\[
56.25 = 9 \cdot 6.25
\]

Based on the two instances above, it seems reasonable to conjecture that if you triple a car’s speed the braking distance needed to brake to a stop is multiplied by nine. The following example shows how to prove this conjecture.

**Example 1** Given the direct variation formula \( d = \frac{1}{16}s^2 \). Prove that if \( s \) is tripled, \( d \) is multiplied by nine.
Solution  Let \( d_1 \) be the original distance (before tripling the speed) and let \( d_2 \) be the distance after tripling. To find \( d_2 \), \( s \) must be tripled. So replace \( s \) by \( 3s \). Here a proof is given in two-column form.

1. \( d_1 = \frac{1}{16} s^2 \)  
given
2. \( d_2 = \frac{1}{16} (3s)^2 \)  
substitution
3. \( = \frac{1}{16} \cdot 9s^2 \)  
Power-of-a-Product Property
4. \( = 9 \cdot \frac{1}{16} s^2 \)  
Associative and Commutative Properties of Multiplication
5. \( = 9 \cdot d_1 \)  
substitution (step 1 into step 4)

Example 1 illustrates the following general theorem.

The Fundamental Theorem of Variation:

a. If \( y \) varies directly as \( x^n \) and \( x \) is multiplied by \( c \), then \( y \) is multiplied by \( c^n \).

b. If \( y \) varies inversely as \( x^n \) and \( x \) is multiplied by a nonzero constant \( c \), then \( y \) is divided by \( c^n \).

Proof:

We give the proof in paragraph form.

a. If \( y \) varies directly as \( x^n \) and \( y_1 \) is the original value of \( y \), then

\[
y_1 = k x^n.
\]

When \( x \) is multiplied by \( c \), a new value \( y_2 \) is generated and

\[
y_2 = k (cx)^n.
\]

Applying the Power-of-a-Product and the Associative and Commutative Properties gives

\[
y_2 = k (c^n x^n) \quad = c^n (kx^n) \quad = c^n y_1.
\]

b. If \( y \) varies inversely as \( x^n \), then

\[
y_1 = \frac{k}{x^n}.
\]

When \( x \) is multiplied by \( c \), a new value \( y_2 \) is generated. By an argument similar to the one above, you can show that

\[
y_2 = \frac{y_1}{c^n}.
\]

You are asked to supply the necessary steps in Question 15 at the end of the lesson.
The Fundamental Theorem of Variation can be applied in many situations.

**Example 2**  In Lesson 2-2 you were given \( d = \frac{k}{w} \) as the law of the lever. Nathan weighs twice as much as his daughter Stephie. Compare their distances from the pivot when they are balanced on a seesaw.

**Solution**  Apply the Fundamental Theorem of Variation. Because Nathan weighs twice as much as Stephie, you are asked to find the effect of replacing \( w \) with \( 2w \) in the variation formula \( d = \frac{k}{w} \). This is an inverse-variation equation with \( n = 1 \).

So when \( w \) is multiplied by 2, \( d \) is divided by 2. Thus Nathan’s distance from the pivot is half that of Stephie’s.

**Questions**

In 1 and 2, refer to the formula \( d = \frac{1}{16}s^2 \) and the table on speeds and braking distances at the start of this lesson.

1. The pairs (20, 25) and (40, 100) illustrate the pattern that if the car’s speed is doubled, the braking distance is multiplied by ___?

2. Find two pairs of numbers that illustrate this result: if the car’s speed is multiplied by four, then its braking distance is multiplied by 16.

3. If \( y = kx^n \) and \( x \) is multiplied by \( c \), then \( y \) is ___?

4. If \( y = \frac{k}{x^n} \) and \( x \) is multiplied by \( c \) \((c \neq 0)\), then \( y \) is ___.

5. Refer to Example 2. Suppose Nathan weighs three times as much as his niece Oprah. Compare Nathan’s and Oprah’s distances from the pivot when they are balanced.

In 6 and 7, \( y = 5x^4 \).

6. Describe the change in \( y \) when \( x \) is tripled.

7. What happens to \( y \) when \( x \) is divided by three?
In 8–10, state the effect that halving the \( x \)-values (multiplying them by \( \frac{1}{2} \)) would have on the \( y \)-values.

8. \( y = 10x \)  
9. \( y = 10x^2 \)  
10. \( y = \frac{10}{x} \)

11. In a sentence or two, explain the difference in the effects on \( y \)-values of doubling the \( x \)-value in a direct variation and doubling the \( x \)-value in an inverse variation.

In 12 and 13, refer to the logos at the right. The radius of the larger logo is twice the radius of the smaller one.

12. What is the ratio of the larger circumference to the smaller?

13. What is the ratio of the larger area to the smaller?

14. The weight \( w \) of an adult animal of a given species is known to vary directly with the cube of its height. If an ancient cat was 1.8 times as tall as a modern one, how many times as great was the weight of the ancient cat as the weight of the modern cat? (This type of equation is important to someone studying relative bone structure or energy requirements of the animals.)

15. Complete the proof of part b of the Fundamental Theorem of Variation.

16. **Multiple choice** Most of the power of a boat motor goes into generating the wake (the track left in the water). The engine power \( P \) used to generate the wake is directly proportional to the seventh power of the boat's speed \( s \). How can you express this relationship? (Lesson 2-1)
   (a) \( P = 7s \)  
   (b) \( s = kP^7 \)  
   (c) \( P = ks^7 \)  
   (d) \( P = k^7s \)

17. Suppose \( r \) varies directly as the 3rd power of \( s \). If \( r = 24 \) when \( s = 8 \), find \( r \) when \( s = 5 \). (Lesson 2-1)
18. Use Newton’s Law of Universal Gravitation, \( W = \frac{k}{r^2} \), where \( W \) = weight of a body and \( r \) = distance from the center of the earth. Suppose a chimpanzee weighs 85 lb on the surface of the earth. How much will it weigh when orbiting in space 1,000 miles above the earth’s surface? (Remember, the radius of the earth is about 4,000 mi.) (Lesson 2-2)

19. Skill sequence Solve. Remember to find two values for \( x \). (Previous course)
   
   a. \( x^2 = 49 \)  
   b. \( 3x^2 = 49 \)  
   c. \( 2x = \frac{49}{2x} \)

20. Use \( a_n = 30 - (n - 1) \). Find the first value of \( n \) which makes 
    \( a_n < -12 \). (Lessons 1-8, 1-9)

21. Type the following BASIC program on your computer. This program finds values for the direct variation \( y = x^n \) \((n > 0)\).

```
10 INPUT "A POSITIVE INTEGER"; N
20 PRINT "VALUES OF Y = X ^ N WHEN N = "; N
30 PRINT "X"", "Y"
40 FOR X = -10 TO 10 STEP .5
50 LET Y = X ^ N
60 PRINT X, Y
70 NEXT X
80 END
```

a. Run the program using \( N = 1 \). Notice that as \( x \) increases, \( y \) increases. When \( x \) doubles, what happens to \( y \)?

b. Run the program for \( N = 2, 3, \) and \( 4 \). For each value of \( N \), what happens to \( y \) when \( x \) doubles?
The purpose of this lesson is to study the properties of the graph of the direct variation formula \( y = kx \).

Recall from the Questions in Lesson 2-1 that the length of time between seeing a flash of lightning and hearing thunder varies directly with the distance from the lightning. The formula \( d = \frac{1}{3}t \) describes this situation for the values given in that lesson. This direct variation can also be represented graphically. Below is a table of some values that satisfy the equation \( d = \frac{1}{3}t \).

<table>
<thead>
<tr>
<th>( t = ) time (in seconds)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d = ) distance (in miles)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Because the equation \( d = \frac{1}{3}t \) is solved for \( d \), we consider \( d \) the dependent variable and \( t \) the independent variable. When making graphs the independent variable is always plotted along the horizontal axis (x-axis) and the dependent variable along the vertical axis (y-axis).

When all real-world solutions to the equation \( d = \frac{1}{3}t \) are plotted in the coordinate plane, the graph is a ray starting at the origin and passing through the first quadrant. Note that neither distance nor time can be negative in this situation, so there are no points on the graph of \( d = \frac{1}{3}t \) in any other quadrants.
In general, the graph of \( y = kx \) is a line through the origin. Recall that the steepness of a line is measured by a number called the **slope**. The slope of a line is the **rate of change** between any two points on the line. Let \((x_1, y_1)\) and \((x_2, y_2)\) be the two points. Then as pictured below, the expression \( y_2 - y_1 \) is the vertical change and \( x_2 - x_1 \) is the horizontal change.

\[
\text{slope} = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}
\]

### Definition:

The slope of the line through two points \((x_1, y_1)\) and \((x_2, y_2)\) equals

\[
\frac{y_2 - y_1}{x_2 - x_1}
\]

### Example

Find the slope of the line with equation \( d = \frac{1}{5}t \), where \( t \) is the independent variable and \( d \) the dependent variable.

**Solution**  
Find two points on the line. Then use the definition of slope.  
Either point may be considered \((x_1, y_1)\).  
Here we use \((x_1, y_1) = (10, 2)\) and \((x_2, y_2) = (25, 5)\).

\[
\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{25 - 10} = \frac{3}{15} = \frac{1}{5}
\]

Regardless of the points chosen, the slope of the line \( d = \frac{1}{5}t \) will be \( \frac{1}{5} \). Check the visual pattern. For every change of 5 horizontal units there is a change of 1 vertical unit. An equivalent way to say this is that for every change of 1 horizontal unit, there is a change of \( \frac{1}{5} \) of a vertical unit.

\[
\text{slope} = \frac{1}{5}
\]

\[
\frac{1}{5} = \frac{1}{5}
\]
Notice the constant of variation in the equation \( d = \frac{1}{3} t \) from Example 1 equals the slope of the line. Each value is \( \frac{1}{3} \).

Below are graphs of four direct-variation equations of the form \( y = kx \).

![Graphs of direct-variation equations](image)

Observe that in each case the graph is a line through the origin with slope \( k \). As examples, the slope of \( y = 2x \) is 2 and the slope of \( y = -\frac{3}{4}x \) is \( -\frac{3}{4} \). This is true for all values of \( k \).

**Theorem:**

The graph of the direct-variation equation \( y = kx \) has constant slope \( k \).

**Proof:**

Let \((x_1, y_1)\) and \((x_2, y_2)\) be two distinct points on \( y = kx \), with \( k \neq 0 \). Then, substitute in the equation

\[
y_1 = kx_1,
\]

and

\[
y_2 = kx_2.
\]

Subtract the equations:

\[
y_2 - y_1 = kx_2 - kx_1
\]

Use the distributive property:

\[
y_2 - y_1 = k(x_2 - x_1)
\]

Solve for \( k \):

\[
\frac{y_2 - y_1}{x_2 - x_1} = k
\]

So \( k \) is the slope.
Questions

1. The slope of a line is found by dividing the change in ? distance by the change in ? distance between any two points on the line.

2. By definition, \( \frac{y_2 - y_1}{x_2 - x_1} \) is the slope of the line through the two points __?__ and __?__.

3. What is the slope of the line \( d = \frac{1}{5} t \)?

4. A slope of \(-\frac{3}{4}\) means that for every change of 4 units to the right there is a change of ? units _. It also means that for every change of 1 unit to the right there is a change of ? units _.

5. The graph of every direct variation equation \( y = kx \) is a ___, with slope ___.

6. Use the equations \( y = 3x \) and \( y = \frac{1}{2}x \).
   a. Complete the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 3x )</th>
<th>( y = \frac{1}{2}x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>-4</td>
<td>-12</td>
<td>-2</td>
</tr>
</tbody>
</table>

   b. On a single set of axes, graph both lines using the values from the table above.
   c. The slope of the line with equation \( y = 3x \) is ___.
   d. The slope of the line with equation \( y = \frac{1}{2}x \) is ___.

In 7–9, find the slope of:

7. a mountain road that rises 3.6 vertical meters for each 60 horizontal meters.

8. the line through the points (6, 42) and (0, 0).

9. a submarine dive if the submarine drops 2000 feet while moving forward 8000 feet.

10. The cost \( c \) of gasoline varies directly with the number of gallons \( g \) bought.
    a. If 15 gallons cost $13.50, find a formula for \( c \) in terms of \( g \).
    b. Make a table of three pairs of solutions to the equation in part a.
    c. Graph these solutions. They should lie on a ray.
    d. What is the slope of the ray in part c?
11. Graphs that slant up as you read from left to right have \( ? \) slope;
graphs that slant down as you read from left to right have \( ? \) slope.

12. Match each graph with its equation. On each graph the \( x \)-axis and the
\( y \)-axis have the same scale.

I. \( y = 3x \)  
II. \( y = -3x \)  
III. \( y = \frac{1}{3}x \)  
IV. \( y = -\frac{1}{3}x \)

13. Refer to the drawing below.

a. Use the definition of slope to calculate the slope of the line. Simplify your answer.

b. What have you proved?

14. State whether the formula is a direct variation, an inverse variation or
neither. (Lessons 2-1, 2-2)

a. \( y = \frac{-8}{x} \)

b. \( y = \frac{-x}{8} \)

c. \( y = x - 11 \)

d. the law of the lever

e. the volume of a regular pyramid related to its base area if the
height is held constant (You may want to refer to the Appendix of
Geometry Formulas.)

LESSON 2-4  The Graph of \( y = kx \)  77
15. In the variation equation $W = \frac{k}{d}$, what is the effect on $W$ if:
   a. $d$ is tripled?
   b. $d$ is halved? *(Lesson 2-3)*

16. Assume the cost of a spherical ball bearing varies directly as the cube of its diameter. What is the ratio of the cost of a ball bearing 6 mm in diameter to the cost of a ball bearing 3 mm in diameter? *(Lesson 2-3)*

17. **Skill sequence** Solve for $x$. *(Lessons 1-7, 1-8)*
   a. $3x = 2$
   b. $3x = 2y$
   c. $3x = 2y + 6$
   d. $3(x + 5) = 2(y + 6)$

18. Find a counterexample to disprove the conjecture: If $a > b$ and $c > d$, then $ac > bd$. *(Lesson 1-6)*

19. Each of the following terms is a synonym for "slope." Find out who might use each term.
   a. marginal cost   b. pitch   c. grade
In Lesson 2-1 you learned that the distance needed to stop a car after applying the brakes varies directly with the square of the car's speed. The formula \( d = \frac{1}{16}s^2 \) describes this relation between braking distance and speed for a certain car. A table of some solutions to this equation is given below.

<table>
<thead>
<tr>
<th>( s )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>0</td>
<td>6.25</td>
<td>25</td>
<td>56.25</td>
<td>100</td>
<td>156.25</td>
<td>225</td>
<td>306.25</td>
</tr>
</tbody>
</table>

The points do not all lie on a straight line. This can be verified by calculating the rate of change between different pairs of points on the graph.
Example 1  Find the rate of change
a. $r_1$ between (20, 25) and (40, 100);
b. $r_2$ between (40, 100) and (60, 225).

Solution  

a. Use the definition of slope:

$$r_1 = \frac{100 \text{ ft} - 25 \text{ ft}}{40 \text{ mph} - 20 \text{ mph}} = \frac{75 \text{ ft}}{20 \text{ mph}} = 3.75 \text{ ft/mph}$$

This means that on the average when driving between 20 mph and 40 mph, for every increase of 1 mph in speed, you need 3.75 more feet of braking distance.

b. Similarly, $r_2 = \frac{225 \text{ ft} - 100 \text{ ft}}{60 \text{ mph} - 40 \text{ mph}} = \frac{125 \text{ ft}}{20 \text{ mph}} = 6.25 \text{ ft/mph}$.

So on the average, between $s = 40$ and $s = 60$ for every change of 1 mph (the horizontal unit), there is a change of 6.25 feet of braking distance (the vertical unit).

Check  
Look at the graph. Is segment $BC$ with slope 6.25 steeper than segment $AB$ with slope 3.75?

Yes, it is.

Because the rate of change between different pairs of points on the graph of $y = \frac{s^2}{16}$ is not constant, two conclusions can be drawn:

1. The graph of $y = \frac{1}{16}s^2$ is not a line.
2. The steepness of the graph cannot be described by a single number.

The equation $d = \frac{1}{16}s^2$ is a direct-variation formula of the form $y = kx^2$. In order to draw conclusions about the graphs of equations of this form, you must examine additional cases.
Example 2  Graph solutions to the following three equations:

\[
\begin{align*}
y &= x^2 \\
y &= 2x^2 \\
y &= \frac{1}{4}x^2
\end{align*}
\]

Solution  Make a table of solutions. To save space, the value of the independent variable is written only once.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = x^2)</th>
<th>(y = 2x^2)</th>
<th>(y = \frac{1}{4}x^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>18</td>
<td>(\frac{9}{4})</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>-3</td>
<td>9</td>
<td>18</td>
<td>(\frac{9}{4})</td>
</tr>
</tbody>
</table>

Observe that the graphs of \(y = x^2\), \(y = 2x^2\), and \(y = \frac{1}{4}x^2\) are curves. These curves are called **parabolas**. Because each of these parabolas coincides with its reflection image over the y-axis, each is **reflection-symmetric**. The y-axis is the **line of symmetry**. Also, each parabola passes through the point \((0, 0)\). The graph of \(y = 2x^2\) goes up faster than the graph of \(y = x^2\); the graph of \(y = \frac{1}{4}x^2\) goes up more slowly.

The graph of \(d = \frac{1}{16}s^2\) plotted at the beginning of this lesson goes up more slowly than any of the parabolas of Example 2. It is half a parabola because speed cannot be negative. There are no points in the second quadrant.
Example 3 shows graphs of \( y = kx^2 \) for two negative values of \( k \).

**Example 3** Graph solutions to the following two equations:

\[
\begin{align*}
  y &= -x^2 \\
  y &= -\frac{1}{2}x^2
\end{align*}
\]

**Solution** Make a table, using integer values of \( x \) from 4 to -4. Recall that \(-x^2\) means "the opposite of \( x^2 \)" or \(-1 \cdot x^2\). Plot the ordered pairs.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = -x^2 )</th>
<th>( y = -\frac{1}{2}x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-16</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-9</td>
<td>-2.25</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-0.25</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-0.25</td>
</tr>
<tr>
<td>-2</td>
<td>-4</td>
<td>-1</td>
</tr>
<tr>
<td>-3</td>
<td>-9</td>
<td>-2.25</td>
</tr>
<tr>
<td>-4</td>
<td>-16</td>
<td>-4</td>
</tr>
</tbody>
</table>

The graphs in Example 3 have shapes similar to those in Example 2. Again, each curve passes through the origin and is symmetric to the \( y \)-axis. However, we say that the curves in Example 3 "open down," while those in Example 2 "open up." In general, for \( y = kx^2 \), when \( k > 0 \) the parabola opens up and when \( k < 0 \) the parabola opens down.

\[ y = kx^2 \] for \( k > 0 \)

\[ y = kx^2 \] for \( k < 0 \]
Questions

In 1–3, refer to the formula \( d = \frac{1}{16}s^2 \) relating speed and braking distance.

**True or false**

1. The graph is a straight line.

2. The rate of change on the graph is constant, regardless of the points used.

3. The replacement set for \( s \) is the set of all real numbers.

4. Name the type of curve that results from graphing \( y = kx^2 \) \((k \neq 0)\).

5. What does it mean to say that the graph of \( y = kx^2 \) is symmetric to the \( y \)-axis?

6. In general, for what values of \( k \) does the graph of \( y = kx^2 \)
   a. open up?
   b. open down?

7. a. Make a table of solutions for \( x = -2, -1, 0, 1, \) and 2 for the following three equations.
   b. Graph the solutions on one set of axes. Use as the domain the set of real numbers between -2 and 2 inclusive.
      \[
      y = x^2 \quad y = 3x^2 \quad y = -3x^2
      \]

8. Let \( N \) represent the number of houses that can be served by a water main of diameter \( d \) centimeters. Suppose \( N = \frac{1}{2}d^2 \).
   a. Make a table of solutions for this equation. For values of \( d \) use 0, 10, 20, 30, 40.
   b. Graph these solutions.
   c. Estimate from your graph the number of homes that can be served by a main of diameter 35 cm.
   d. According to the variation equation, how many homes can be served by a main of diameter 35 cm?

9. Match each graph with the proper equation. Each graph has the same scale.
   \[
   y = \frac{1}{2}x^2 \quad y = -2x \quad y = -x^2 \quad y = 3x^2
   \]
   a. \hspace{2cm} b. \hspace{2cm} c. \hspace{2cm} d.
10. Refer to the table and graph of $y = x^2$ in Example 2. Find the rate of change between the points.
   a. (0, 0) and (1, 1)
   b. (1, 1) and (2, 4)
   c. (2, 4) and (3, 9)
   d. (3, 9) and (4, 16)
   e. Use your results from parts a to d to make a conjecture about the rate of change between the points $(n, n^2)$ and $(n + 1, (n + 1)^2)$.
   f. Prove your conjecture by calculating the rate of change for the points in part e.

11. You do not need a computer for this question. Here is a computer program in BASIC that takes a value of $k$ and prints a list of solutions to $y = kx^2$.

```
10 PRINT "WHAT IS K?"
20 INPUT K
30 PRINT "SOLUTIONS TO Y = K * X ^ 2"
40 PRINT "X", "Y"
50 FOR X = -5 TO 5
60 LET Y = K * X ^ 2
70 PRINT X,Y
80 NEXT X
90 END
```

a. How many ordered pairs of solutions will be printed?
   b. What is the first pair to be printed?
   c. What is the last pair to be printed?
   d. Describe the output that would result from changing line 50 to

   ![FOR X = -5 TO 5 STEP .5]

12. The Fahrenheit and Celsius scales indicate temperature. Temperature can also be measured in kelvins. This measurement is sometimes called measuring on the Kelvin scale, in degrees Kelvin. At a given altitude, the volume $V$ of a fixed amount of air varies directly with its Kelvin temperature $t$. The lowest possible temperature occurs when $t$ is zero, about $-273^\circ$ C. Suppose that a balloon contains 7.5 liters of air at 300 kelvins (about room temperature). (Lesson 2-1)
   a. Write a specific variation formula for $V$ in terms of $t$.
   b. Use this formula to predict the volume of air in the balloon at temperatures of 400, 500, 600, and 1000 kelvins.

Review

Architects designing auditoriums use the fact that sound intensity $I$ is inversely proportional to the square of the distance $d$ from the sound source. (Lessons 2-2, 2-3)
   a. Write the variation equation that represents this situation.
   b. A person moves to a seat 4 times farther from the source.
      The sound will be heard ___ as intensely.
14. What is another name for slope? (Lesson 2-4)

15. Graph the following four equations on one set of axes.
   \( l: y = 4x \)  \( m: y = \frac{1}{4}x \)  \( n: y = -\frac{1}{4}x \)  \( o: y = -4x \)  
   (Lesson 2-4)

16. Solve for \( x \): \( y = -\frac{1}{4}x \). (Lesson 1-8)

17. a. Which of the triangles pictured below can be proved congruent to \( \triangle ABC \)? Name them, with vertices in correct order.
   b. Give the justification for each triangle congruence you find. (Previous course)

18. Accurately graph the curve \( y = x^2 \) from \( x = 0 \) to \( x = 3 \). The area between the curve and the \( x \)-axis is an integer number of square units. What is this number?
Graphs of equations are so helpful to have that there exist calculators and programs for personal computers that will automatically display graphs. Because computer screens are larger than calculator screens they can more clearly show more of a graph; but graphing calculators are less expensive, more portable and sometimes are easier to use.

Graphing calculators and computer graphing-programs work in much the same way; so we call them **automatic graphers** and do not distinguish between them. Of course, no grapher is completely automatic. Each has particular keys to press that you must learn from a manual. Here we discuss what you need to know in order to use any automatic grapher. Consult your calculator owner’s manual or your **function grapher’s** documentation for specific information about your grapher.

The part of the coordinate grid that is shown is called a **window**. The screen below displays a window in which

\[-2 \leq x \leq 13\]

and

\[-3 \leq y \leq 7.\]

On calculators, the intervals for \(x\) and \(y\) may be left unmarked. Usually you need to pick the \(x\)-values at either end of the window. Some graphers automatically adjust and choose \(y\)-values so that your graph will fit, but often you also need to choose the \(y\)-values. If you do not do this, the grapher will usually make use of a **default window**, that is, a window that is used whenever you do not specify the intervals on which to plot \(x\) and \(y\).

On almost all graphers, the equation to be graphed must be a formula for \(y\) in terms of \(x\).

\[
\begin{align*}
y &= 3x^2 \quad \text{and} \quad y = \frac{5}{9}(x - 32) \quad \text{can be handled.} \\
x &= 4y \quad \text{and} \quad x + y = 17 \quad \text{cannot be handled.}
\end{align*}
\]

On many graphers you enter equations by using the keys *, /, and \(^\wedge\) to indicate multiplication, division, and powering, respectively. For instance, to graph \(y = 3x^2\) or \(y = \frac{5}{9}(x - 32)\) you may need to enter

\[
y = 3 \times x \wedge 2 \quad \text{or} \quad y = (5/9) \times (x - 32).
\]
Automatic graphers generally follow the standard rules for order of operations stated in Lesson 1-2. The steps needed to graph an equation with an automatic grapher are:

1. Solve the equation you wish to graph for \( y \), and enter it into your grapher.
2. Determine a window and key it in.
3. Give instructions to graph.

**Example 1**

Use an automatic grapher to sketch solutions to \( \frac{y}{x} = 10 \) in the following windows:

a. \(-15 \leq x \leq 15, -10 \leq y \leq 10\)

b. \(-3 \leq x \leq 3, -30 \leq y \leq 30\)

c. \(-1.5 \leq x \leq 3.5, -40 \leq y \leq 40\)

**Solution**

Rewrite the equation as \( y = 10x \). How you enter it will vary from one machine to another. Typically, you might type \( y = 10 \times x \). Follow the instructions for your grapher to input the size of the window. Typical output is shown below.

Notice that although the graphs appear to have different steepness, each is a line with a slope of 10. For every horizontal change of 1 unit, there is a vertical change of 10 units. The size of the viewing window on an automatic grapher may change your impression of the shape of a graph, but it does not change the mathematical properties of the graph.

Most automatic graphers can plot solutions to more than one equation at a time. Some allow you to plot many graphs simultaneously.
Example 2

a. Graph \(y = ax^2\) when \(a = \frac{1}{2}, 1, 2, \) and \(3.\)
b. What happens to the graph as \(a\) gets larger?

Solution

a. The question asks you to graph
\[ y = \frac{1}{2}x^2 \]
\[ y = x^2 \]
\[ y = 2x^2 \]
\[ y = 3x^2 \]

Each equation is already solved for \(y\). Enter one at a time, following the instructions on your grapher. We use the window \(-6 \leq x \leq 6\), knowing that these parabolas are symmetric to the y-axis. The interval \(-2 \leq y \leq 10\) is reasonable for comparing the parabolas.

b. As the value of \(a\) increases, the parabola looks thinner and thinner. The thinnest parabola is \(y = 3x^2\). The parabola that looks widest is \(y = \frac{1}{2}x^2\).

Some graphers have a zoom feature like those found on cameras. This feature enables you to change the window of a graph without retying intervals for \(x\) and \(y.\) In general, there is no "best window." Usually a good window for a graph is one in which you can estimate the coordinates of the \(x\)- and \(y\)-intercepts (if any), and any other points you need in the problem. For instance in Example 2 above if you want to study the behavior of the four parabolas near the vertex, you may want to zoom by a factor of 10. (Typically, graphers zoom around the origin.) This means that the viewing rectangle now is determined by \(-0.6 \leq x \leq 0.6\) and \(-0.2 \leq y \leq 1.\) The result is shown below.
Again the thinnest parabola is the graph of 
\[ y = 3x^2, \text{ and the widest is the graph of } y = \frac{1}{3}x^2. \] To some people the two lowest graphs do not "look like" parabolas. However this is an illusion created by the window used. Each graph is a parabola; each has exactly the same mathematical properties—namely, vertex, symmetry line, rate of change between points—as it has when pictured in Example 2.

Many function graphers on computers will print hard copy, that is, a paper copy of a graph shown on the screen or stored on disk. If that is the case with your automatic grapher your teacher will probably accept such graphs in answer to homework questions. If hard copy is unavailable or unacceptable in your class, you must copy graphs from the grapher’s screen to paper. When copying graphs always show:

1. the size of the window and the scales on the axes;
2. key features of the graph such as x- or y-intercepts or vertices;
3. its approximate shape.

Questions

Covering the Reading

1. On an automatic grapher, to what does the window refer?

2. a. What is a default window?
   b. Does your automatic grapher have a default window? If so, describe it.

3. Describe the window pictured below.

In 4–6, suppose that a grapher works only if an equation is input as a formula for \( y \) in terms of \( x \). Decide whether the equation is in a form in which it can be graphed with that grapher.

4. \( x = 3y \)  
5. \( y = 1.3x^2 \)  
6. \( y = (4 - x)/2 \)
7. In Example 1, which window
   a. appears to have the steepest line?
   b. appears to show the line with the greatest slope?

In 8, refer to Example 2.

8. a. Which value of $a$ gives the widest parabola?
   b. Which value of $a$ gives the thinnest parabola?
   c. Make a copy of the graphs, and add a sketch of the graph of $y = 4x^2$ to it.

In 9–12, use an automatic grapher.

9. a. Graph $y = -6x$ using the following windows.
   (i) your default window, if any.
   (ii) $-15 \leq x \leq 15$, $-10 \leq y \leq 10$
   (iii) $-3 \leq x \leq 3$, $-20 \leq y \leq 20$
   b. What is the slope of each line drawn?

10. a. Graph on one set of axes $y = ax^2$ when $a = -\frac{1}{2}$, -1, -2, and -3.
    Use the window $-6 \leq x \leq 6$, $-10 \leq y \leq 2$.
    b. What happens to the graph as $a$ gets smaller?
    c. Compare and contrast these graphs to those in Example 2.

11. a. Use any convenient window. Graph on one set of axes:
    \[ y = 100x \]
    and \[ y = 75x + 25x. \]
    b. Sketch what appears.
    c. Predict what the graph of $y = (113x + 87x)/2$ will look like. Then use this formula in your automatic grapher to test your prediction.

12. The cost $c$ of an above-ground swimming pool 6-ft deep varies directly as the square of its diameter $d$. Suppose a pool with diameter 12 ft costs $720.
   a. Write an equation for the relation between $c$ and $d$.
   b. Plot solutions to part a over a reasonable domain for $d$.
   c. Use your graph to estimate the cost of a pool with an 18-ft diameter.
   d. Check your estimate in part c by using the equation in part a.
In 13 and 14, the graph below was drawn using the equations:

\[ y = 4.9x^2 \]

and \[ y = -4.9x^2 \]

and the window \(-1 \leq x \leq 1\) and \(-7 \leq y \leq 7\).

Suppose the window was changed as given below.

a. Sketch what you think the screen would show.

b. Check your work by using an automatic grapher.

![Graph](image)

13. \(-5 \leq x \leq 5\) and \(-100 \leq y \leq 100\)

14. \(-0.1 \leq x \leq 0.1\) and \(-1 \leq y \leq 1\)

15. **Skill sequence** Solve and check. (Lessons 1-6, 1-7)
   a. \(12n = 18\)
   b. \(\frac{n}{12} = 18\)
   c. \(\frac{18}{n} = 12\)
   d. \(\frac{n}{12} + 3 = \frac{n}{18}\)

16. A tortoise is walking at a rate of \(3 \frac{a}{\text{minute}}\). Assume this rate continues.
   a. How long will it take the tortoise to travel 60 feet?
   b. How long will it take the tortoise to travel \(f\) feet? (Lesson 1-1)

17. Simplify.
   a. \((2x + 3) + (4x + 5)\)
   b. \((2x + 3) - (4x + 5)\)
   c. \((2x + 3)(4x + 5)\) (Previous course, Lesson 1-5)

18. If \(y\) varies directly as the cube of \(x\), and \(y = \sqrt[3]{24}\) when \(x = 2\), what is the average rate of change of \(y\) from \(x = 2\) to \(x = 3\)? (Lessons 2-1, 2-5)

19. Consider graphing \(y = x^2\) with an automatic grapher using the window \(-a \leq x \leq a\), \(-b \leq y \leq b\).
   a. Let \(a = 3\) and \(b = 10\) and graph.
   b. Select a large enough value of \(a\) so that the graph will seem to coincide with the nonnegative \(y\)-axis. What value of \(a\) will do this on your grapher?
   c. For \(a = 3\), what value of \(b\) is so large that the graph seems to coincide with the \(x\)-axis?
The Metro Car Sales example of Lesson 2-2 is one instance of inverse variation. Recall that the number of students \( s \) hired to wash cars and the number of hours \( t \) each will need to work are related by the equation \( t = \frac{36}{s} \). A graph of this relation is shown here.

This graph has several properties. One is that it is made up of unconnected points. Such graphs are called discrete graphs. It would not make sense to connect the points of this graph because \( s \), the number of students, can only be a whole number. A discrete graph is one type of discontinuous graph. A discontinuous graph cannot be drawn without picking up your pencil. In contrast, the graphs in Lessons 2-4 and 2-5, which can be drawn without picking up your pencil, are called continuous graphs.

A second property is that the graph never crosses the \( t \)-axis. If \( s = 0 \), then \( \frac{36}{s} \) is undefined.
A third property is that the rate of change between any two points is always negative. For instance, between \((4, 9)\) and \((8, 4\frac{1}{2})\), the rate of change is
\[
\frac{4\frac{1}{2} - 9}{8 - 4} = -1.125.
\]

Between \((8, 4\frac{1}{2})\) and \((12, 3)\), the rate of change is
\[
\frac{3 - 4\frac{1}{2}}{12 - 8} = -.375.
\]

That these rates of change are different implies that the points do not lie on a line. The rate of change between any two points on the graph is negative, so each point on this discrete graph is lower on the graph as you read from left to right. This is similar to the idea that when the slope of the line \(y = kx\) is negative, the line is falling.

Other properties of the graph of \(y = \frac{k}{x}\) can be seen if \(x\) is assigned negative values.

---

**Example 1**

Draw the graphs of \(y = \frac{16}{x}\) and \(y = -\frac{16}{x}\) for \(x \neq 0\).

**Solution**

At the left below is a table of solutions. To save space, the independent variable \(x\) is written only once. The graphs are at the right.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = \frac{16}{x})</th>
<th>(y = -\frac{16}{x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>-16</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>3</td>
<td>5(\frac{1}{3})</td>
<td>-5(\frac{1}{3})</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>6</td>
<td>2(\frac{2}{3})</td>
<td>-2(\frac{2}{3})</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>12</td>
<td>1(\frac{1}{3})</td>
<td>-1(\frac{1}{3})</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-16</td>
<td>16</td>
</tr>
<tr>
<td>-2</td>
<td>-8</td>
<td>8</td>
</tr>
<tr>
<td>-3</td>
<td>-5(\frac{1}{3})</td>
<td>5(\frac{1}{3})</td>
</tr>
<tr>
<td>-4</td>
<td>-4</td>
<td>4</td>
</tr>
<tr>
<td>-6</td>
<td>-2(\frac{2}{3})</td>
<td>2(\frac{2}{3})</td>
</tr>
<tr>
<td>-8</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-12</td>
<td>-1(\frac{1}{3})</td>
<td>1(\frac{1}{3})</td>
</tr>
<tr>
<td>-16</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>
The type of curve graphed in Example 1 is called a hyperbola. A hyperbola is not continuous, because you must pick up your pencil to draw the two separate parts, or branches. Also, a hyperbola is not discrete because all the points are connected to some other points.

You also have studied inverse-square variation. What does the graph of \( y = \frac{k}{x^2} \) look like?

**Example 2**

Graph \( y = \frac{16}{x^2} \) and \( y = -\frac{16}{x^2} \).

**Solution**

Again a table of solutions is below at the left and the graphs are at the right. The values in the table were produced by an automatic grapher.

\[
\begin{array}{c|c|c}
 x & y = \frac{16}{x^2} & y = -\frac{16}{x^2} \\
-8 & .25 & -.25 \\
-7 & .326531 & -.326531 \\
-6 & .444444 & -.444444 \\
-5 & .64 & -.64 \\
-4 & 1 & -1 \\
-3 & 1.77778 & -1.77778 \\
-2 & 4 & -4 \\
-1 & 16 & -16 \\
0 & *** & *** \\
1 & 16 & -16 \\
2 & 4 & -4 \\
3 & 1.77778 & -1.77778 \\
4 & 1 & -1 \\
5 & .64 & -.64 \\
6 & .444444 & -.444444 \\
7 & .326531 & -.326531 \\
8 & .25 & -.25 \\
\end{array}
\]

(Note that for \( x = 0 \), no value of \( y \) is given. Instead *** is printed in the table. Our automatic grapher uses this symbol to indicate that 0 is not an element of the domain of \( x \). That is, \( \frac{16}{0^2} \) and \( -\frac{16}{0^2} \) are not defined. Some function graphers state ERROR or some other message to indicate that a number is not part of the domain for the independent variable.)
The graph of an inverse-square variation does not have a special name, so we shall just call it an **inverse-square graph**. The inverse-square graph is symmetric to the y-axis. Notice that the inverse-square graph, like a hyperbola, has two distinct branches. However, the two branches do *not* form a hyperbola because the shape of each branch, as well as the relative location of the branches, differs from a hyperbola.

Neither the hyperbola with equation \( y = \frac{k}{x} \) (\( k \neq 0 \)) nor the inverse-square curve \( y = \frac{k}{x^2} \) intersects the coordinate axes. You can verify these results by zooming or rescaling to look more closely at the graphs near \( x = 0 \) or for very large or very small values of \( x \). For instance, below you see three views of \( y = \frac{16}{x} \) for different windows with large \( x \) values. Note that for all positive numbers \( x \), \( \frac{16}{x} > 0 \).

![Graphs of y = k/x and y = k/x^2](image)

In general, when \( x = 0 \), \( y = \frac{k}{x} \) and \( y = \frac{k}{x^2} \) are undefined. So neither curve crosses the y-axis. Also, when \( k \neq 0 \) neither \( \frac{k}{x} \) nor \( \frac{k}{x^2} \) can ever equal 0. Thus neither curve crosses the x-axis.

**Questions**

1. A ___ graph is made up of unconnected points.
2. A ___ graph cannot be drawn without picking up the pencil.
3. Refer to the graph of \( t = \frac{36}{s} \) in this lesson.
   a. What is the rate of change between (4, 9) and (12, 3)?
   b. What is the rate of change between (12, 3) and (4, 9)?
4. What is the graph of \( y = \frac{k}{x} \) called?
5. Find the rate of change between the points on the graph of \( y = \frac{-16}{x^2} \) for which \( x = 4 \) and \( x = 8 \).

6. *Multiple choice* Which equation has a graph that is symmetric to the \( y \)-axis? \( (k \neq 0) \)
   
   \( \text{(a) } y = \frac{k}{x} \quad \text{(b) } y = \frac{k}{x^2} \quad \text{(c) } y = kx \)

7. In which quadrants are the branches of \( y = \frac{-16}{x} \)?

8. In which quadrants are the branches of \( y = \frac{k}{x^2} \): 
   a. if \( k \) is positive? 
   b. if \( k \) is negative?

   In 9 and 10, which of the following words describe the graph?
   a. continuous 
   b. discontinuous 
   c. discrete

9. 

   ![Graph](image)

   ![Graph](image)

10. 

11. a. Draw a graph of \( y = \frac{24}{x} \).
   b. Find the rate of change from \( x = 2 \) to \( x = 6 \) for \( y = \frac{24}{x} \).
   c. Draw a graph of \( y = \frac{24}{x^2} \).
   d. Find the rate of change from \( x = 2 \) to \( x = 6 \) for \( y = \frac{24}{x^2} \).
   e. Which of the two graphs is falling faster from \( x = 2 \) to \( x = 6 \)?

12. Sam is once again on a seesaw. He weighs 40 pounds and is sitting 5 feet from the pivot. (Remember the law of the lever is \( d = \frac{k}{w} \).)

   a. Find \( k \) and then write the formula for the variation in this situation.
   b. Draw a graph of weights and distances from the pivot that would balance Sam.
13. Examine the graph of \( y = \frac{16}{x} \) on page 93.
   a. How many symmetry lines does the graph have?
   b. Write an equation for each symmetry line.
   c. Does the graph of \( y = \frac{-16}{x} \) have the same symmetry lines? If not, what are equations for its symmetry line(s)?

14. a. Use an automatic grapher to graph on one set of axes the four curves \( y = \frac{k}{x} \), where \( k = 1, 2, 5, \) and 10. Use a window \(-5 \leq x \leq 5, -10 \leq y \leq 10\).
   b. What happens to the graph of \( y = \frac{k}{x} \) as \( k \) gets larger?

15. In the figure at the left below, parabolas \( a \) and \( b \) are congruent. If parabola \( a \) has equation \( y = 6x^2 \), what is an equation for parabola \( b \)? (Lesson 2-5)

![](image1)

16. Why is the line graphed below not an example of a direct variation? (Lesson 2-4)

![](image2)

17. When trying to solve the equation \( \frac{1}{3}x + \frac{1}{2}x + 5 = 10 \), Mikki's first step was \( 3x + 2x + 30 = 60 \). What two properties did she apply? (Lessons 1-5, 1-6)
18. Solve for $x$:
\[
\frac{1}{4}x + \frac{2}{5}x + 9 = 10 \quad (Lesson \ 1-7)
\]

19. In the graph below, the grid lines are 1 unit apart.
Each labeled point is at an intersection of grid lines.

a. Are triangles $ABC$ and $EDC$ congruent?

b. If so, why? If not, why not?

(Previous course)

20. a. Draw the graph of $y = \frac{36}{x^3}$.

b. Draw the graph of $y = \frac{36}{x^4}$.

c. Use your answer to parts a and b to predict which one of the following equations will have a graph symmetric to the y-axis.
\[
(i) \quad y = \frac{36}{x^3} \quad (ii) \quad y = \frac{36}{x^6}
\]

d. Use an automatic grapher to graph the equations in part c to test your prediction. What property of exponents justifies the result you observed?
You may know that the water pressure on a deep sea diver increases as the diver goes deeper. How is the pressure related to the depth?

The following table gives the water pressure (in pounds per square inch, or psi) exerted on a diver at various depths (in ft).

<table>
<thead>
<tr>
<th>Depth of diver (ft)</th>
<th>10</th>
<th>25</th>
<th>40</th>
<th>55</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure of diver (psi)</td>
<td>4.3</td>
<td>10.8</td>
<td>17.2</td>
<td>23.7</td>
<td>32.3</td>
</tr>
</tbody>
</table>

This information is graphed below. Because the pressure on the diver depends on the diver’s depth, pressure is the dependent variable and is placed on the vertical axis. Depth is the independent variable and is graphed on the horizontal axis. The points seem to lie on a line through the origin. It makes sense that the origin is on this line because on the surface—that is, 0 feet under water—there is 0 pounds per square inch of water pressure. Therefore, it seems appropriate to describe the relation between the variables by saying the pressure varies directly as the depth.
If $p$ represents the pressure and $d$ represents the depth, the formula for this variation is $p = kd$. The constant $k$ can be determined from one of the data points. For instance, substitute $p = 4.3$ psi and $d = 10$ ft into the equation to get

\[
4.3 = k \cdot 10
\]

\[
k = 0.43. \quad \text{(The unit is \(\text{psi} \text{ ft}\).)}
\]

This relation between $p$ and $d$ can be expressed as

\[
p = 0.43d.
\]

It is important to check that this formula holds for all the data in the table. You should see whether each data point satisfies the equation. For instance, if $d = 25$ ft, then

\[
p = (0.43)(25) = 10.75
\]

which is close to the value of 10.8 psi in the table.

The equation $p = 0.43d$ is a mathematical model of the real-life relation between pressure and depth. A **mathematical model** is a graph or a sentence that describes data or a relation between variables. The formula $p = 0.43d$ holds true for all the values in the table. A good model is one that holds true for all the given information. In this book there are many examples of mathematical models.

The model $p = 0.43d$ makes it possible to predict the pressure on a diver at depths other than those given in the table. At a depth of 125 ft, for instance, the model predicts that the pressure on a diver would be

\[
p = (0.43)(125) = 53.75
\]

\[
\approx 53.8 \text{ psi.}
\]

Here is another situation whose mathematical model involves variation.

**Example** Perri Menter was investigating the relation between the volume and pressure of a gas in her laboratory. While she held the temperature in the laboratory constant, she varied the pressure (the independent variable) and measured the volume (the dependent variable) to obtain the following data.

<table>
<thead>
<tr>
<th>Pressure (psi)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume (ft³)</td>
<td>83</td>
<td>55</td>
<td>42</td>
<td>33</td>
<td>28</td>
<td>24</td>
<td>21</td>
</tr>
</tbody>
</table>
The laboratory results are graphed below. The shape of the graph suggests two possible models: \( V \) varies inversely as \( P \) or inversely as the square of \( P \).

a. Does \( V = \frac{k}{P^2} \) model the data?

b. Does \( V = \frac{k}{P} \) model the data?

c. Predict the volume of gas if the pressure is 45 psi.

**Solution**

a. To test \( V = \frac{k}{P^2} \), first substitute the coordinates of one data point to find \( k \). For instance, if \( P = 20 \) psi, then \( V = 83 \text{ ft}^3 \); so

\[
83 = \frac{k}{(20)^2} \\
k = 33.200.
\]

Next, decide whether the equation

\[
V = \frac{33.200}{P^2}
\]

is valid by substituting the coordinates of another data point. For instance, substitute \( P = 30 \) psi into this equation.

\[
V = \frac{33.200}{(30)^2} \approx 37
\]

This is not close to the value of 55 \( \text{ft}^3 \) found in the table. This counterexample shows that \( V = \frac{k}{P^2} \) is not a correct model for the data.
b. To test if \( V = \frac{k}{P} \) is a correct model, again substitute 20 psi for \( P \) and 83 ft\(^3\) for \( V \). For these values, \( k = 1660 \text{ ft}^3\text{-psi} \) and the model is

\[ V = \frac{1660}{P}. \]

Now check whether this equation is valid for all the data of the experiment. For instance, if \( P = 30 \text{ psi} \), then

\[ V = \frac{1660 \text{ ft}^3\text{-psi}}{30 \text{ psi}} \approx 55 \text{ ft}^3. \]

This is the value in the table. It can likewise be shown that all the data satisfy the equation. Thus \( V = \frac{1660}{P} \) is a good model for Perri Menter's data.

c. Substitute \( P = 45 \text{ psi} \) into the model \( V = \frac{1660}{P} \). Then \( V = \frac{1660}{45} = 36.8 \), or about 37. So the model predicts a volume of 37 ft\(^3\) at 45 psi.

Check Use the graph. The point (45, 37) is on the hyperbola.

Questions

In 1 and 2, refer to the example about deep-sea diving.

1. Describe in English the variation between the pressure and the depth.

2. Use the model to predict the pressure on a diver who is 130 ft below the surface.

3. Define: mathematical model.

In 4–7, refer to the example.

4. True or false As the pressure on the gas is increased, the volume of the gas is increased.

5. Use the point (40, 42) to show that \( V = \frac{33,200}{P^2} \) is not a good model for the volume and pressure data.

6. Verify that the point (40, 42) satisfies the formula \( V = \frac{1660}{P} \).

7. Use the good model to predict the volume of the gas under a pressure of 18 psi.

8. Which of the following words describe the graph of depths and pressures at the beginning of this lesson?

   a. discrete  
   b. continuous  
   c. discontinuous
9. a. Graph all solutions to \( y = \frac{1660}{P} \).
   
   b. How does this graph compare with that of \( V = \frac{1660}{P} \) in the example?

10. Refer to the graph at the right.
    a. Which of the following equations could be a model for this graph?
       I: \( y = kx \)
       II. \( y = kx^2 \)
       III: \( y = \frac{k}{x} \)
       IV: \( y = \frac{k}{x^2} \)
    
    b. Find the constant \( k \) for your model.
    c. Test your model to see if \( y \) is 8 when \( x \) is 4.

11. Multiple choice Which formula best models the graph at the right?
    (a) \( P = kh \)    (b) \( P = kh^2 \)
    (c) \( P = \frac{k}{h} \)    (d) \( P = \frac{k}{h^2} \)

12. A scientist dropped a ball from a cliff and used a slow motion film to determine the distance it fell over different periods of time. The data are summarized below.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (m)</td>
<td>4.9</td>
<td>19.6</td>
<td>44.1</td>
<td>78.4</td>
<td>122.5</td>
</tr>
</tbody>
</table>

   a. Draw a graph to represent these data. Let \( t \) be the independent variable and \( d \) be the dependent variable. (Use a large enough scale on the \( d \)-axis to handle 122.5.)
   b. Pick the variation equation from those of Question 10a that best models this situation. Use one data point to calculate \( k \) and check the model with the other data points.
   c. Predict how far the ball would fall in 4.5 sec.
13. Find the rate of change between the points on \( y = \frac{15}{x} \) where \( x = 1 \) and \( x = 11 \). (Lesson 2-4)

14. Suppose that the value of \( x \) is halved. Find how the value of \( y \) is changed if \( y \) is directly proportional to
   a. \( x \)
   b. \( x^2 \)
   c. \( x^4 \) (Lesson 2-1)

In 15–18, match the graph to the most likely equation. (Lessons 2-4, 2-5, 2-7)

a. \( y = 3x \)       b. \( y = -\frac{3}{x} \)       c. \( y = \frac{3}{x^2} \)

d. \( y = -\frac{x}{3} \) e. \( y = -\frac{1}{3}x^2 \)

15.  

16.  

17.  

18.  

19. How long does it take to travel \( k \) kilometers at a rate of \( r \) kilometers per hour? (Lesson 1-1)

20. The maximum pressure that a deep-sea diver can withstand without using special equipment is about 65 psi.
   a. Find how deep a diver can go below the surface without special equipment.
   b. How can divers go beneath the depth found in part a?
All through this chapter you have seen situations in which two quantities vary. In many real life situations there are more than two variables. Consider, for instance, the situation presented on the very first page of the chapter, where the problem is to determine how much weight can be supported by a board. Three quantities which influence this are the width w (front to back), the thickness t of the board, and the distance d between supports. What model describes the maximum weight MAXWT that can be supported in terms of the other three variables? The model cannot be described by a single graph in two dimensions because there are four variables to be considered. The goal is to find an equation relating w, t, d, and the dependent variable MAXWT.

One way to find a model is to investigate separately the relationship between the dependent variable, the weight, and each independent variable. This is done by keeping constant all but one independent variable.

We show this with a story. The data are made up, but the idea is not. Our heroine is again Perri Menter. She found the model as follows. First, she held two independent variables constant: d and t. She did this by choosing boards 2 in. thick and setting the supports 10 ft apart. Then she varied the widths of these boards and measured how much weight could be supported before the boards broke. Perri obtained the following data.

<table>
<thead>
<tr>
<th>Width of board (in.)</th>
<th>w</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Weight (lb)</td>
<td>MAXWT</td>
<td>27</td>
<td>53</td>
<td>80</td>
<td>107</td>
<td>133</td>
<td>160</td>
</tr>
</tbody>
</table>
The graph above shows how the maximum weight \( \text{MAXWT} \) depends on the width \( w \). Because the points seem to lie on a line through the origin she concluded that \( \text{MAXWT} \) varies directly as \( w \).

Perri then investigated the relationship between \( \text{MAXWT} \) and the thickness \( t \). She held the distance \( d \) between supports constant at 10 ft and the width \( w \) constant at 3 in. She varied the thicknesses of the boards and measured the maximum weight that could be supported. The following table presents her findings.

<table>
<thead>
<tr>
<th>Thickness (in.)</th>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum weight (lb)</td>
<td>( \text{MAXWT} )</td>
<td>20</td>
<td>80</td>
<td>180</td>
<td>320</td>
<td>500</td>
<td>720</td>
</tr>
</tbody>
</table>

The graph above shows how \( \text{MAXWT} \) depends on \( t \). The points seem to lie on a parabola through the origin. This implies that \( \text{MAXWT} \) varies directly as the square of \( t \).

She investigated the relationship between \( \text{MAXWT} \) and \( d \) by holding \( t \) and \( w \) constant. She chose boards for which \( t \) was 2 in. and \( w \) was 3 in. Perri obtained the following data.

<table>
<thead>
<tr>
<th>Distance (ft)</th>
<th>( d )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum weight (lb)</td>
<td>( \text{MAXWT} )</td>
<td>800</td>
<td>400</td>
<td>267</td>
<td>200</td>
<td>160</td>
<td>133</td>
</tr>
</tbody>
</table>
The graph shows how $MAXWT$ depends on $d$. It is not immediately clear whether $MAXWT$ varies inversely as $d$ or inversely as $d^2$. However, it can be shown by the method of the last section that $MAXWT$ varies inversely as $d$.

Ms. Menter summarized her findings as follows:

$MAXWT$ varies directly as $w$ and the square of $t$;
$MAXWT$ varies inversely as $d$.

These relations can be expressed in a single formula as

$$MAXWT = \frac{kwt^2}{d},$$

where $k$ is a constant.

Notice that each independent variable that varies directly as $MAXWT$ is in the numerator, and the independent variable that varies inversely as $MAXWT$ is in the denominator. The formula tells you that the greater the width and depth and the shorter the distance between supports, the stronger the board will be.

In the next lesson you will calculate the constant of variation $k$ in this type of relationship.

**Questions**

1. *True or false* The variables $MAXWT$, $d$, $t$, and $w$ can all be graphed on one set of axes.

2. How can one investigate the relationship between a dependent variable and more than one independent variable?

3. How did Perri Menter determine that $MAXWT$ varies directly as $w$?

4. What was the maximum weight supported by a board 10 ft long, 3 in. wide, and 5 in. thick?

5. What is the shape of the graph of the relationship between $MAXWT$ and $d$?

6. In the formula $MAXWT = \frac{kwt^2}{d}$, any variable which varies directly as $MAXWT$ is in the $\underline{\text{?}}$ of the expression.

7. Use the method of Lesson 2-8 to show that $MAXWT$ varies inversely as $d$ and not $d^2$. 

LESSON 2-9  Fitting a Model to Data II  107
8. **Multiple choice** The two graphs below show the relationships between a dependent variable \( y \) and two independent variables \( x \) and \( z \). Which equation best models this situation?

(a) \( y = kx \)  
(b) \( y = \frac{kx}{z} \)  
(c) \( y = kxz^2 \)  
(d) \( y = kx^2z \)

9. Cyrus Nathan Tist attempted to find how the volume of a gas (the dependent variable) depends on the temperature and pressure of the gas (the independent variables).

a. When he held the pressure fixed at 250 millibars (abbreviated mbar), he obtained the following results. (The temperature is in degrees Kelvin, where Kelvin temperature = 273° + Celsius temperature.)

<table>
<thead>
<tr>
<th>Temperature (°K)</th>
<th>250</th>
<th>275</th>
<th>300</th>
<th>325</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume (cm³)</td>
<td>417</td>
<td>458</td>
<td>500</td>
<td>542</td>
<td>583</td>
</tr>
</tbody>
</table>

Graph these data points. On the V-axis, start at 400 and increase the scale by 20s; that is, make marks at 400, 420, ..., 600.

b. How does \( V \) vary with \( T \)?

c. When Cy held the temperature fixed at 300°K, he obtained the following results.

<table>
<thead>
<tr>
<th>Pressure (mbar)</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>350</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume (cm³)</td>
<td>625</td>
<td>500</td>
<td>417</td>
<td>357</td>
<td>313</td>
</tr>
</tbody>
</table>

Graph these data points. On the V-axis start at 300 and increase the scale by 50's.

d. How does \( V \) vary with \( P \)?

e. Write an equation of variation to show how \( V \) depends on \( T \) and \( P \). Do not solve for \( k \).

10. Cy was trying to determine how the pressure exerted on the floor by the heel of a shoe depends on the width of the heel and the weight of the person wearing the shoe. He started by measuring the pressure (in psi) exerted by several people wearing a shoe with a heel width of 3.5 in. The data are summarized below:

<table>
<thead>
<tr>
<th>Weight (lb)</th>
<th>62</th>
<th>85</th>
<th>100</th>
<th>128</th>
<th>154</th>
<th>180</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure (psi)</td>
<td>5.7</td>
<td>7.8</td>
<td>9.1</td>
<td>11.7</td>
<td>14.1</td>
<td>16.5</td>
</tr>
</tbody>
</table>
He then had his niece Ego, who weighs 142 lb, wear shoes with different heel widths, and he measured the pressure exerted. The data are summarized below:

<table>
<thead>
<tr>
<th>Heel width (in.)</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure (psi)</td>
<td>159.0</td>
<td>70.7</td>
<td>39.8</td>
<td>25.4</td>
<td>17.7</td>
<td>13.0</td>
</tr>
</tbody>
</table>

Assuming that PRESS (the pressure), \( w \) (the weight), and \( h \) (the heel width) are related by a variation model, find an equation to describe that relationship. Do not solve for \( k \).

11. Use \( y = \frac{-20}{x^2} \).
   a. What real number is excluded from the domain of \( x \)?
   b. Multiple choice Which could be the graph of the equation? (Lesson 2-7)

![Graph options]

12. Which of the following equations have graphs that are continuous? (Lessons 2-4, 2-5, 2-7)
   a. \( y = \frac{4x}{7} \)
   b. \( y = .08x^2 \)
   c. \( y = \frac{-10}{x} \)
   d. \( y = \frac{3}{x^2} \)

13. The braking distance needed to stop a car is directly proportional to the square of the car’s speed. (Lessons 2-1, 2-5)
   a. If it takes 50 meters to brake a car that was traveling 88 kph, how many meters will it take to brake a car traveling 100 kph?
   b. Find the rate of change for the braking distance between 88 kph and 100 kph.

14. The graph of the equation \( y = \frac{3x}{4} \) is a ___ with slope ___. (Lesson 2-4)

15. The perimeter of a rectangle is to be less than 12.4 meters. Its length is 3.2 meters. What are the possible values of the width? (Lessons 1-1, 1-9)

16. The ability of a board to support a weight also depends on the type of wood. In other words, the constant of variation \( k \) in the formula

\[
MAXWT = \frac{kwt^2}{d}
\]

depends on the type of wood.
   a. For a stronger kind of wood, is \( k \) larger or is \( k \) smaller?
   b. Which is strongest: oak, birch, or pine?
In this chapter you have seen several situations where a dependent variable varies directly with some variables and inversely with others. When direct and inverse variations occur together, the situation is one of combined variation. Perhaps the simplest equation of combined variation is

\[ y = \frac{kx}{z} \]

where \( k \) is the constant of variation. The equation can be translated as "\( y \) varies directly as \( x \) and inversely as \( z \)."

A combined-variation situation can have more than two variables, and the independent variables can have any positive exponent. You saw an instance of this in Lesson 2-9:

\[ \text{MAXWT} = \frac{kwT^2}{d} \]

This formula gives the maximum weight in pounds \( \text{MAXWT} \) that can be supported by a board of width \( w \) in., thickness \( t \) in., and distance between supports of \( d \) ft. But it does not give an explicit value for the constant \( k \). As you know, mathematical models can be used to make predictions, but this model cannot be used until the constant \( k \) is determined.

You can find \( k \) the same way you found the constant for direct and inverse variation. You need to find one instance that relates all the variables simultaneously. In Lesson 2-9 there are eighteen possible instances which can be used to find \( k \). (Each of the three graphs used in deriving the formula for \( \text{MAXWT} \) has six instances. For example, the first graph relating \( \text{MAXWT} \) and \( w \) gives six possible pairs of numbers for \( w \) and \( \text{MAXWT} \). For each of these pairs, \( t = 2 \) in. and \( d = 10 \) ft.) One instance you can use is \( \text{MAXWT} = 27 \) lb. \( w = 1 \) in., \( t = 2 \) in., and \( d = 10 \) ft. When you substitute into the formula, you get

\[ 27 \text{ lb} = \frac{k(1 \text{ in.})(2 \text{ in.})^2}{10 \text{ ft}} \]

Solve this for \( k \) to get

\[ 67.5 \frac{\text{ft-lb}}{\text{in.}^2} = k. \]

This value for \( k \) should be checked by using other data points; you will do this in the questions at the end of the lesson. Thus, the formula becomes

\[ \text{MAXWT} = \frac{67.5 \text{ wt}^2}{d} \quad \text{or} \quad \text{MAXWT} = 67.5 \frac{\text{wt}^2}{d}. \]

Now it is possible to use this model to make predictions.
Example 1  Find the maximum weight that can be supported by a board 1.5 in. wide and 11.5 in. deep, with supports 20 ft apart.

Solution  Use the preceding formula with \( d = 20 \text{ ft}, \ w = 1.5 \text{ in.}, \) and \( t = 11.5 \text{ in.} \) Then

\[
MAXWT = \frac{67.5(1.5)(11.5)^2}{20} = 669.5.
\]

The board can support about 670 lb.

Often a situation involving combined variation is expressed in English and must be translated into a mathematical statement.

Example 2  The time \( T \) that it takes a parade to pass a reviewing stand varies directly as the length \( L \) of the parade and inversely as the speed \( s \) of the parade.

Write a general equation to model this situation.

Solution  Because \( T \) is described in terms of \( L \) and \( s \), \( T \) is the dependent variable. Because \( T \) varies directly as \( L \), \( L \) will be in the numerator. Because \( T \) varies inversely as \( s \), \( s \) will be in the denominator. The equation is

\[
T = \frac{kL}{s}.
\]

Sometimes one quantity varies directly as the product of two or more independent variables, but not inversely as any variable. This is called joint variation. Perhaps the simplest equation of joint variation is

\[
y = kxz,
\]

where \( k \) is the constant of variation. The equation can be translated as "\( y \) varies jointly as \( x \) and \( z \)" or "\( y \) varies directly as the product of \( x \) and \( z \)."

As in combined variation, a joint variation situation can have more than two independent variables, and the independent variables can have any positive exponent. Recall from geometry the formula for the volume of a cone:

\[
V = \frac{1}{3}\pi r^2 h.
\]
This can be expressed as "the volume varies jointly with the height and the square of the radius of the base." The constant of variation is \( \frac{\pi}{3} \).

Example 3

The amount of heat \( H \) lost through a single pane window varies jointly as the area \( A \) of the pane and the difference \( T_i - T_o \) in temperatures on either side of the window. Suppose when the indoor temperature is \( T_i = 70^\circ \text{F} \) and the outdoor temperature is \( T_o = 0^\circ \text{F} \), the heat lost through a 12 ft\(^2\) window is 950 BTUs (British thermal units). Find the amount of heat lost through a 16 ft\(^2\) window if the indoor temperature is 75\(^\circ\) F and the outdoor temperature is -5\(^\circ\) F.

Solution

First, write the general equation: \( H, A, \) and \( T \) are related by

\[
H = kA(T_i - T_o).
\]

Now find \( k \).

When \( A = 12 \) and \( H = 950 \), \( T_i = 70^\circ \) and \( T_o = 0^\circ \).

Substitute.

\[
950 = k \cdot 12(70 - 0)
\]

Solve for \( k \).

\[
k \approx 1.13
\]

Now rewrite the formula with the calculated value of \( k \).

\[
H \approx 1.13 A(T_i - T_o)
\]

Finally, substitute \( A = 16 \), \( T_i = 75^\circ \) and \( T_o = -5^\circ \).

\[
H \approx (1.13) (16) (75 - -5)
\]

\[
\approx 1446.4
\]

The heat lost is about 1450 BTU, which is about as much heat as a small space heater provides.

Questions

1. **True or false** Combined variation involves both direct and inverse variations together.

   In 2–4, refer to the example about the maximum weight that can be supported by a board.

2. \( MAXWT \) varies directly as \( \_\_\_\_ \) and \( \_\_\_\_ \) and inversely as \( \_\_\_\_ \).

3. Find \( k \) by using this data point from Lesson 2-9: \( MAXWT = 80 \text{ lb}, \) \( w = 3 \text{ in.}, t = 2 \text{ in.}, \) and \( d = 10 \text{ ft} \). This checks that the value of \( k \) found in the text is reasonable.
4. Find the maximum weight that can be supported by a board with supports 16 ft apart, 11.5 in. wide, and 1.5 in. deep. (Use \( k = 67.5 \), as found in the text.)

5. Translate into a single formula: \( R \) varies directly as \( L \) and inversely as \( d^2 \).

6. Translate the formula \( V = \frac{1}{3} \pi r^2 h \) into English, using the language of variation.

7. Refer to Example 3. Find how much heat is lost through a 10-ft\(^2\) window when the indoor temperature is 72°F and the outdoor temperature is 34°F.

8. Translate into a single formula: The time \( t \) it takes to finish algebra homework varies directly as the number of questions assigned \( a \) and inversely as the number \( d \) that can be solved with the aid of a calculator.

9. One general equation for a combined variation is

\[
y = k \frac{xz}{w}.
\]

Solve for \( k \) in terms of the other variables.

10. Use the formula for \( \text{MAXWT} \). Suppose that the maximum load that can be supported by a board is 2250 lb, and that the constant of variation is 67.5 \( \text{ft-lb in}^{-3} \). If the board is 10 in. deep and the supports are 12 ft apart, how wide is it?

11. Refer to Example 2. A parade 600 ft long walking at 2.5 mph needs 90 min to pass the reviewing stand. How long would it take a parade 500 ft long walking at 3 mph to pass the reviewing stand?

12. The wind force \( F \) on a vertical surface varies jointly as the area \( A \) of the surface and the square of the wind speed \( S \). The force is 75 lb on a vertical surface of area 10 ft\(^2\) when the wind blows at 40 mph.
   a. Using the given variables, translate the first statement into an equation of variation.
   b. Find the constant of variation.
   c. Rewrite the equation of variation using the constant found in part b.
   d. Find the force exerted by a wind of 80 mph on a vertical surface of area 25 ft\(^2\).

13. \( y \) varies directly as \( x \) and inversely as \( z \). Find how \( y \) changes when \( x \) and \( z \) are both doubled.
14. The resistance $R$ in an electrical circuit is related to the diameter $d$ of the wire and the length $L$ of the wire. (Lessons 2-7, 2-8)
   a. In an experiment, Perri Menter obtained the following data with a 50 ft wire.

<table>
<thead>
<tr>
<th>Diameter (in.)</th>
<th>.05</th>
<th>.08</th>
<th>.11</th>
<th>.14</th>
<th>.17</th>
<th>.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance (ohms)</td>
<td>9.0</td>
<td>3.5</td>
<td>1.9</td>
<td>1.1</td>
<td>0.8</td>
<td>0.6</td>
</tr>
</tbody>
</table>

   Graph these data points.
   b. How does $R$ vary with $d$?
   c. With a wire of diameter .05 in. she obtained the following data.

<table>
<thead>
<tr>
<th>Length (ft)</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance (ohms)</td>
<td>4.5</td>
<td>9</td>
<td>13.5</td>
<td>18</td>
<td>22.5</td>
<td>27</td>
</tr>
</tbody>
</table>

   Graph these data points.
   d. How does $R$ vary with $L$?
   e. Write an equation that relates $R$, $d$, and $L$. You do not need to find the constant of variation.

15. Use the equation $y = 10x^2$. (Lesson 2-5)
   a. Graph the solution for $-3 \leq x \leq 3$.
   b. What is the name of this curve?
   c. Find the rate of change between $x = 1$ and $x = 2$.
   d. Should you expect that the answer to part c would be the same for any two points on the graph?

In 16 and 17, use the graph of $y = \frac{20}{x^2}$ at the right. (Lessons 2-6, 2-7)

16. Sketch a graph of this equation on the window $-10 \leq x \leq 10$.
   $-10 \leq y \leq 10$.

17. True or false The graph of $y = \frac{20}{x^2}$ is a hyperbola.

In 18–20, an instance of a general property is given. Write the general property. (Lessons 1-5, 1-6, 1-9)

18. $40y - y = 39y$
19. If $2 < 3 - z$, then $2 + z < 3$.

20. $\frac{2}{3} + \frac{x}{3} = \frac{2 + x}{3}$

In 21–22, solve. (Lesson 1-9)

21. $5 - 3x \leq 86$
22. $.05(y - 3) - (.2y - 5) > y + 1$

23. a. Find out what unit is used for heat in the metric system.
   b. How is this unit related to the BTU?
In a formula where \( y \) is given in terms of \( x \), it is natural to ask how changing \( x \) (the independent variable) affects the value of \( y \) (the dependent variable). The rate of change \( \frac{y_2 - y_1}{x_2 - x_1} \) between the two points \((x_1, y_1)\) and \((x_2, y_2)\) is the slope of the line connecting them.

Two types of formulas studied in this chapter are direct variation and inverse variation. When \( k \neq 0 \) and \( n > 0 \), formulas of the form \( y = kx^n \) represent direct variation, and those of the form \( y = \frac{k}{x^n} \) represent inverse variation.

In direct or inverse variation, simple changes occur in \( y \) when \( x \) is multiplied by a constant. When \( x \) is multiplied by \( c \): if \( y \) varies directly as \( x^n \), then \( y \) is multiplied by \( c^n \), and if \( y \) varies inversely as \( x^n \), then \( y \) is divided by \( c^n \). Four special cases commonly occur, and their graphs have special names.

**Direct-variation formulas**

\[
\begin{align*}
y &= kx \\
y \text{ varies directly as } x. & \quad y &= kx^2 \\
k > 0 & \quad k > 0 & \quad k < 0 \\
\end{align*}
\]

![Line graph](image)

![Parabola graph](image)

**Inverse-variation formulas**

\[
\begin{align*}
y &= \frac{k}{x} \\
y \text{ varies inversely as } x. & \quad y &= \frac{k}{x^2} \\
k > 0 & \quad k > 0 & \quad k < 0 \\
\end{align*}
\]

![Hyperbola graph](image)

![Inverse-square graph](image)
Formulas may involve three or more variables. If all the independent variables are multiplied, then joint variation occurs. If they are not all multiplied, the situation is one of combined variation. Variation formulas can be derived from real data by examining two variables at a time and comparing their graphs with those given above. We call this idea modeling, or forming a mathematical model of the data. Automatic graphers, such as graphing calculators or computers with graphing programs, are useful tools to help in graphing and in comparing graphs.

The applications of slope, variation, and modeling are numerous. They include many perimeter, area, and volume formulas; the inverse square laws of sound and gravity; and a variety of relationships among physical quantities such as distance, time, force, and pressure.

---

**Vocabulary**

Below are the most important terms and phrases for this chapter. You should be able to state each in words and give a specific example. For the starred (*) terms, you should be able to supply a good definition.

**Lesson 2-1**

- direct variation*, directly proportional to, varies directly as, constant of variation, dependent variable, independent variable

**Lesson 2-2**

- inverse variation*, is inversely proportional to, varies inversely as, inverse-square variation

**Lesson 2-3**

- Fundamental Theorem of Variation

**Lesson 2-4**

- *rate of change, *slope

**Lesson 2-5**

- parabola, reflection-symmetric line of symmetry

**Lesson 2-6**

- automatic grapher, function grapher, window, default window, zoom feature, hard copy

**Lesson 2-7**

- discrete, discontinuous, continuous hyperbola, branches of a hyperbola, inverse-square curve

**Lesson 2-8**

- mathematical model

**Lesson 2-10**

- combined variation, joint variation
Take this test as you would take a test in class. Use graph paper and a ruler. Then check your work with the solutions in the Selected Answers section in the back of the book.

In 1–3, translate into a variation formula.
1. \( y \) varies inversely as \( x \).
2. The number \( n \) of trees that can be planted per acre varies inversely as the square of their distance \( d \) apart.
3. The weight \( w \) that a column of a bridge can support varies directly as the fourth power of its diameter \( d \) and inversely as the square of its length \( L \).
4. If \( S \) varies directly as the square of \( p \) and \( S = 10 \) when \( p = 3 \), find \( S \) when \( p = 8 \).
5. For the variation equation \( y = 3x^2 \), what is the change in the \( y \)-value when an \( x \)-value is doubled?
6. For the variation equation \( y = \frac{6}{x} \), what is the change in the \( y \)-value when an \( x \)-value is multiplied by \( c \) \((c \neq 0)\)?
7. Find the rate of change of the line through the points \((12, 18)\) and \((20, 30)\).
8. True or false  All graphs of variation pass through the origin.
9. The graph of \( y = kx^2 \) is called a ___ and opens up if ___.

10. Which word or phrase does not describe the graph below?
   a. continuous  b. discrete  c. symmetric about the \( y \)-axis

11. Fill in the blank with the word "inversely," "directly," or "neither inversely nor directly."
   a. The surface area of a sphere varies ___ as the cube of its radius.
   b. The number of different shares you can buy varies ___ as the cost of each share, if you invest exactly $10,000.

In 12 and 13, graph on a coordinate plane.
12. \( y = -5x \)  13. \( y = \frac{5}{x} \)

In 14 and 15, multiple choice
14. Find the equation whose graph looks the most like the graph shown below.
   (a) \( y = -3x \)
   (b) \( y = -\frac{3}{x} \)
   (c) \( y = -\frac{3}{x^2} \)
   (d) \( y = -\frac{x}{3} \)

The Progress Self-Test continues on page 118.
15. Below is a graph of \( y = x^2 \) on the window \(-4 \leq x \leq 4, 0 \leq y \leq 10\). Which cannot be a graph of this equation on some other window?

![Graph of \( y = x^2 \)](image)

16. A worker removing the bolts from the back of a large cabinet knew that it was easier to turn a bolt with a long wrench than with a short one. He decided to investigate the force required with various wrenches. He obtained the following data.

<table>
<thead>
<tr>
<th>Length of wrench (in.)</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force (lb)</td>
<td>620</td>
<td>372</td>
<td>310</td>
<td>233</td>
<td>207</td>
</tr>
</tbody>
</table>

a. Graph these data points.
b. Which variation equation is a better model for this situation, \( F = \frac{k}{L} \) or \( F = \frac{k}{L^2} \)?
c. How much force would be required to turn one of these bolts with a 12-in. wrench?

17. Suppose that variables \( V \), \( h \), and \( g \) are related as illustrated in the graphs below. The points on the graph at the left lie on or near a parabola. The points on the graph at the right lie on a line through the origin.

![Graphs of \( V \) vs. \( h \) and \( g \)](image)

Write a general equation approximating the relationship among \( V \), \( h \), and \( g \).

18. Poiseuille’s Law states that the speed \( S \) at which blood flows through arteries and veins varies directly with the blood pressure \( P \) and the fourth power of the radius \( r \) of the blood vessel. Suppose that blood flows at a rate of 0.09604 cm³/sec through an artery of diameter 0.14 cm when the blood pressure is a normal 100 units. What would be the blood pressure if cholesterol reduced the artery to 0.1 cm in diameter, and the speed stayed the same?
SPUR stands for Skills, Properties, Uses, and Representations.
The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers.

**Objective A:** Translate variation language into formulas. (Lessons 2-1, 2-2, 2-10)

In 1–8, translate into a variation equation.

1. y varies directly as the square of x.
2. s varies inversely with p.
3. The number n of congruent marbles that fit into a box is inversely proportional to the cube of the radius r of each marble.
4. The area A of an image on a movie screen is directly proportional to the square of the distance d from the projector to the screen.
5. The rate of vibration U of a stretched string varies directly with the square root of the tension T and inversely with the product of its length L and diameter D.
6. z varies jointly as x and t.
7. The gravitational pull P of a star on a mass m varies directly as the mass and inversely as the square of the distance d from the star.
8. At a given speed, the distance traveled is directly proportional to the time traveled.
9. In the formula \( r = kstu \), r varies \( \frac{?}{?} \) with \( \frac{?}{?} \).
10. If \( V = k\pi r^2 \), then V varies \( \frac{?}{?} \) as \( \frac{?}{?} \).

**Objective B:** Solve variation problems. (Lessons 2-1, 2-2, 2-10)

11. y varies directly as x. If \( x = 4 \), then \( y = -12 \). Find y when \( x = -7 \).
12. y varies directly as the square of x. When \( x = -5 \), \( y = 75 \). Find y when \( x = 8 \).
13. y varies inversely as the cube of x. If \( x = 4 \), \( y = -\frac{1}{16} \). Find y when \( x = \frac{1}{2} \).
14. z varies directly as the square of x and inversely as y. When \( x = 3 \) and \( y = 5 \), \( z = 4.5 \). Find z when \( x = -2 \) and \( y = -1.5 \).

**Objective C:** Find slopes (rates of change). (Lessons 2-4, 2-5, 2-7)

15. Find the slope of the line through the points (15, 27) and (20, 36).

In 16 and 17, \( y = 5x^2 \).

16. Find the rate of change between \( x = -2 \) and \( x = -1 \).

17. Find the rate of change between \( x = -3 \) and \( x = -2 \).

In 18 and 19, find the rate of change between \( x = 3 \) and \( x = 4 \).

18. \( y = \frac{9}{x} \)

19. \( y = \frac{9}{x^2} \)

20. What is the slope of the line below?

![Graph](image)
PROPERTIES deal with the principles behind the mathematics.

**Objective D:** Determine the effects of changes in the values of variables in a variation formula. (Lessons 2-3, 2-10)

In 21 and 22, suppose that in a variation problem the value of $x$ is tripled. Tell how the value of $y$ changed for each type of variation.

21. $y$ varies directly as $x$.
22. $y$ varies directly as $x^3$.

In 23–24, suppose that $p$ varies inversely as the square of $q$. How does the value of $p$ change if $q$ is

23. doubled?
24. multiplied by ten?
25. If $y = \frac{k}{x^n}$ and $x$ is multiplied by any nonzero constant $c$, then $y$ is ___.
26. If $y = kx^n$ and $x$ is divided by any nonzero constant $c$, then $y$ is ___.

USES deal with applications of mathematics in real situations.

**Objective F:** Recognize variation situations. (Lessons 2-1, 2-2)

In 34–38, complete with "directly," "inversely," or "neither directly nor inversely."

34. The number of adults invited to dinner varies ___ as the number of pieces of silverware used.
35. The number of people invited to dinner varies ___ as the amount of space each guest has at the table.
36. The temperature in a house varies ___ as the number of hours the air-conditioner has been on.
37. The volume of a cylinder of height 10 cm varies ___ as the square of its radius.
38. Your height on a ferris wheel varies ___ as the number minutes you have been on it.

27. If $y = \frac{kx^n}{z^n}$ and both $x$ and $z$ are multiplied by any nonzero constant $c$, then $y$ is ___.

**Objective E:** Identify the properties of variation graphs. (Lessons 2-4, 2-5, 2-7)

28. The graph of the equation $y = kx$ is a ___ having slope ___.
29. Graphs of all direct variation formulas go through the point ____.

In 30–33, refer to these four equations:

(a) $y = kx$  (b) $y = kx^2$  (c) $y = \frac{k}{x}$  (d) $y = \frac{k}{x^2}$.

30. Which equations have graphs that are symmetric to the y-axis?
31. The graph of which equation is a parabola?
32. True or false All four equations have graphs which are continuous everywhere.
33. True or false When $k > 0$, all four equations have points in quadrant 1.

**Objective G:** Fit an appropriate model to data. (Lessons 2-8, 2-9)

In 39 and 40, do steps a to d.

a. Draw a graph to represent the situation.
b. Find a general variation equation to represent the situation.
c. Find the value of the constant of variation and rewrite the variation equation.
d. Answer the question stated in the problem.

39. Officer Friendly measured the length of car skid marks when the brakes were applied at different speeds. He obtained the following data.

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of skid (ft)</td>
<td>18</td>
<td>41</td>
<td>72</td>
<td>113</td>
<td>162</td>
</tr>
</tbody>
</table>

How far would a car skid if the brakes are applied at 70 mph?
40. A man weighs 200 lb on the surface of Earth. The following table gives his weight at various distances from the center of Earth. (Remember: the radius of Earth is approximately 4000 mi.)

<table>
<thead>
<tr>
<th>Distance (miles)</th>
<th>4000</th>
<th>4500</th>
<th>5000</th>
<th>5500</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (lb)</td>
<td>200</td>
<td>158</td>
<td>128</td>
<td>106</td>
<td>89</td>
</tr>
</tbody>
</table>

How much would the man weigh on the top of Mt. Everest, which is about 4005.5 miles from Earth’s center?

41. Cyrus N. Tist tried to discover how the power in an electric circuit is related to the strength of the current and the resistance of the wire. When he held the current constant at 5 amps, he obtained the following data relating power $P$ and resistance $R$.

<table>
<thead>
<tr>
<th>Resistance (ohms)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power (watts)</td>
<td>125</td>
<td>500</td>
<td>1125</td>
<td>2000</td>
<td>3125</td>
<td>4500</td>
</tr>
</tbody>
</table>

a. Graph these data points.
b. How does $P$ vary with $R$?

Then Cy held the resistance constant at 10 ohms. He obtained the following data relating power $P$ and current $C$.

<table>
<thead>
<tr>
<th>Current (amps)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power (watts)</td>
<td>500</td>
<td>1000</td>
<td>1500</td>
<td>2000</td>
<td>2500</td>
<td>3000</td>
</tr>
</tbody>
</table>

c. Graph these data points.
d. How does $P$ vary with $C$?
e. Write an equation of variation relating $P$, $R$, and $C$. Do not find the constant of variation.

42. Perri Menter performed an experiment to determine how the pressure $P$ of a liquid on an object is related to the depth $d$ of the object and the density $D$ of the liquid. She obtained the graph on the left by keeping the depth constant and measuring the pressure on an object in solutions with different densities. She obtained the graph on the right by keeping the density constant and measuring the pressure on an object in a solution at various depths.

Write a general equation relating $P$, $d$, and $D$. Do not find the constant of variation.

Objective H: Solve problems using joint and combined variation models. (Lessons 2-1, 2-2, 2-10)

43. Suppose the price of a pizza varies directly with the square of its diameter. At Vic Yee’s pizza parlour an 8” pizza costs $6.00. How much would a 12” pizza cost?

44. The refund $r$ you get varies directly with the number $n$ of cans you recycle. If you get a $7.50 refund for 150 cans, how much should you get for 400 cans?

45. One of Murphy’s Laws is that the time $t$ a committee spends debating a budget item is inversely proportional to $d$, the number of dollars involved. If a committee spends 10 minutes debating a $300 item, how much time is spent debating a $1000 item?

46. Recall that Newton’s Law of Universal Gravitation is $W = \frac{k}{r^2}$. If Ms. Smith’s son Ian weighs 75 lb on the surface of Earth, how much will he weigh in space 50,000 miles from Earth’s surface? (The radius of Earth is approximately 4000 miles.)

47. The force needed to keep a car from skidding on a curve varies directly as the weight of the car and the square of the speed and inversely as the radius of the curve. Suppose 3960 lb of force is required to keep a 2200-lb car traveling at 30 mph from skidding on a curve of radius 500 ft. How much force is required to keep a 3000-lb car traveling at 45 mph from skidding on a curve of radius 400 ft?
48. An object is tied to a string and then twirled in a circular motion. The tension in the string varies directly as the square of the speed and inversely as the radius. When the radius is 5 ft and the speed is 4 ft/sec, then tension in the string is 90 lb. If the radius is 3.5 ft and the speed is 4.4 ft/sec, find the tension in the string.

**Representations** deal with pictures, graphs, or objects that illustrate concepts.

**Objective I:** Graph variation equations and identify equations from graphs. (Lessons 2-4, 2-5, 2-6, 2-7)

In 49–54, graph each equation.

49. \( y = \frac{1}{2}x \)
50. \( y = \frac{1}{2}x^2 \)
51. \( y = -2x \)
52. \( y = -2x^2 \)
53. \( y = \frac{36}{x} \)
54. \( y = \frac{36}{x^2} \)

**Multiple choice** In 55–58, select the equation whose graph is most like that shown below. Assume the scales on the axes are equal.

55. (a) \( y = 4x \) (b) \( y = -4x^2 \) (c) \( y = -\frac{1}{4}x \) (d) \( y = -\frac{1}{4} \)

56. (a) \( y = 10x^2 \) (b) \( y = -x^2 \) (c) \( y = -10x \) (d) \( y = -\frac{10}{x^2} \)

57. (a) \( y = \frac{x^2}{6} \) (b) \( y = \frac{6}{x} \) (c) \( y = \frac{-6}{x} \) (d) \( y = \frac{-6}{x^2} \)

58. (a) \( y = \frac{x^2}{6} \) (b) \( y = \frac{6}{x} \) (c) \( y = \frac{-6}{x} \) (d) \( y = \frac{-6}{x^2} \)
Objective J. Recognize the effects of a change in scale or viewing window on a graph of a variation equation. (Lesson 2-6)

In 59 and 60, a graph of \( y = 4x \) is drawn below at left using the window \(-5 \leq x \leq 5\), \(-25 \leq y \leq 25\).

59. Sketch a graph of this equation on the window shown above at the right.

60. Does the slope of the line \( y = 4x \) change when the viewing window is changed? If so, how?

61. In the graph of \( y = kx^2 \) shown below, which cannot be the value of \( k \)?
(a) 2  (b) 1  (c) 0.1  (d) -1

Objective K: Read and interpret graphs of joint and combined variation. (Lessons 2-8, 2-9, 2-10)

Multiple choice

62. Which of the following equations could model the relationship graphed in Question 59?
(a) \( y = kx \)  (b) \( y = \frac{k}{x} \)
(c) \( y = kx^2 \)  (d) \( y = \frac{k}{x^2} \)

63. Which of the following equations could model the relationship shown by the two equations \( y = k_1x^2 \) and \( y = k_2z^2 \)?
(a) \( y = kx \)  (b) \( y = kxz^2 \)
(c) \( y = kx^2z \)  (d) \( y = kx^2z^2 \)

64. Which of the following equations could model the relationship shown by the three equations \( P = k_1Q \), \( P = k_2R^2 \), and \( P = k_3S^2 \)?
(a) \( P = kQR^2S^2 \)  (b) \( P = kQ^2R^2S^2 \)
(c) \( P = \frac{kRS^2}{Q} \)  (d) \( P = \frac{k}{R^2S^2Q} \)
CHAPTER 3

Linear Relations

3-1: Constant Increase or Decrease
3-2: The Graph of \( y = mx + b \)
3-3: Linear Combinations
3-4: The Graph of \( Ax + By = C \)
3-5: Finding an Equation of a Line
3-6: Arithmetic Sequences: Explicit Formulas
3-7: Arithmetic Sequences: Recursive Formulas
3-8: Piecewise Linear Graphs
3-9: Linear Inequalities
In Chapter 2 you studied direct variation which was modeled by equations of the form \( y = kx \). The graph of \( y = kx \) is a line, so \( y = kx \) is called a linear equation. Many other situations can be modeled by linear equations. Here are three types.

**Constant Increase**
A crate weighs 30 kilograms when empty. It is filled with oranges weighing 0.2 kilogram each. Find the weight \( W \) of a crate containing \( n \) oranges.
Answer: \( W = 30 + .2n \)

**Linear Combination**
A group bought \( A \) adult tickets at $7 each and \( S \) student tickets at $3 each. The group spent $42. What equation relates \( A, S \), and the total amount spent?
Answer: \( 7A + 3S = 42 \)

**Point–Slope**
Stuart Dent is conducting an experiment with a spring and a weight. The spring is 15 centimeters long when a 10-gram weight is attached, and its length increases 0.8 centimeter with each additional gram weight. Write an equation relating spring length \( L \) and weight \( W \).
Answer: \( L - 15 = .8(W - 10) \)

In this chapter you will learn how to determine linear equations and inequalities used to model situations similar to these. You will also discover some efficient and powerful techniques for graphing lines and linear relations.
Consider these situations:

A. The temperature at 8:00 A.M. is 5° Celsius and increases 2° per hour over a five-hour period;
B. A medical laboratory charges each patient an initial fee of $30 for consultation and an additional $10 per test;
C. A 150-kg man goes on a diet and loses 1 kg per week;
D. At the beginning of the month, Katie buys a 50-pound sack of wild-bird feed. She puts $\frac{2}{3}$ of a pound in the bird feeder each morning.

In each of the above situations there is a constant change applied to an initial condition. In A and B that change is a constant increase. In C and D the change is a constant decrease. These situations can all be modeled by linear equations. The following examples show how.

Example 1  The temperature is 5° Celsius and is increasing 2° an hour. What is the temperature after $h$ hours?

Solution  Write the temperature for several hours to find a general pattern.

<table>
<thead>
<tr>
<th>Hours</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5 + 0 \cdot 2 = 5</td>
</tr>
<tr>
<td>1</td>
<td>5 + 1 \cdot 2 = 7</td>
</tr>
<tr>
<td>2</td>
<td>5 + 2 \cdot 2 = 9</td>
</tr>
<tr>
<td>3</td>
<td>5 + 3 \cdot 2 = 11</td>
</tr>
<tr>
<td>4</td>
<td>5 + 4 \cdot 2 = 13</td>
</tr>
<tr>
<td>5</td>
<td>5 + 5 \cdot 2 = 15</td>
</tr>
</tbody>
</table>

If $T$ is temperature and $h$ is the number of hours, then the equation relating $T$ and $h$ is

$5 + h \cdot 2 = T$

or

$T = 2h + 5$. 
Because \( T \) is expressed in terms of \( h \), \( T \) is the dependent variable and \( h \) is the independent variable. Recall that the independent variable is plotted along the horizontal axis and the dependent variable is plotted along the vertical axis. Thus solutions to the equation \( T = 2h + 5 \) are all the ordered pairs \((h, T)\) whose values satisfy the equation. These pairs are shown on the graph below.

For hours starting with the initial reading, the graph is a ray. Points farther along the ray represent times and temperatures farther in the future. If you think of negative values of \( h \) as "hours ago," then the graph of \( T = 2h + 5 \) is a line including points to the left of the vertical axis and below the horizontal axis.

Recall that an intercept of a graph is the coordinate of a point where the graph intersects an axis. Here the graph crosses the \( T \)-axis at \((0, 5)\), so the \( T \)-intercept is 5. The graph contains \((-2.5, 0)\), so its \( h \)-intercept is -2.5.

The slope of the line is the rate of change, \( 2^\circ \) per hour. To test this, find the slope between two points on the line. Trying \((0, 5)\) and \((3, 11)\) with the slope formula gives

\[
\frac{11 - 5}{3 - 0} = \frac{6}{3} = 2.
\]

Notice that in \( T = 2h + 5 \), the slope is 2 and 5 is the \( y \)-intercept.

LESSON 3-1  Constant Increase or Decrease  127
In general, any constant increase or constant decrease situation can be modeled by an equation of the form $y = mx + b$. The graph of $y = mx + b$ is a line with slope $m$ and $y$-intercept $b$. The slope $m$ corresponds to the rate of change in the situation. The $y$-intercept $b$, which is the value of $y$ when $x = 0$, corresponds to the initial value of the dependent variable. The form $y = mx + b$ is called the slope-intercept form of an equation for a line.

Example 2 describes a situation of constant decrease.

Example 2
At the beginning of the month, Katie buys a 50-pound sack of wild-bird feed. She puts $\frac{2}{3}$ pound in the bird feeder each morning.

a. Let $y$ (the dependent variable) be the number of pounds left in the sack after $x$ days. Write an equation relating $y$ to $x$ in slope-intercept form.

b. Graph the equation from part a.

c. How long will it be until the supply runs out?

Solution

a. This is an instance of constant decrease. So the equation is

$$y = mx + b,$$

and $m$ and $b$ need to be found. The rate of change $m$ is $\frac{2}{3}$ pound per day. Because the amount of feed in the sack is decreasing, $m = -\frac{2}{3}$. The initial amount of food is 50 pounds. Because 50 is the value of $y$ when $x = 0$, the $y$-intercept is 50, and the equation is

$$y = -\frac{2}{3}x + 50.$$

b. Make a table with some of the solutions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>50</td>
<td>48</td>
<td>46</td>
<td>44</td>
<td>42</td>
</tr>
</tbody>
</table>

![Graph of the equation $y = -\frac{2}{3}x + 50$.]
c. The supply runs out when \( y = 0 \). Substitute this into the equation and solve for \( x \).

\[
\begin{align*}
\text{Substitute} & \quad 0 = -\frac{2}{3}x + 50 \\
\text{Multiply by 3 to clear fractions} & \quad 0 = -2x + 150 \\
\text{Add } 2x \text{ to both sides.} & \quad 2x = 150 \\
\text{Multiply both sides by } \frac{1}{2}. & \quad x = 75
\end{align*}
\]

The supply will last 75 days.

Check Although our graph does not extend far enough to show it, the point \((75, 0)\) is on the graph of this equation. The rate of change between any two points should be the slope, \(-\frac{2}{3}\). Using \((0, 50)\) and \((6, 46)\) gives a slope of

\[
\frac{50 - 46}{0 - 6} = \frac{4}{-6} = -\frac{2}{3}.
\]

In cases of constant increase, as in Example 1, the graph of the line slants up from left to right, indicating a positive slope. In cases of constant decrease, as in Example 2, the graph slants down from left to right, indicating a negative rate of change. This makes it easy to tell at a glance whether a graph represents a linear increase or decrease.

Questions

In 1–4, refer to Example 1.

1. In \( T = 2h + 5 \), name:
   a. the independent variable
   b. the dependent variable

2. What is the temperature after \( 3\frac{1}{2} \) hours?

3. In the equation \( T = 2h + 5 \), 5 represents the \( \_? \) on the graph and the \( \_? \) in the problem.

4. True or false The \( 2^o \) increase per hour is the slope of the line.

5. True or false All instances of constant increase can be modeled by the equation \( y = mx + b \).

In 6 and 7, refer to the equation \( y = mx + b \).

6. The coefficient of \( x \) tells you the \( \_? \) of the line.

7. In cases of constant increase or constant decrease, the \( y \)-intercept \( b \) corresponds to \( \_? \).
In 8 and 9, refer to Example 2.

8. How many pounds of bird feed are left after 10 days?

9. How long will it take for the supply of bird feed to get below 15 pounds?

10. The graph below represents an example of constant ___.

![Graph](image)

In 11-14, identify: (a) the slope; (b) the y-intercept.

11. $y = 6x - 5$
12. $y = \frac{3}{5} - \frac{3}{4}x$
13. $y = x + 3$
14. $y = kx$

15. The equation $y = \frac{3}{4}x + 7$ is graphed below.
   a. Verify that $(4, 10)$ is on the graph.
   b. From the equation, what should the slope be?
   c. Use the points $(0, 7)$ and $(4, 10)$ to verify your answer to part b.
   d. What is the y-intercept?

![Graph](image)

16. Refer to situation C, about the man on a diet, on page 126.
   a. Find an equation relating weight $K$ and number of weeks $w$.
   b. Graph your equation from part a.

17. Suppose $y$ varies directly as $x$, and that $y$ is 7 when $x$ is 2.
   a. Find the constant of variation and write an equation describing the variation.
   b. Make a table of values and graph the equation.
   c. Verify that this direct variation equation fits the $y = mx + b$ model by identifying the slope and y-intercept.
   d. Does this variation represent constant increase or constant decrease?
18. **Skill sequence** Solve for \( x \). (Lessons 1-7, 1-9)
   
   a. \(-3x = 1.8\)  
   b. \(9 - 3x = 1.8\)  
   c. \(2 - (9 - 3x) \leq 1.8\)  
   d. \(2 - (9 - 3x) \leq 1.8 - x\)

19. Solve for \( y \): \(x + 2y = 5\). (Lesson 1-7)

20. Suppose \( B \) ounces of blended fruit juice is 10% apple juice. How many ounces of juices other than apple juice are in the blend? (Lesson 1-5)

21. a. Find the area of a circle inscribed in a square with side 6 cm long.  
   b. Find the area of a circle inscribed in a square with side \( x \) cm long.  
   (Previous course)

22. Given \( a_n = 2 + 5(n - 1) \). What is the first value of \( n \) which makes \( a_n \geq 51 \)? (Lessons 1-3, 1-9)

23. Simplify \( \frac{1}{a} \). (Lesson 1-5)  

24. Simplify \( \frac{y}{x} \). (Lesson 1-5)

25. What place in the world would you most like to visit? Find out how much it would cost to go there by air, and estimate how much your average daily expenses would be. Write an equation that can be used to calculate the total cost \( T \) of your visit if you stayed for \( n \) days.
As you saw in the last section, the solutions to an equation of the form \( y = mx + b \) lie on a line with slope \( m \) and \( y \)-intercept \( b \). The slope and \( y \)-intercept give you a powerful and efficient way to graph any equation in this form.

**Example 1**

Graph the line \( y = 4x + 7 \) using its slope and \( y \)-intercept.

**Solution**

The \( y \)-intercept is 7, so the line contains \((0, 7)\). Use the slope to locate another point. The slope 4 means that every horizontal change of one unit to the right corresponds to a vertical change of four units up. Starting at \((0, 7)\), count 1 unit right and 4 up. This gives the new point \((0 + 1, 7 + 4) = (1, 11)\). Plot \((1, 11)\) and draw the line.

**Check 1**

The point \((1, 11)\) satisfies the equation \( y = 4x + 7 \), since \( 11 = 4 \cdot 1 + 7 \). Since the two points \((0, 7)\) and \((1, 11)\) determine a line, the graph must be correct.

**Check 2**

The point \((-1, 3)\) satisfies the equation. This point also lies on the line determined by \((0, 7)\) and \((1, 11)\).
The line $y = 4x - 2$ is graphed below, along with $y = 4x + 7$. Both lines have slope 4. On each line, as you move 1 unit to the right, the line moves up 4 units. Right triangles $ABC$ and $DEF$ are congruent by SAS Congruence, so these lines form congruent angles at $A$ and $D$ with the $y$-axis. Consequently, $\overline{AB}$ and $\overline{DE}$ are parallel. This argument can be repeated with any two lines that have the same slope. Thus, the following theorem can be proved.

**Theorem:**

If two lines have the same slope, then they are parallel.

You can also prove the converse of this theorem. Parallel lines are drawn below, with slopes $m_1$ and $m_2$, and transversal $l$ is parallel to the $y$-axis. Recall that corresponding angles formed by parallel lines and a transversal are congruent. So the corresponding angles, $\angle 1$ and $\angle 2$, are congruent. Also, line $l$ forms right angles with $\overrightarrow{GH}$ and $\overrightarrow{JK}$. So the triangles are congruent by the AAS Congruence. Consequently, $m_1 = m_2$ and the slopes are equal. Since this proof used lines that intersect the $y$-axis, we have proved the converse, which is stated at the top of page 134.
Theorem:

If two nonvertical lines are parallel, then they have the same slope.

Graphing a line by using its slope and y-intercept can be much faster than first constructing a table of solutions. However, at times an equation for a line may need to be rewritten before it is in slope-intercept form. Example 2 illustrates this.

**Example 2**  Graph the line $2y = -3x + 10$ using its slope and y-intercept.

**Solution**  The equation is solved for $2y$, thus it is not in slope-intercept form. To solve for $y$, divide both sides by 2.

$$y = -\frac{3}{2}x + 5$$

From this form you can see that the slope is $-\frac{3}{2}$ and the y-intercept is 5. Again, first plot the y-intercept. A slope of $-\frac{3}{2}$ means a vertical change of $-\frac{3}{2}$ unit for every horizontal change of 1 unit, which is the same as 3 units down for every 2 units to the right.

Start at $(0, 5)$ to get the new point $(0 + 2, 5 - 3) = (2, 2)$.

**Check**  Substitute $(2, 2)$ into the equation.

Does $2 = -\frac{3}{2}(2) + 5$? Yes.

Lines with negative slope go down to the right. Lines with positive slope go up to the right. Lines with slope 0 are horizontal. Vertical lines are a different matter; they are discussed in Lesson 3-4.
Example 3  Graph the line \( y = -2 \).

Solution  The equation \( y = -2 \) is the same as \( y = 0x - 2 \). This shows that the y-intercept is -2. So the line contains \((0, -2)\) and the slope is 0. A slope of 0 means that for a horizontal change of 1 unit there is a vertical change of 0 units. In other words, there is no vertical change, and the graph is a horizontal line.

Check  Any value of \( x \) in \( y = 0x - 2 \) yields a y-value of -2. In other words, all points on the line have -2 as their second coordinate.

In general, a line is horizontal if and only if it has an equation of the form \( y = b \). Its slope is 0 and its y-intercept is \( b \).

Questions

Covering the Reading

1. The equation \( y = mx + b \) is called the \( ? \) form of an equation for a line.

2. A slope of 7 means a \( ? \) change of \( ? \) units for every horizontal change of one unit.

3. Multiple choice  A slope of \( -\frac{5}{6} \) means
   (a) a vertical change of -6 units for a horizontal change of 5 units.
   (b) a vertical change of \( -\frac{5}{2} \) unit for every horizontal change of 1 unit.
   (c) a vertical change of 6 units for a horizontal change of -5 units.
   (d) a vertical change of 1 unit for a horizontal change of \( -\frac{5}{6} \) units.

4. Refer to the line of Example 1. Start at the point \((1, 11)\).
   a. Going 1 unit to the right and 4 units up puts you at what point?
   b. Verify that your answer to part a lies on the line.

5. Refer to the line of Example 2. It appears that \((4, -1)\) lies on the line. Verify that this is true using the given equation for the line.
6. If two lines are parallel, what can be said about their slopes?

7. A line is parallel to \( y = \frac{1}{3}x - 2 \) and contains \( (1, 5) \). What is the slope of this line?

In 8 and 9, refer to triangles in this lesson.

8. Name the corresponding sides and angles that show \( \triangle ABC \) to be congruent to \( \triangle DEF \).

9. Name the corresponding sides and angles that show \( \triangle GHI \cong \triangle JKL \).

10. Given the equation \( 4y = -7x - 20 \).
   a. Rewrite the equation in slope-intercept form.
   b. Identify the slope and the \( y \)-intercept.
   c. Graph the equation.

11. Graph the line whose equation is \( y = 1 \).

12. The equation \( y = b \) represents a \underline{?} line with slope \underline{?}.

13. a. Draw the line with \( y \)-intercept \(-6\) and slope \( \frac{2}{3} \).
   b. Write the equation of this line in slope-intercept form.
   c. Use the equation to predict \( x \) when \( y \) is 3. Check to see if the point is on the line.

14. Graph the lines \( y = 2 \) and \( y = 2x \) on the same set of axes.

15. Consider the equation \( 5x + 2y = 24 \).
   a. Put the equation in slope-intercept form.
   b. Identify the slope and the \( y \)-intercept.
   c. Graph the line using the slope and intercept.

16. A line has no \( x \)-intercept and goes through the point \( (17, -68) \). Give an equation for the line.

17. a. Graph the line \( y = -3x + 1 \).
   b. Plot the point \((3, 2)\), and draw a line through it parallel to the line \( y = -3x + 1 \).

18. Write an equation for the line with \( y \)-intercept \(11 \) that is parallel to \( y = \frac{2}{3}x + 7 \).

In 19–21, tell whether the line has a positive or negative slope. (Lesson 3-1)

19. \( y \) \hspace{2cm} 20. \( y \) \hspace{2cm} 21. \( y \)
22. A tank has a slow leak. The water level starts at 100 inches and falls $\frac{1}{2}$ inch per day.
   a. What kind of situation is this: constant increase or decrease?
   b. Write an equation relating day $d$ and the water level $L$.
   c. After how many days will the tank be empty? (Lesson 3-1)

23. Find $Q_6$ if $Q_n = 4000(1.05)^n$. (Lesson 1-3)

24. Find $S_4$ if $S_1 = 2$ and $S_n = 3 \cdot S_{n-1}$. (Lesson 1-4)

25. Solve: $-3x - 5(x - 9) > -6x$. (Lesson 1-9)

26. a. How much alcohol is in a 9-oz solution of water and alcohol that is 20% alcohol?
   b. How much alcohol is in an x-oz solution of water and alcohol that is 20% alcohol?
   c. How much water is in an x-oz solution of water and alcohol that is 20% alcohol? (Previous course)

27. A function grapher can save time on this question. Consider the lines with equations $y = \frac{1}{4}x$ and $y = 4x$.
   a. Graph both lines on the same pair of coordinate axes.
   b. Find the slope of each line.
   c. Determine an equation for the bisector of the acute angles formed by these lines.
   d. Repeat parts a–c, using lines with equations $y = \frac{1}{3}x$ and $y = 3x$.
   e. Make a conjecture generalizing this problem and its results.
   f. Make and test a conjecture with an example where the lines have negative slopes.
Consider the following problem.

Milton has $C$ 20¢ stamps and $E$ 25¢ stamps. Find the total value of his stamps.

The rate-factor model of multiplication gives $C$ stamps at 20¢ per stamp, for a total cost of $20C$ cents. Likewise, $E$ stamps at 25¢ per stamp cost $25E$ cents. Then the total value of the stamps is

$$20C + 25E.$$

This expression is called a **linear combination** of $C$ and $E$. In a linear combination, all variables are to the first power and are not multiplied or divided by each other.

Linear combinations occur in a wide variety of real situations.

**Example 1**

In professional hockey a win is worth 2 points, a tie is worth 1 point, and a loss is worth 0 points. The Eagle hockey team has earned a total of 35 points.

a. Write an equation to express the relationship between the number of wins $W$, ties $T$, losses $L$, and the total points of the Eagle team.

b. If the team had 12 wins, how many ties did it have?
Solution

a. Each win is worth 2 points, so W wins are worth 2W points. A tie is worth 1 point, so T ties are worth 1T points. Because losses are worth 0 points, L losses add 0L to the total. This total is 35, so

\[ 2W + 1T + 0L = 35. \]

Simplify.

\[ 2W + T = 35 \]

b. Substituting 12 for W into the equation found in part a gives

\[ 2 \cdot 12 + T = 35. \]

Solve for T.

\[ T = 11 \]

So when the team had 12 wins, it also had 11 tie games.

Check The 12 wins are worth 24 points, and 11 ties are worth 11 points. This is 35 points altogether.

Example 2

A chemist mixes x ounces of a 20% alcohol solution with y ounces of a 30% alcohol solution. The final mixture contains 9 ounces of alcohol.

a. Write an equation relating x, y, and the total number of ounces of alcohol.

b. How many ounces of the 30% alcohol solution must be added to 2.7 ounces of the 20% alcohol solution to get 9 ounces of alcohol in the final mixture?

Solution

a. A 20% alcohol solution means that 20% of the x ounces are alcohol and 20% of x is 0.2x. Similarly, 30% of the y ounces are alcohol, which is 0.3y. The linear combination 0.2x + 0.3y gives the total number of ounces of alcohol. There are 9 ounces of alcohol, so an equation is 0.2x + 0.3y = 9.

b. Substitute 2.7 for x.

\[ 0.2(2.7) + 0.3y = 9 \]
\[ 0.54 + 0.3y = 9 \]
\[ 0.3y = 8.46 \]
\[ y = 28.2 \]

So the mixture contains 28.2 ounces of the 30% alcohol solution.

The equation 0.2x + 0.3y = 9 can be graphed. Solving for y shows that the graph is a line and puts the equation in slope-intercept form.

Multiply both sides by 10 to clear fractions. 2x + 3y = 90
Subtract 2x from both sides. 3y = -2x + 90
Divide both sides by 3. y = -\( \frac{2}{3} \)x + 30
Thus, the slope of the line is $\frac{-2}{3}$ and the y-intercept is 30. The graph is shown below.

The graph is continuous because the number of ounces of either solution may be any nonnegative real number. It is a segment because the amount of each solution cannot be negative.

Each point on the segment refers to a different mixture of the alcohol solutions. The point (2.7, 28.2) stands for 2.7 oz of 20% solution and 28.2 oz of 30% solution. The point (36, 6) means that 36 oz of the 20% solution could be mixed with 6 oz of the 30% solution to yield 9 oz of alcohol.

Any linear-combination situation in two variables is modeled by an equation whose graph is a line or a part of a line. This fact is the origin of the phrase "linear combination."

Questions

1. The expression $20C + 25E$ is called a __ of $C$ and $E$.

2. At a sale Greta Diehl bought $B$ blouses at $7 each, $S$ skirts at $14 each, and $H$ pairs of shoes at $19 each. Write a linear combination to find the amount spent at the sale.

In 3 and 4, refer to Example 1.

3. The team has $W$ wins, $T$ ties and $L$ losses. How many points were earned by the team?

4. With $T$ as the dependent variable, $(4, 27)$ is a solution to $2W + T = 35$. This solution means the team won __ games, tied __ games, and earned a total of __ points.

5. $40x + 8y$ is a __ of $x$ and $y$.

6. The graph of $Ax + By = C$ is a ___.

Covering the Reading

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7. Suppose that $S$ ounces of a solution that is 60% alcohol are combined with $N$ ounces of a 90% alcohol solution.
   a. How many ounces of alcohol are in the 60% solution?
   b. How many ounces of alcohol are in the 90% solution?
   c. How many total ounces of alcohol are in the combination?
   d. If Alice Seawell wants 18 ounces of alcohol in the final mixture, what equation relates $S$, $N$, and the 18 total ounces of alcohol?
   e. Solve the equation of part d for $N$. Graph the solutions to the equation, plotting $S$ on the horizontal axis.
   f. How many ounces of the 90% solution must be added to 9 ounces of the 60% solution to get 18 ounces of alcohol in the final mixture?

8. In a store, lettuce sells for 89¢ a head and tomatoes for 59¢ per pound.
   a. What will be the cost of 6 heads of lettuce and 8 pounds of tomatoes?
   b. What will be the cost of $H$ heads of lettuce and $P$ pounds of tomatoes?
   c. Write an equation indicating the amounts of lettuce $H$ and tomatoes $P$ you can buy for $5.00.

9. William Bates Green spent Saturday mowing lawns. He charged $5 for small lawns and $10 for large lawns and earned $70. Let $S$ be the number of small lawns and $L$ be the number of large lawns.
   a. What type of numbers make sense for $S$ and $L$ in this context?
   b. Write an equation relating $S$, $L$, and the amount of money earned.
   c. Graph the equation of part b.
   d. Give all possible pairs of numbers of large and small lawns William could have mowed.

10. The Ironman triathlon is a sporting event made up of a 2.4-mile swim, a 112-mile bicycle race, and a marathon run of 26.2 miles. If a competitor takes $S$ minutes per mile swimming, $B$ minutes per mile biking, and $R$ minutes per mile running, what will be the competitor's total time for the triathlon?
11. For the line graphed below, determine:
   a. its slope   b. its y-intercept   c. an equation (Lessons 3-1, 3-2)

![Graph of a line](image)

12. A line is parallel to $5y = 20 - 30x$. What is its slope? (Lesson 3-2)

In 13–18, find the distance between the given points. (Previous course)

13. (1, 3) and (6, 15)  
14. (4, 5) and (-4, 5)  
15. (2, 11) and (2, -7)  
16. (-3, -2) and (6, 0)  
17. $(a, b)$ and $(c, d)$  
18. $(k, 0)$ and $(0, k)$

In 19 and 20, graph on a coordinate plane. (Lesson 3-2)

19. $y = 3$  
20. $y = \frac{3}{4}x - 3$

21. The math department at a school has 100 reams of paper at the start of the school year (a ream of paper contains 500 sheets). Each school day the department uses about $\frac{2}{3}$ of a ream.
   a. Let $d$ be the number of school days from the start of the year and $R$ be the number of reams remaining. Write a formula for $R$ in terms of $d$.
   b. When the supply gets down to 10 reams, a new supply of paper needs to be ordered. After how many school days will paper need to be ordered? (Lesson 3-1)

22. 20% of 80 is what percent of 200? (Previous course)

23. In many schools, a student’s grade-point average is calculated using linear combinations. Some schools give 4 points for each A, 3 points for each B, 2 points for each C, and 1 point for each D. Suppose a student gets 7 As, 3 Bs, and 2 Cs.
   a. Calculate this student’s total number of points.
   b. Divide your answer in part a by the total number of classes (12) to get the grade point average.
   c. Calculate your own grade point average for last year using this scheme.
The set of all points with $x$-coordinate equal to 2 is a vertical line, as the graph below shows. This line can be described by the equation $x = 2$. Sometimes it is useful to think of the equivalent form

$$x + 0 \cdot y = 2$$

to stress that $y$ can take on any value.

What is the slope of the line $x = 2$? Calculating the slope by using the points $(2, 0)$ and $(2, 3)$ results in a denominator of 0.

$$m = \frac{3 - 0}{2 - 2} = \frac{3}{0}$$

You know that division by 0 is undefined, so the slope is said to be undefined. By the same argument, the slope of any vertical line $x = a$ is undefined.
Here is a summary of some important properties of lines with which you are familiar.

Vertical lines cannot have equations of the form \( y = mx + b \) because for a vertical line, the coefficient of \( y \) must be 0. But there is an equation form which includes all these instances. The form is \( Ax + By = C \), where \( A \) and \( B \) are not both zero. This is called the **standard form of a linear equation**. The following argument shows why the standard form of a linear equation describes all possible lines.

**When \( B \neq 0 \):** The equation \( Ax + By = C \) can be rewritten in slope-intercept form.

\[
\begin{align*}
\text{Add} \ -Ax. & \quad By = -Ax + C \\
\text{Multiply by } \frac{1}{B} (B \neq 0). & \quad y = \frac{A}{B}x + \frac{C}{B}
\end{align*}
\]

This is an equation of a line with slope \( \frac{A}{B} \) and \( y \)-intercept \( \frac{C}{B} \).

1. If \( A \neq 0 \), then the slope is not 0 and the line is **oblique**.
2. If \( A = 0 \), then \( y = \frac{C}{B} \).

This is an equation of a **horizontal** line with slope 0 and \( y \)-intercept \( \frac{C}{B} \).

**When \( B = 0 \):** The equation \( Ax + By = C \) can be written in the following form:

\[
\begin{align*}
Ax = C \\
x = \frac{C}{A}
\end{align*}
\]

This is an equation of a **vertical** line. A vertical line has no slope and has \( x \)-intercept \( \frac{C}{A} \).
The above argument proves that if an equation is of the form $Ax + By = C$ ($A$ and $B$ not both 0), then it represents a line. The converse of that statement is “If an equation represents a line, then it is of the form $Ax + By = C$ ($A$ and $B$ not both 0).” This converse can be proved by reversing the steps in the argument. Both statements together can be expressed as a theorem.

**Theorem:**

The graph of the equation $Ax + By = C$ ($A$ and $B$ not both 0) is a line.

The standard form of a linear equation arises in linear-combination situations. There is a shortcut in graphing an equation in this form by using its intercepts. The next example shows how to use the shortcut.

**Example**  
Graph the equation $6x - 3y = 12$ by using its intercepts.

**Solution**  
The $x$-intercept is the value of $x$ at the point where the line crosses the $x$-axis. This point has second coordinate 0. So substitute 0 for $y$ and solve for $x$.

$$6x - 3(0) = 12$$
$$x = 2$$

Thus the $x$-intercept is 2.

To find the $y$-intercept, substitute 0 for $x$ and solve for $y$.

$$6(0) - 3y = 12$$
$$y = -4$$

Thus the $y$-intercept is -4.

Plot (2, 0) and (0, -4). Draw the line between them, as shown below.

![Graph of 6x-3y=12](image)

**Check**  
The point (1, -2) appears to be on the graph. Substitute to see if the coordinates satisfy the equation.

Does $6(1) - 3(-2) = 12$?
Does $6 + 6 = 12$? Yes
Questions

Covering the Reading

In 1–3, match the line with a description of its slope.
1. horizontal line               (a) 0 slope
2. vertical line                (b) non-zero slope
3. oblique line                 (c) slope undefined

4. The line whose equation is of the form \( y = mx + b \) is oblique when \( m \neq 0 \).

5. How many intercepts does an oblique line have?

6. The graph of \( y = 7 \) is a \( \square \) line with \( \square \)-intercept \( \square \).

7. a. Graph the line with equation \( x = -6 \).
    b. What can be said about the slope of this line?

In 8–10, refer to the general discussion of the graph of \( Ax + By = C \).
8. a. If neither \( A \) nor \( B \) is 0, the equation can be written in slope-intercept form as \( \square \).
    b. The slope of the line is \( \square \).
    c. The \( y \)-intercept of the line is \( \square \).

9. If \( A = 0 \) and \( B \neq 0 \), then the line is \( \square \) and has \( \square \) slope.

10. If \( A \neq 0 \) and \( B = 0 \), then the line is \( \square \) and has \( \square \) slope.

11. To find the \( x \)-intercept of a line, for which variable should you substitute 0?

12. Consider the graph of the equation \( 4x - 3y = 24 \).
    a. Find its \( x \)-intercept.
    b. Find its \( y \)-intercept.
    c. Graph the line using the points from parts a and b.

In 13–15, (a) tell whether each line is vertical, horizontal, or oblique; (b) give all intercepts for each line; (c) graph each equation.
13. \( y = 4 \)                     14. \( 2x - 3y = 18 \)          15. \( 2x = 16 \)

16. Meg combines \( N \) oz of a solution that is 10% alcohol with \( Y \) oz of a solution that is 20% alcohol. She ends up with a mixture that contains 1.2 oz of alcohol.
    a. Write an equation relating \( N \), \( Y \), and the amount of alcohol in the mixture.
    b. Graph the equation you obtained in part (a) by finding the \( N \)- and \( Y \)-intercepts. Consider \( N \) the independent variable.
    c. Use your graph to find out how many ounces of the 20% solution must be added to 8 oz of the 10% solution to get the final mixture.

17. Give an equation in standard form for the line with \( y \)-intercept \(-\frac{1}{2}\) and slope 2.
18. A 3-line classified advertisement in a local paper costs $10.20 for five weekdays and $12.35 for a weekend edition. What is the cost of \( x \) ads during the week and \( y \) ads on the weekend? (Lesson 3-3)

19. You have some money saved from your job. You invest \( S \) of it in a savings account that pays 8% interest and the rest \( R \) in a checking account that pays 6%. You earn $84 interest in one year. (Lesson 3-3)
   a. Write an equation relating \( S \), \( R \), and the total amount of interest.
   b. Give three possible pairs of values for \( R \) and \( S \).

20. The city police department pays police officers $1500 per month and pays their supervisors $2400 per month. The total payroll for the month is $60,000. (Lesson 3-3)
   a. Write an equation relating the number of police officers \( P \), the number of supervisors \( S \), and the total monthly payroll.
   b. If there are 10 supervisors, how many police officers are there?

21. A slope of \(-\frac{4}{3}\) means which of the following? (Lessons 2-4, 3-2)
   (a) A vertical change of -3 units for a horizontal change of 4 units
   (b) A vertical change of -4 units for a horizontal change of 3 units
   (c) A vertical change of -\frac{4}{3} units for a horizontal change of 1 unit
   (d) A vertical change of 1 unit for a horizontal change of -\frac{4}{3} units

22. Solve for \( x \). \( x - 11 = \frac{7}{5}(x + 3) \) (Lesson 1-7)

23. Suppose you can get as many 22¢ stamps and 3¢ stamps as you want. You could make 24¢ using eight 3¢ stamps, but you cannot make 23¢ postage exactly. What is the largest value that cannot be made with these stamps?

24. a. Find the \( x \)- and \( y \)-intercepts of \( \frac{x}{2} + \frac{y}{7} = 1 \).
   b. Find the \( x \)- and \( y \)-intercepts of \( \frac{x}{5} + \frac{y}{6} = 1 \).
   c. Based on parts a and b above, make a conjecture about the \( x \)- and \( y \)-intercepts of \( \frac{x}{a} + \frac{y}{b} = 1 \). Either prove your conjecture or give a counterexample to it.
Finding an Equation of a Line

Two points determine a line. You use this idea every time you draw the line through two points with a ruler. It is a postulate from geometry. In algebra, this idea raises the question: What is an equation of the line through two given points? The next two examples show one way to get such an equation.

Example 1  Find an equation of the line $L$ through $(3, 5)$ and $(6, -1)$.

Solution  Line $L$ is oblique so it has an equation in slope-intercept form $y = mx + b$. First calculate the slope $m$.

$$m = \frac{-1 - 5}{6 - 3} = \frac{-6}{3} = -2$$

Substitute -2 for $m$:

$$y = -2x + b$$

Only the value of $b$ is left to find. We can use either of the points to get $b$. Choose $(3, 5)$ and substitute for $x$ and $y$.

$$5 = -2 \cdot 3 + b$$
Solve for \( b \):

\[ b = 11 \]

The work is done. Since \( m = -2 \) and \( b = 11 \), an equation for \( L \) is \( y = -2x + 11 \).

**Check** From the graph above you can see that the \( y \)-intercept must be greater than 8 and that the slope is indeed negative. This is a quick check.

For an exact check, substitute the other given point \((6, -1)\) in the equation to test if it works. Does \(-1 = -2 \cdot 6 + 11\)? Yes, so the equation is correct.

If one of the given points is on the \( y \)-axis, then the \( y \)-intercept \( b \) is already given, and all that is needed is \( m \). This situation is illustrated in Example 2.

**Example 2** Suppose you remember that \( 0^\circ C = 32^\circ F \) and \( 100^\circ C = 212^\circ F \), but you have forgotten the conversion formula. You know that the formula is linear. Reconstruct the formula with \( C \) as the first coordinate and \( F \) as the second.

\[ \begin{align*}
\text{Solution} & \quad \text{With} \ C \text{ as the independent variable, the formula will be of the form } F = mC + b. \text{ Values of } m \text{ and } b \text{ are needed. First find the slope, using the given points } (0, 32) \text{ and } (100, 212). \\
& \quad m = \frac{212 - 32}{100 - 0} = \frac{180}{100} = 1.8
\end{align*} \]

Because the line crosses the \( F \)-axis at \((0, 32)\), \( b = 32 \). Substitute these values into \( F = mC + b \). The desired formula is \( F = 1.8C + 32 \).

**Check** Substitute \((100, 212)\) in the equation. Does \(212 = 1.8(100) + 32\)? Yes.
Recall Playfair’s Parallel Postulate from geometry: Through a point in a plane, there is exactly one line parallel to a given line. This line can be found algebraically if you know the slope of the given line.

**Example 3**

A line \( l \) passes through the point \((12, -5)\) and is parallel to the line \( y = 4x \).

a. Graph \( l \).

b. Find an equation for \( l \) in slope-intercept form.

**Solution**

a. Recall that parallel lines have the same slope. Because the line \( y = 4x \) has slope 4, the line \( l \) through \((12, -5)\) must also have slope 4. At the right, this slope is used to graph \( l \). Notice that its intercepts are not easily read.

b. The line \( l \) contains \((12, -5)\) and has slope 4. Let \((x, y)\) be any other point on \( l \). Substitute into the slope formula.

\[
\frac{y - (-5)}{x - 12} = 4
\]

To put the equation into slope-intercept form, multiply both sides by \((x - 12)\).

\[
y - (-5) = 4(x - 12)
y + 5 = 4x - 48
y = 4x - 53
\]

**Check** Is the point \((13, -1)\) on the line? Does \(-1 = 4(13) - 53\)? Yes.

Each of the equations \( y - (-5) = 4(x - 12) \) and \( y = 4x - 53 \) represents the line through \((12, -5)\) with slope 4. The slope-intercept form is handy because it is a formula that you can use to calculate values of \( y \) quickly if you know \( x \)-values. However, the form \( y - (-5) = 4(x - 12) \) is nice because the three constants in it are the given information: the coordinates of a point and the slope.
The method of Example 3 can be generalized.

**Point-Slope Theorem:**
If a line contains \((x_1, y_1)\) and has slope \(m\), then it has equation
\[
y - y_1 = m(x - x_1).
\]

**Proof:**
Let \(L\) be the line with slope \(m\) containing \((x_1, y_1)\). If \((x, y)\) is any other point on \(L\), then by the definition of slope,
\[
m = \frac{y - y_1}{x - x_1}.
\]
Multiplying both sides by \(x - x_1\) gives
\[
m(x - x_1) = y - y_1.
\]
This is the desired equation of the theorem.

The equation \(y - y_1 = m(x - x_1)\) is called a **point-slope equation** for a line. The most convenient form to use for writing an equation depends on the information given. If you know the slope and \(y\)-intercept, use \(y = mx + b\). If you know the slope and some other point, use \(y - y_1 = m(x - x_1)\). If you know two points, find the slope and then use either the point-slope form or the slope-intercept form.

**Questions**

1. __ points determine a line.
2. A line contains \((6, 4)\) and \((2, 8)\).
   a. Find its slope.
   b. Use the point \((6, 4)\) together with the slope to find the \(y\)-intercept.
   c. Write an equation for the line and check.
   d. Does the point \((0.8, 9.8)\) satisfy the equation of part c?
3. Why is the slope-intercept form the most convenient one to use in Example 2?
4. State Playfair’s Parallel Postulate.
5. *True or false* A line is determined by its slope and any point on it.
6. The point-slope form of the equation for a line with slope \(m\) and passing through point \((x_1, y_1)\) is ___.

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7. Which is easier to use, the point-slope form or the slope-intercept form for the given information?
   a. given the slope and a point other than the y-intercept
   b. given the y-intercept and the slope
   c. given two points

In 8 and 9, use the most convenient form of a linear equation to write an equation for the line with the given information.
8. slope 6 and y-intercept -1
9. slope $\frac{2}{3}$ and passing through (7, 1)

In 10 and 11, a line passes through the points (7, 12) and (5, 16).
10. a. Find the slope of the line.
    b. Use the slope and the point (7, 12) in the point-slope form to find an equation for the line.
    c. Put your solution to part b in standard form.

11. Repeat Questions 10b and 10c with the point (5, 16) and the slope from 10a. Do you get an equivalent equation? If not, why not?

12. Find an equation for the line through (-4, 5) parallel to $y = 6x + 10$.

13. Scientists often use kelvins to measure temperature. On this scale, $32^\circ F \approx 273.15$ kelvins and $212^\circ F \approx 373.15$ kelvins. Let $F$ represent Fahrenheit temperature and $K$ represent the number of kelvins. The relationship is linear. Find an equation relating temperature in kelvins (dependent variable) and the Fahrenheit temperature.

14. A printer finds that it costs $1290 to print 30 books and $1335 to print 45 books. Let $c$ be the cost of printing $b$ books. Assume $c$ is linearly related to $b$.
   a. Find an equation relating cost to the number of books printed.
   b. How much will it cost to print 100 books?
   c. How much will it cost to print 0 books? (This is the set-up cost.)

15. Let $P = (3, 4)$, $Q = (3, -5)$, $R = (-2, -5)$, and $S = (-2, 4)$.
   a. Graph rectangle $PQRS$.
   b. Give equations for the four sides.
   c. Find the area of $PQRS$. (Lesson 3-4, previous course)

16. If $3x + 8 = 40$, find the value of $6x + 16$. (Previous course)

In 17 and 18, refer to the following situation. Jamie is driving along a deserted country road. Her car uses one gallon of gas for every 27 miles she travels. Her gas tank holds 18 gallons. Let $m$ be the number of miles she travels. Let $g$ be the number of gallons of gas used. Let $L$ be the amount left in the tank.
17. Find an equation relating $g$ and $L$.

18. Find an equation relating $g$ and $m$. (Lessons 3-1, 2-1, 1-1)
19. As of November 1988, the greatest combined number of points ever scored in a professional basketball game was 370 by Denver and Dallas in 1983. A free throw is worth 1 point, a field goal 2 points, and there are 3-point shots. Suppose two opponents break this record with A free throws, B field goals and C 3-point shots. Write an equation or inequality that expresses this idea. (Lesson 3-3)

20. Triangle $MOP$ is shown below. Graph the reflection image over the $x$-axis of $\triangle MOP$. Label the points $M'$, $O'$, and $P'$ and state their coordinates. (Previous course)

21. In 1957, the world record in 800-meter freestyle swimming was about 10 minutes 30 seconds for women and 9 minutes 15 seconds for men. Between 1957 and 1980, these records had been decreasing at a rate of about 4 sec/yr for women and 3 sec/yr for men.
   a. According to this information, what should the records have been in 1988?
   b. According to this information, what will the records be in the year 2000?
   c. Check a book of records and see if the predictions for 1988 were true.
Some sequences are described by linear equations. Recall that the domain for a sequence is the set of natural numbers. If you substitute the natural numbers 1, 2, 3, 4, 5, ... for $x$ in the linear equation

$$y = 4x - 5$$

you will generate the following sequence of values for $y$:

$$-1, 3, 7, 11, 15, ...$$

By substituting $n$ for $x$ and $a_n$ for $y$ you can generate a formula for $a_n$, the $n$th term of this sequence.

$$a_n = 4n - 5$$

Remember that $n$ is now restricted to the set of natural numbers, \{1, 2, 3, ...\}.

Both the linear equation and the sequence equation are graphed below. On the graph of the linear equation, the ordered pairs are of the form $(x, y)$, while on the graph of the sequence equation, the ordered pairs are of the form $(n, a_n)$. The graph of $a_n = 4n - 5$ is part of the graph of $y = 4x - 5$. The sequence generates a set of discrete points because its domain is restricted to the natural numbers. Just as the line continues forever, so the graph of the sequence equation continues for all natural-number values of $n$. 

\[\text{linear equation} \quad y = 4x - 5\]

\[\text{sequence formula} \quad a_n = 4n - 5\]
In the sequence \( a_n = 4n - 5 \) there is a constant difference of 4 between successive terms. This difference is the slope of the graph. As a check, the ordered pairs (2, 3) and (3, 7) from the sequence graph give the slope:

\[
m = \frac{7 - 3}{3 - 2} = \frac{4}{1} = 4
\]

A sequence with a constant difference is called a linear sequence or arithmetic sequence. (Here the word arithmetic is used as an adjective; it is pronounced arithmetic.)

If you are given the first term and the constant difference for an arithmetic sequence, you can find an explicit formula for the \( n \)th term of that sequence. Finding this formula is just like finding an equation of a line using the point-slope form. Example 1 shows how this can be done.

**Example 1** Find a formula for the \( n \)th term of the arithmetic sequence 4, 11, 18, 25, ....

**Solution** The constant difference between terms is 11 - 4, or 18 - 11, or 25 - 18; it is the slope 7. The first term \( a_1 \) is 4, which gives the point \((1, 4)\).

The point-slope form is

\[ y - y_1 = m(x - x_1). \]

Substitute \( a_n \) for \( y \), 4 for \( y_1 \), 7 for \( m \), \( n \) for \( x \), and 1 for \( x_1 \).

\[ a_n - 4 = 7(n - 1) \]

Solve for \( a_n \).

\[ a_n - 4 = 7n - 7 \]

\[ a_n = 7n - 3 \]

This is a formula for the \( n \)th term of the sequence.

**Check** Substitute values for \( n \) into the formula for \( a_n \). Do you get the correct term? Try \( n = 4 \). Do you get \( a_4 = 25? \) \( a_4 = 7 \cdot 4 - 3 = 25 \). Yes.

The solution of Example 1 can be generalized to find an explicit formula relating the \( n \)th term, \( a_n \), and the first term, \( a_1 \), of an arithmetic sequence. Each term is of the form \((n, a_n)\) and the first term is the ordered pair \((1, a_1)\). The slope is just the constant difference, which we call \( d \). Use these values in the point-slope form of the linear equation:

\[ a_n - a_1 = d(n - 1) \]

Solve for \( a_n \):

\[ a_n = a_1 + (n - 1)d \]
This short argument proves the following theorem.

**Theorem:**

The $n$th term $a_n$ of an arithmetic sequence with first term $a_1$ and constant difference $d$ is given by the explicit formula

$$a_n = a_1 + (n - 1)d.$$

**Example 2** Find the 40th term of the arithmetic sequence 100, 97, 94, 91, ..., ...

**Solution** The first term $a_1 = 100$, and the constant difference is -3. Since the 40th term is to be found, substitute $n = 40$ into the formula of the theorem:

$$a_{40} = 100 + (40 - 1) \cdot -3 = 100 + 39 \cdot -3 = -17$$

The 40th term is -17.

**Example 3** In a concert hall the first row has 10 seats in it, and each subsequent row has two more seats than the row in front of it. If the last row has 64 seats, how many rows are in the concert hall?
Solution Because each succeeding row has two additional seats, the number of seats in each row generates the sequence

10, 12, 14, 16, ..., 64.

Thus, you know that \( a_1 = 10, \) \( d = 2, \) and \( a_n = 64, \) where \( n \) is the number of rows. To find \( n, \) substitute the known values into

\[
a_n = a_1 + (n - 1)d
\]

and solve.

\[
64 = 10 + (n - 1)2 \\
54 = 2(n - 1) \\
27 = n - 1 \\
28 = n
\]

There are 28 rows of seats in the concert hall.

Check 1 Substitute \( a_1 = 10, \) \( d = 2, \) and \( n = 28 \) into

\[
a_n = a_1 + (n - 1)d.
\]

\[
a_{28} = 10 + (28 - 1)2 = 10 + 54 = 64
\]

The last row has the correct number of seats.

Check 2 You could also check your answer by writing the first 28 terms of the sequence to verify that \( a_{28} = 64. \)

Questions

1. Suppose \( a_n = 5n + 2. \)
   a. The domain for \( n \) is _?_.
   b. Graph the sequence.

In 2 and 3, (a) write the first three terms of the sequence; (b) find the constant difference between the terms; (c) graph.

2. \( a_n = 3n \)

3. \( a_n = \frac{1}{2}n - 7 \)

4. How is the constant difference of a linear sequence related to the slope of its graph?

5. The \( n \)th term of an arithmetic sequence with first term \( a_1 \) and constant difference \( d \) is _?_.

6. a. Find a formula for the \( n \)th term of the arithmetic sequence
   \( 13, \ 15, \ 17, \ 19, \ 21, \ldots. \)
   b. Calculate the 51st term of this sequence.

In 7 and 8, (a) find a formula for the \( n \)th term of the arithmetic sequence; (b) find the 100th term.

7. \( 6, \ 15, \ 24, \ 33, \ldots \)

8. \( 16, \ 14.5, \ 13, \ 11.5, \ldots \)
9. Refer to Example 3. Suppose that in some other concert hall the first row has 40 seats, each subsequent row has two more seats than the row in front of it, and the last row has 70 seats. How many rows of seats are there?

10. *Multiple choice* Which sequence is *not* an arithmetic sequence?
(a) 5, 9, 13, 17, …
(b) 3, 6, 12, 24, …
(c) $\frac{3}{2}, 1, \frac{3}{2}, 2, \frac{3}{2}, …$
(d) 0, -1, -2, -3, -4, …

11. a. Does the graph of an arithmetic sequence have an intercept?
b. If yes, how can you find it from a formula for the $n$th term? If not, why not?

12. a. What numbers will the following BASIC program print when run, if you input 15 for $A$ and 3 for $D$ in line 10?

```
10 INPUT A, D
20 FOR N = 1 TO 6
30  AN = A + D * (N - 1)
40  PRINT AN
50 NEXT N
60 END
```
b. If line 20 is changed to *FOR N = 1 TO 100*, and the user inputs 15 for $A$ and 3 for $D$, what will be the last number printed?

13. Stu and Perri start biking each week for training. They start by biking 14 miles the first week. By the twenty-fifth week they want to bike 74 miles a week. If the number of miles biked each week is to form an arithmetic sequence, what should be their weekly increase?

In 14 and 15, a local radio station is holding a contest to give away cash. The announcer calls a number and if the person who answers guesses the correct amount of money in the pot, he or she wins the money. If the resident misses, $20 is added to the money pot.

14. On the 12th call, a contestant won $675. How much was in the pot at the beginning?

15. Suppose the pot starts with $150. On what call would the winner receive $1110?
16. A business finds that it costs $950 to make 300 pillows and $1475 to make 650 pillows. Assuming a linear relationship between cost and number of pillows, find the cost of making 5000 pillows. (Lesson 3-5)

17. Find an equation for the line that goes through (-6, -8) and (9, 2). (Lesson 3-5)

18. Find an equation for the line graphed at the right. (Lesson 3-5)

19. Find an equation of the line passing through (-7, 8) and parallel to \( y = \frac{3}{5}x - 2 \). (Lesson 3-5)

20. Give the distance between (0, 0) and (-3, -4). (Previous course)

Multiple choice In 21 and 22, refer to the graphs below. (Lessons 2-4, 2-5, 2-7)

21. Which graph does not represent a direct variation?

22. Which graph represents a situation in which the constant of variation is positive?

23. Here is a sequence that is not a linear sequence.

\[ 3, 5, 9, 17, 33, 65, 129, 257, 513, \ldots \]

a. What is the next term?
b. Find a recursive formula for the \( n \)th term.
c. Find an explicit formula for the \( n \)th term.
Recall from Lesson 1-4 that a recursive formula gives a rule for finding the \(n\)th term of a sequence from one or more of the previous terms. Consider this recursive formula:

\[
\begin{align*}
a_1 &= -5 \\
a_n &= a_{n-1} + 3, \text{ for } n > 1
\end{align*}
\]

The first few terms of this sequence are 
-5, -2, 1, 4, 7, 10, ... .

This is an arithmetic sequence because there is a constant difference of 3 between successive terms.

The key that this is an arithmetic sequence is in the second line. Suppose 3 is replaced by \(d\). Then

\[a_n = a_{n-1} + d, \text{ for } n > 1.\]

Subtracting \(a_{n-1}\) from each side gives \(a_n - a_{n-1} = d\). That is, \(d\) is the difference between the \(n\)th term and the \((n - 1)\)st term. If \(d\) is constant, then the sequence has a constant difference and is arithmetic. This proves the following theorem.

**Theorem:**

If \(d\) is constant, the recursive formula

\[
\begin{align*}
a_1 \\
a_n &= a_{n-1} + d, \text{ for } n > 1
\end{align*}
\]

generates the arithmetic sequence with first term \(a_1\) and constant difference \(d\).

**Example 1**  
What sequence is generated by \(\begin{align*}a_1 &= 1000 \\
a_n &= a_{n-1} - 40, \text{ for } n > 1\end{align*}\)?

**Solution**  
The first term is 1000. The difference is -40. So the sequence is  
1000, 960, 920, 880, ... .

It is useful to be able to find both explicit and recursive formulas for arithmetic sequences.
Example 2  Steve borrowed $370 from his parents for airfare to visit his sister in college. He will pay them back at the rate of $30 each month. Let \( a_n \) be the amount he still owes after \( n \) months. Find (a) a recursive formula and (b) an explicit formula for \( a_n \).

Solution

a. After 1 month he owes $340, so \( a_1 = 340 \). Each month he pays back $30, so \( d = -30 \). The recursive formula is then

\[
\begin{align*}
  a_1 &= 340 \\
  a_n &= a_{n-1} - 30, \quad n > 1.
\end{align*}
\]

b. For an explicit formula, \( a_n = a_1 + (n - 1)d \)

\[
\begin{align*}
  a_n &= 340 + (n - 1) \cdot -30 \\
  a_n &= 340 - 30n + 30 \\
  a_n &= 370 - 30n
\end{align*}
\]

Check  Each formula generates the sequence 340, 310, 280, 250, ... .

Computers can generate sequences using either explicit or recursive formulas. The BASIC programs below each generate the first 10 terms of the sequence of Example 2.

**Recursive**

10 LET A = 340
20 FOR N = 1 TO 10
30 PRINT A
40 A = A - 30
50 NEXT N
60 END

**Explicit**

11 FOR N = 1 TO 10
21 LET AN = 340 - 30 * (N - 1)
31 PRINT AN
41 NEXT N
51 END
Questions

Covering the Reading

1. A recursive formula gives you a rule for finding the \(n\)th term if you already know \(a_1\).

2. A recursive formula for an arithmetic sequence with first term \(a_1\) and constant difference \(d\) is \(a_n = a_{n-1} + d\).

3. a. What are the first four terms of the sequence generated by this formula?
\[
\begin{align*}
a_1 & = 1 \\
a_n & = a_{n-1} + 6, \text{ for } n > 1
\end{align*}
\]
b. Write a computer program to generate the first 25 terms of this sequence.

4. Refer to Example 2. Suppose Steve borrowed $325 and repaid $25 per month. Find (a) a recursive formula and (b) an explicit formula for the amount \(a_n\) owed \(n\) months after payment begins.

5. Write (a) a recursive formula and (b) an explicit formula for the arithmetic sequence 13, 19, 25, 31, \ldots.

6. The BASIC program below generates several terms of a sequence using a recursive formula.

```
10 LET A = 15
20 FOR N = 1 TO 7
30 PRINT A
40 A = A + 3.5
50 NEXT N
60 END
```

a. What sequence is printed when the program is run?
b. Change this program so that the sequence is defined explicitly.

Applying the Mathematics

7. Write a recursive formula for the arithmetic sequence with explicit formula \(a_n = 10.8 + 2.4n\).

In 8 and 9, rewrite each recursive formula in explicit form.

8. \[
\begin{align*}
a_1 & = 8.1 \\
a_n & = a_{n-1} + 1.7, \text{ for } n > 1
\end{align*}
\]

9. \[
\begin{align*}
a_1 & = x \\
a_n & = a_{n-1} + 3x, \text{ for } n > 1
\end{align*}
\]

10. Jennifer’s grandparents opened a savings account for their granddaughter. They started the account with $500 on her first birthday, and each subsequent year on her birthday they deposited $150.

a. Write an explicit formula for the amount \(a_n\) in the account after \(n\) birthdays.
b. Write a recursive formula for \(a_n\).
In 11–13, write a computer program to generate:
11. the first 1000 odd numbers.
12. the first 500 multiples of 11.
13. the sequence of Question 4.

14. Find the 400th positive integer in the arithmetic sequence 9, 19, 29, 39, \ldots . \ (Lesson 3-6)

15. If $t_n = 10 + 7(n - 1)$ find $t_{89}. \ (Lesson 3-6)$

16. In a contest the first-place winner gets $100,000$ and the tenth-place winner gets $23,500$. If the winning amounts form an arithmetic sequence, find the cash difference between prizes. \ (Lesson 3-6)

17. Find an equation of the line containing (-2, 4) parallel to $2x + y = 12. \ (Lesson 3-5)$

18. Graph $6(-3x + 50y) = 2400$. Be careful! You will need to choose your scales carefully in order to show the intercept and slope clearly. \ (Lesson 3-4)

19. Use $y = 3 - x \ (Lessons 3-2, 1-1$: Previous course$)$
   a. Graph only the part of the graph between the x- and y-intercepts.
   b. What is the domain?
   c. Find the midpoint of the segment you graphed in part a.

In 20 and 21, let $A = (\frac{1}{2}, \frac{2}{3})$ and $B = (-\frac{5}{2}, -\frac{11}{3}). \ (Lessons 1-4, 1-5, 3-1)$

20. Find the length of $AB$. \ 21. Find the slope of $AB$.

22. Find all the positive integers that meet all three of these conditions at the same time: (a) They leave a remainder of 1 when divided by 2. (b) They leave a remainder of 3 when divided by 4. (c) They leave a remainder of 5 when divided by 6. (Hint: 35 is one such number.)
As you know, graphs can convey a great deal of information in a small amount of space. Horace made the graph below to describe his bicycle trip, which included two stops. It gives his distance $D$, in miles from home, $t$ hours after leaving home.

This graph is made up of five segments. Because each of these five segments is a piece of a line, the graph is called piecewise linear. The graph shows that Horace started from home and bicycled at a constant rate for 2 hours, winding up 28 miles from home. After a half-hour stop, he traveled for another hour and a half, until he was 48 miles from home. Then he stopped again, this time for two hours. Finally, he returned home traveling at a constant rate and reached home 9 hours after starting out.

Sometimes it is efficient to have equations for each part of a piecewise linear graph. Such situations can occur when large numbers of items are bought or used. If the cost or rate changes, the graph may be piecewise linear. This can happen with food or utility bills.
Example 1  An electric company calculates bills for its residential customers based on these charges:

$10.00 monthly service fee;
$.07 per kwh energy charge for the first 400 kwh;
$.04 per kwh energy charge for each kwh over 400.

a. Calculate the monthly electric bill for a family that uses 300 kwh.
b. Calculate the monthly electric bill for a family that uses 600 kwh.
c. Draw a graph showing how cost $C$ (in dollars) is related to usage $k$ (in kwh).

Solution
a. Add the service fee to the charge for the electricity used.

\[ C = 10 + .07(300) \]
\[ C = 10 + 21 \]
\[ C = 31 \]

The family must pay $31 for 300 kwh.

b. The cost changes when more than 400 kwh are used. There are three things to be considered.

\[ C = 10 + .07(400) + .04(200) \]
\[ C = 10 + 28 + 8 = 46 \]

The family must pay $46 for 600 kwh.

c. For the first 400 kwh the increase is constant. This piece of the graph is a segment with one endpoint (0, 10) and a slope of $.07 per kwh. The other endpoint is the point at $k = 400$. When $k = 400$, \[ C = 10 + .07(400) = 38 \].

At (400, 38) a new constant rate of increase begins. Because the slope of $.04 per kwh is less for this piece of the graph, it increases more slowly. This part of the graph is a ray, since the rate of $.04 per kwh applies to all values of $k$ greater than 400. The graph is shown below.

\[ \text{Monthly Electric Costs} \]

\[ \text{Cost (dollars)} \]
\[ \text{(300, 31)} \]
\[ \text{(400, 38)} \]
\[ \text{(600, 46)} \]

\[ \text{Energy Use (kwh)} \]

\[ \text{LESSON 3-8 Piecewise Linear Graphs} \]

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Piecewise linear graphs can be described algebraically by using the following procedure. First, write an equation for each segment or ray that is part of the graph. Then list the equation for each piece of the graph together with the domain over which that equation is defined.

Example 2  In the situation of Example 1, describe algebraically how the cost $C$ depends on $k$.

Solution  The slope of the segment from the point $(0, 10)$ to the point $(400, 38)$ is .07.
Thus,

$$C = .07k + 10 \text{ for } 0 \leq k \leq 400.$$  

An equation for the ray through $(400, 38)$ with slope .04 is given in point-slope form:

$$C - 38 = .04(k - 400)$$
$$C = .04(k - 400) + 38$$
$$C = .04k + 22$$

In summary, this situation is described algebraically as follows:

$$\begin{align*}
C &= .07k + 10, \text{ for } 0 \leq k \leq 400 \\
C &= .04k + 22, \text{ for } k > 400.
\end{align*}$$

Notice that the final description contains an equation for each segment or ray together with the domain over which that equation is defined.

Questions

1. A graph that is composed of pieces of lines is called ____.

In 2–6, refer to the graph of Horace’s Bicycle Trip.

2. How many line segments are drawn in this graph?

3. What is the total amount of time that Horace stopped during his trip?

4. What was the farthest Horace was away from home on this trip?

5. How fast was Horace going during the first two hours of his trip?

6. How fast was Horace going during the last three hours of his trip?
Applying the Mathematics

7. Find the monthly electric bill for a family that uses 500 kwh.

8. **Multiple choice** The formula \( C = 0.07k + 10 \) gives the cost of \( k \) kwh of electricity
   (a) for \( k > 0 \). (b) for \( 0 \leq k \leq 400 \). (c) for \( k \geq 400 \).

In 9–12, refer to the following graph. Carmen walks to school, to her job after school, and then home. Let \( M \) be the number of miles she is from home at time \( T \).

9. How far is school from home?

10. During what time period is Carmen at work?

11. How long does it take Carmen to walk to school?

12. **a.** Find the slope of the segment from \((8:00, 0)\) to \((8:30, 1.2)\) using hours as the unit for time.
    **b.** What does this slope represent?

13. Draw a graph of the following situation:
    \[
    \begin{align*}
    y &= x, \text{ for } x \leq 0 \\
    y &= -x, \text{ for } x > 0
    \end{align*}
    \]

In 14 and 15, use the following phone rates: A customer is allowed to use a maximum of 120 message units per month for a $10.15 fee. Each local call after the 120th is billed at a rate of $0.035 per message unit.

14. Find the monthly phone bill for a family using
    **a.** 100 message units
    **b.** 120 message units
    **c.** 160 message units.

15. **a.** Draw a graph of the relation between the cost \( C \) and the number of message units \( m \).
    **b.** Describe this situation algebraically by finding an equation for each segment or ray.

16. Draw a graph of the following situation. When \( x < -2 \), \( y = 4 \). When \(-2 \leq x \leq 2 \), \( y = x^2 \). When \( x > 2 \), \( y = 4 \).
17. For the arithmetic sequence 2, -3, -8, -13, ... .
   a. Write an explicit formula. (Lesson 3-6)
   b. Write a recursive formula. (Lesson 3-7)

18. Find a recursive formula for the sequence described by \( a_n = -1.5 + 2.5n \). (Lesson 3-7)

19. A store displays cans stacked in rows. The top row has 1 can, each subsequent row has 3 more cans than the row above it, and the bottom row has 37 cans. How many rows of cans are there? (Lesson 3-6)

20. Find an equation for the line through \( (\frac{1}{2}, \frac{3}{4}) \) parallel to the line through \( (5, 7) \) and \( (6, -8) \). (Lesson 3-5)

21. Solve for \( y \): \( 5x - 8y < -18 \). (Lesson 1-9)

In 22–27, use the figures below. The triangles are congruent and segments that look parallel are. Fill in the blank with one of the words translation, rotation, or reflection. (Previous course)

22. B is a __ image of A.

23. C is a __ image of B.

24. A is a __ image of B.

25. D is a __ image of B.

26. B is a __ image of C.

27. D is a __ image of C.

28. Find the gas or electric rates for a home or apartment in your local area. Make a graph like that in Example 1 for the rates you find.

29. Most function graphers allow more than one graph to be shown at a time. Graph the monthly electric costs from Example 1, using a function grapher, by splitting the graph into its two linear parts.
Take a linear equation and replace the $=$ sign with $<$, $\le$, $>$, or $\ge$. The result is a linear inequality. Here are some examples.

$$x \ge 5 \quad y > 2x + 7 \quad x + 3y \le 12$$

There are usually more solutions to these sentences than are practical to list. However, the solutions can readily be seen when they are graphed.

In any graph of a line in a plane, the line separates the plane into two distinct regions. They are called half-planes. The line itself is the boundary of the two regions. In a linear inequality, the solutions are located in one of the half planes and sometimes include points on the boundary.

Inequalities with horizontal or vertical boundaries can be quickly graphed.

**Example 1** Graph the linear inequality $x \ge 5$ in the coordinate plane.

**Solution** This half-plane is the collection of all points where the $x$-coordinate is equal to or greater than 5. The graph includes the line $x = 5$ and every point to the right of that line. These points are shaded in the graph.

Check Pick a point in the shaded region. Its coordinates should satisfy $x \ge 5$. We pick $(7, 1)$. Is $7 \ge 5$? Yes. So $(7, 1)$ is in the solution set.

If you pick a point in the non-shaded region its coordinates should not satisfy the inequality. We pick $(1, 4)$. Is $1 \ge 5$? No. Therefore $(1, 4)$ is not in the solution set.
Example 2 illustrates graphing inequalities with oblique boundaries.

**Example 2**

Graph the linear inequality \( y > 2x + 7 \).

**Solution**

Many ordered pairs satisfy this inequality. To picture them, first locate the boundary \( y = 2x + 7 \). This equation is in slope-intercept form. Plot the y-intercept 7 and use the slope to get another point \((1, 9)\). These points must be connected with a dashed line because the sign is >, not ≥. This means that the boundary points do not satisfy the inequality. The boundary line is shown below at the left.

To see which half-plane contains the solutions, test a point. Usually \((0, 0)\) is an easy point to test. Substitute \((0, 0)\) into \( y > 2x + 7 \). Is \(0 > 2(0) + 7\)? No. So \((0, 0)\) is not included in the points that are possible solutions to \( y > 2x + 7 \). The graph should be shaded in the half-plane on the other side of the line as shown at the right above.

**Check**

Pick a point in the shaded region. We pick \((-3, 6)\). Do the coordinates satisfy \( y > 2x + 7 \)? Is \(6 > 2(-3) + 7\)? Yes.

Linear inequalities often arise from real situations.

**Example 3**

A ferry boat transports cars and buses across a river. It has space for 12 cars, and a bus takes up the space of 3 cars. Draw a graph showing how many cars and buses can be taken in one crossing.

**Solution**

This is a linear-combination situation. A car occupies 1 space, so \( x \) cars need \( x \) spaces. A bus occupies 3 spaces, so \( y \) buses need \( 3y \) spaces. The ferry has only 12 spaces; so a sentence describing the situation is

\[ x + 3y \leq 12. \]
First, locate the boundary \( x + 3y = 12 \). This equation is in standard form; so use the intercepts, \((0, 4)\) and \((12, 0)\). Some points on the boundary go through lattice points (points with integer coordinates) and do satisfy the inequality.

To determine in which half-plane the solutions lie, try an ordered pair. Substituting \((0, 0)\) gives \(0 + 3(0) \leq 12\). This is true; so \((0, 0)\) is in the solution set, and the half-plane of solutions is below the boundary line. Because \(x\) and \(y\) represent numbers of cars and buses, they must be whole numbers. Thus the graph consists of only those points on or below the line \(x + 3y = 12\) whose coordinates are whole numbers. Because this is a discrete set, the solutions are represented by the dots in the graph. On the boundary, dots are shown only for the points with whole-number coordinates. The solution set is shown in the graph above. Notice that there are 35 combinations of cars and buses that can be taken by the ferry.

**Check** To check that the correct half-plane was chosen, pick a point in the other half-plane. It should not work in the inequality. We choose \((10, 5)\). Substitute into the inequality. Is \(10 + 3(5) \leq 12\)? No; so the solution checks.

In Example 3 the solutions can be identified individually. But in Example 2, the number of solutions is infinite. So the graph for Example 2 must be shaded because the solutions cannot be individually identified.

In summary, to graph a linear inequality:

1. Graph the appropriate boundary line, either dotted or solid for the corresponding linear equation.
2. Test a point in one half-plane to see if the point satisfies the inequality. The point \((0, 0)\) is often used, if possible.
3. Shade the half-plane that satisfies the inequality, or plot points if the situation is discrete.
Questions

Covering the Reading

1. A linear inequality is formed by replacing the = sign in a linear equation with ___, ___, ___, or ___.

2. A line separates a plane into two distinct regions called ___ . The line itself is called the ___ of these regions.

3. The graph of all solutions to \( x < -2 \) consists of points to the ___ of the line \( x = -2 \).

4. Graph the set of all ordered pairs that satisfy \( y \geq 3 \).

In 5–7, refer to Example 2. Justify your answer.

5. True or false The ordered pair \((-4, 3)\) is a solution to the inequality.

6. Why is the boundary line dotted rather than solid?

7. How would the graph change if the inequality were \( y < 2x + 7 \)?

In 8–10, refer to Example 3.

8. Why are there no points in the solution set in the second, third, or fourth quadrants?

9. a. In how many ways can the ferry cross the river full?
   b. List them.
   c. On which part of the graph are these solutions found?

10. a. In how many ways can the ferry cross the river with empty space?
    b. List four of them.
    c. Where are these solutions found on the graph?

In 11 and 12, graph each inequality.

11. \( 3y < 4x + 6 \)  
12. \( 2x - 5y \geq 20 \)

13. A person wants to buy \( x \) pencils at 10¢ each and \( y \) erasers at 15¢ each. The total spent must be less than 90¢.
   a. What inequality must \( x \) and \( y \) satisfy?
   b. Graph all solutions.
   c. How many solutions are there?

14. The players on the Dunkers basketball team must win at least twice as many games as they lose to make the playoffs. Graph the set of points \((L, W)\) that satisfy these conditions.

15. Write an inequality that describes the region graphed below.

![Graph](image)
16. Philip David Bayusun fills the bathtub slowly at a constant rate. He turns off the water, then gets in the tub and bathes. After a few minutes he gets out of the tub and pulls the plug. The water drains quickly. Which of the graphs below shows the relation between the volume $V$ of water in the tub and time $t$? *Lesson 3-8*

![Graphs](a), (b), (c), (d)

17. Draw a graph of the following situation:

\[
\begin{align*}
y & = 6, \text{ for } x > 2 \\
y & = 3x, \text{ for } 0 \leq x \leq 2 \\
y & = -\frac{1}{2}x, \text{ for } x \leq 0
\end{align*}
\]

*Lesson 3-8*

18. Write a recursive formula for the sequence described by $a_n = 3n + 11$. *Lesson 3-7*

19. Write equations for the lines graphed below. *Lessons 3-4, 3-5*

![Graphs](a), (b)

20. Two figures have equal areas. Must they be congruent if the figures are: *(Previous course)*

a. squares  
b. triangles  
c. octagons  
d. circles?

21. In this chapter you have graphed many lines that came from real situations.

a. Pick one of the situations and reword it so that it leads to a linear inequality, as in Example 3.

b. Graph the inequality.
A linear equation in two variables is one that is equivalent to an equation of the form $Ax + By = C$. The graph of a linear equation is a line. If the line is not vertical, then its equation can be put into the form $y = mx + b$, with slope $m$ and $y$-intercept $b$. Horizontal lines have slope 0 and equations of the form $y = b$. Slope is not defined for vertical lines, which have equations of the form $x = a$.

Linear equations result from two basic kinds of situations: constant increase or decrease, and linear combination. Sequences with a constant increase or decrease have a constant difference between terms. Their graphs are collinear points, and they are called linear, or arithmetic, sequences. If $a_n$ is the $n$th term of an arithmetic sequence with constant difference $d$, then the sequence can be described explicitly as

$$a_n = a_1 + (n - 1)d$$

or recursively as

$$\begin{cases} a_1 \\ a_n = a_{n-1} + d, \text{ for } n > 1. \end{cases}$$

A graph that is the union of segments and rays is called piecewise linear. Piecewise linear graphs result from situations in which rates are constant for a while but change at known points.

A linear inequality in two variables is one that is equivalent to $Ax + By < C$ or $Ax + By \leq C$. The graph of a linear inequality is a half-plane, the set of points on one side of a line. Linear inequalities can arise from any of the situations that lead to linear equations.

### Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description or a specific example.

**Lesson 3-1**
constant-increase, constant-decrease situation
slope-intercept form of a linear equation: $y = mx + b$
* $y$-intercept

**Lesson 3-3**
linear-combination situation

**Lesson 3-4**
vertical line, oblique line, horizontal line
standard form of a linear equation: $Ax + By = C$
* $x$-intercept

**Lesson 3-5**
point-slope form of a linear equation: $y - y_1 = m(x - x_1)$

**Lesson 3-6**
* linear sequence, arithmetic sequence
* explicit formula for an arithmetic sequence:

$$a_n = a_1 + (n - 1)d$$

**Lesson 3-7**
recursive formula for an arithmetic sequence:

**Lesson 3-8**
piecewise linear graph

**Lesson 3-9**
linear inequality
half-plane
boundary
Progress Self-Test

Take this test as you would take a test in class. Use graph paper, a ruler, and a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. Graph the line with equation $y = 3x - 5$.
2. Graph the line with equation $y = 40$.
3. Graph the set of points satisfying $x + 2y > 6$.
4. Consider the line with equation $4x - 5y = 12$.
   a. What is its slope?
   b. What are its $x$- and $y$-intercepts?
5. The equation $y = mx + b$ models a constant-decrease situation for what values of $m$?
6. Give an equation for the line through $(4, 2)$ and $(-5, 3)$.
7. Give an equation of the line parallel to $y = \frac{5}{3}x + 4$ that goes through $(5, -1)$.
8. a. For what kind of lines is slope not defined?
    b. Which lines have a slope of zero?

In 9 and 10, a company makes 36" and 48" shoe-laces by cutting off lengths from a spool of cord. Let $S$ be the number of 36" laces and $L$ be the number of 48" laces made.

9. How much cord will be used in making $S$ short and $L$ long laces?
10. If a spool has 3000 inches of cord and 50 short laces are made, how many long laces can be made?

In 11 and 12, a scuba diver is 40 m below the surface. She ascends at a constant rate of 0.8 m/sec.
11. What will be her depth after $t$ seconds?
12. How long will it take to reach a depth of 10 m?

13.  
   
   ```plaintext
   10 LET A = 1
   20 FOR N = 1 TO 12
   30 PRINT A
   40 A = A + 5
   50 NEXT N
   60 END
   ```
   a. What sequence is generated?
   b. Is the sequence arithmetic?

14. Rewrite $y = \frac{1}{3}x + 2$ in standard form.

15. *Multiple choice* A store charges for copies:
   
   For 1–50 copies 5¢ each
   51–200 copies 4¢ each
   more than 200 copies 3¢ each
   Which graph most closely describes the total cost $C$ for printing $n$ copies?

   ![Graphs](image)

   In 16 and 17, -7, -10, -13, -16, ... is an arithmetic sequence.

16. Write an explicit formula for the sequence.
17. Write a recursive formula for the sequence.

18. a. Is the origin in the solution set for the graph of $y > -3x$? Justify your answer.
    b. Is $(2, -1)$ in the graph of $3x - 5y < 8$?
    Justify your answer.

19. Carlos began by swimming 20 min every day for the first week. Each week he increased his daily swim time by 15 min. After how many weeks was he swimming 110 min daily?

20. The graph below represents the height and horizontal distance moved by a ski lift.

   ![Graph](image)
   a. What is the slope of the section whose horizontal distance goes from 600 to 900 feet?
   b. Write an equation for the section in part a.
SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers.

- **Objective A:** Determine the slope and intercepts of a line given its equation. (Lessons 3-1, 3-2, 3-4)
  
  In 1–3, give: (a) the slope; (b) the y-intercept.
  
  1. \( y = 7x - 2 \)
  2. \( 500x + 700y = 1200 \)
  3. \( y = 4 \)
  
  In 4–6, find: (a) the x-intercept; (b) the y-intercept.
  
  4. \( 3x + 5y = 45 \)
  5. \( x = -4.7 \)
  6. \( 6y = 8x \)

- **Objective B:** Find an equation for a line given two points on it, or given a point on it and its slope. (Lesson 3-5)
  
  7. Find an equation for the line with slope 8 containing (40, 75).
  8. Find an equation for the line with slope -0.25 through the origin.
  9. Find an equation for the line through (2, 4) and (-1, 6).
  10. Find an equation for the line through (5,-9) and (5, 14).
  11. Find an equation for the line parallel to \( 3x + 2y = 9 \) and passing through (-1, 2).
  12. Find an equation for the line parallel to \( y = 4x \), containing (11, 0).

- **Objective C:** Convert linear equations from standard form to slope-intercept form, and vice-versa. (Lessons 3-2, 3-4)
  
  In 13 and 14, put into slope-intercept form.
  
  13. \( 2x + 6y = 12 \)
  14. \( x - y = 4 \)

  In 15 and 16, put into standard form \( Ax + By = C \) with \( A, B, \) and \( C \) integers.
  
  15. \( y = \frac{2}{3}x - \frac{5}{3} \)
  16. \( 2y - 4 = 5x \)

- **Objective D:** Describe arithmetic sequences, both explicitly and recursively. (Lessons 3-6, 3-7)
  
  In 17 and 18, describe the \( n \)th term of each arithmetic sequence: (a) in explicit form; (b) in recursive form. Then (c) find the 75th term of the sequence.
  
  17. 7, 12, 17, 22, ... 
  18. 8, 2, -4, -10, ...

  19. Give a recursive description of the sequence \( a_n = 2n - 11 \).

  20. Find an explicit formula for the \( n \)th term of the sequence

  \[
  \begin{align*}
  a_1 &= \frac{1}{2} \\
  a_n &= a_{n-1} + 4, \text{ for } n > 1.
  \end{align*}
  \]

  The Chambered Nautilus shows recursive growth.
In 21 and 22, (a) find how many terms will be printed, and (b) give the first five terms the computer will print when the program is run.

21.

<table>
<thead>
<tr>
<th>Line</th>
<th>Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>LET A = 100</td>
</tr>
<tr>
<td>30</td>
<td>FOR N = 1 TO 1000</td>
</tr>
<tr>
<td>40</td>
<td>PRINT A</td>
</tr>
<tr>
<td>50</td>
<td>A = A + 1</td>
</tr>
<tr>
<td>60</td>
<td>NEXT N</td>
</tr>
<tr>
<td>70</td>
<td>END</td>
</tr>
</tbody>
</table>

22. 25 FOR N = 1 TO 10
35 LET AN = 2*N + 3
45 PRINT AN
55 NEXT N
65 END

**PROPERTIES** deal with the principles behind the mathematics.

- **Objective E:** Recognize properties of graphs of linear relations. (Lessons 3-2, 3-4, 3-5, 3-9)
  In 23–25, state whether the line is vertical, horizontal, or oblique.
  23. \( y = -4 \)
  24. \( 2x - 3y = 8 \)
  25. \( 4x = 12 \)
  26. Parallel lines have the same ___.
  27. Multiple choice Which of the following does not mean a slope of \(-\frac{3}{4}\)?
     (a) a vertical change of -3 units for a horizontal change of 4 units
     (b) a vertical change of -4 units for a horizontal change of 3 units
     (c) a vertical change of \(-\frac{4}{3}\) units for a horizontal change of 1 unit
     (d) a vertical change of \(\frac{3}{2}\) units for a horizontal change of -1 unit
  28. True or false The line with the equation \( y - 5 = 3(x - 2) \) goes through the point (5, 2).
  29. How does the graph of \( y > 2x - 7 \) differ from the graph of \( y = 2x - 7 \)?

In 30–32, tell whether the point (3, 3) is in the solution set of the inequality.

  30. \( y \geq x \)
  31. \( y < x \)
  32. \( y < 2x - 7 \)

- **Objective F:** Identify properties of the three general forms of linear relations. (Lessons 3-2, 3-4, 3-5)
  In 33–35, use the equation \( y = mx + b \).
  33. It models a constant increase if \( m \) is ____.
  34. What constant represents the initial amount?
  35. What is this form of a linear equation called?

In 36 and 37, use the equation \( Ax + By = C \).

  36. To find the \( x \)-intercept, substitute ____ for ____.
  37. If \( A = 0 \) and \( B \neq 0 \), the graph of this equation is a ____ line.
  38. Give the point-slope form of a linear equation.

- **Objective G:** Recognize properties of arithmetic sequences. (Lessons 3-6, 3-7)
  39. Arithmetic sequences are formed by ____ a ____ to the previous term.

  40. Multiple choice Which is not true of the graph of an arithmetic sequence?
     (a) The graph consists of discrete points.
     (b) All points on the graph are collinear.
     (c) The graph is a half-plane.

In 41–44, tell whether the numbers could be the first four terms of an arithmetic sequence.

  41. \( 1.2, 1.4, 1.6, 1.8, \ldots \)
  42. \( \pi + 1, \pi + 2, \pi + 3, \pi + 4, \ldots \)
  43. \( -9, -11, -13, -15, \ldots \)
  44. \( 4, 2, \frac{1}{2}, \frac{1}{3}, \ldots \)
In 45–48, does the formula generate an arithmetic sequence?

45. \[
\begin{cases}
  a_1 = 1.5 \\
  a_n = a_{n-1} + 13, \text{ for } n > 1
\end{cases}
\]

46. \[
\begin{cases}
  a_1 = 9 \\
  a_n = 2a_{n-1} \text{ for } n > 1
\end{cases}
\]

47. \[a_n = 3n^2 + 2\]

48. \[a_n = 11n + 4\]

**USES** deal with applications of mathematics in real situations.

**Objective H:** Model constant increase or constant decrease situations. (Lesson 3-1)

In 49 and 50, a crate weighs 3 kilograms (kg) when empty. It is filled with grapefruit weighing 0.2 kg each.

49. Write an equation relating the weight \(w\) and the number \(n\) of grapefruit.

50. Find the weight when there are 22 grapefruit in the crate.

In 51 and 52, a math teacher has a ream of 500 sheets of graph paper. Each week the advanced algebra class uses about 30 sheets.

51. About how many sheets are left after \(w\) weeks?

52. After how many weeks will there be 50 sheets left?

**Objective I:** In a real-world context, find an equation for a line containing two given points. (Lesson 3-5)

53. Woody Bench finds that it costs his business $7,600 to make 30 desks and $16,000 to make 100 desks. Assuming a linear relationship between the cost and the number of desks, how much will it cost to make 1000 desks?

54. Celsius temperature and Réaumur temperature are related by a linear equation. Two pairs of corresponding temperatures are \(0^\circ C = 0^\circ R\) and \(100^\circ C = 80^\circ R\). Write a linear equation relating \(R\) and \(C\), and solve it for \(R\).

55. Charlotte’s business finds that the cost of making shoes is linearly related to the number of shoes it makes. It costs $1450 to make 150 pairs of shoes and $1675 to make 225 pairs of shoes.

56. Lubbock Lumber sells 6-foot 2-by-4s for $1.70 each and 8-foot 2-by-6s for $2.50 each. Last week they sold $250 worth of these boards. Let \(F\) be the number of 2-by-4s and \(S\) be the number of 2-by-6s.

a. Write an equation to model this situation.

b. If 100 2-by-4s were sold, how many 2-by-6s were sold?

57. A maintenance engineer of a swimming pool combines \(A\) gallons of water that is 6% chlorine and \(B\) gallons of water that is 8% chlorine.

a. How much water is there altogether?

b. How much chlorine is there altogether?

c. At least 2 gallons of chlorine are needed in the pool. Write an inequality that describes this situation.

**Objective K:** Solve real-world problems using arithmetic sequences. (Lessons 3-6, 3-7)

58. The number of feet traveled during each second of free fall is given by the formula \(a_n = 16 + 32(n - 1)\). What distance is traveled during the eighth second?

59. When Florence Flask joined a laboratory she was given a $26,000 salary and promised at least an $1800 raise each year. What is the longest it could take for her salary to reach $35,000?
**Objective L:** Graph linear equations and linear inequalities. (Lessons 3-2, 3-4, 3-9)

60. Graph the line with slope 5 and y-intercept 7.

61. Graph the line $4x - 6y = 36$.

62. Graph $x = 3$ in the coordinate plane.

63. Graph $y = -1$ in the coordinate plane.

64. Graph the set of points that satisfy $5x + 4y \leq 40$.

65. Graph the set of points that satisfy $y < 2x$.

In 66–68, tell whether the slope of the line is positive, negative, zero, or undefined.

66. 

67. 

68. 

69. What is an equation for the line graphed below?

**Objective M:** Graph and describe piecewise linear situations. (Lessons 3-8)

In 70–72, refer to the graph below. Cory traveled from her cousin’s house to her grandmother’s and then back home.

70. How far from home did Cory start?

71. How fast did Cory travel during the first two hours?

72. What was the total distance Cory traveled this day?

73. Graph the situation described by:

\[
\begin{cases}
  y = x + 3, & \text{for } x > 0 \\
  y = -x + 3, & \text{for } x \leq 0
\end{cases}
\]

74. A cheetah trots along at 5 mph for a minute, spies a small deer and speeds up to 60 mph in just 6 seconds. After chasing the deer at this speed for 30 seconds, the cheetah gives up and, over the next 20 seconds, slows to a stop. Graph this situation plotting time on the horizontal axis and speed on the vertical axis.
CHAPTER 4

Matrices

4-1: Storing Data in Matrices
4-2: Matrix Multiplication
4-3: Size Changes
4-4: Scale Changes
4-5: Reflections

4-6: Transformations and Matrices
4-7: Rotations
4-8: Perpendicular Lines
4-9: Matrix Addition
4-10: Translations
A matrix is a rectangular arrangement of objects, each of which is called an element of the matrix. The plural of "matrix" is "matrices." One use of matrices is to store data. In the matrix below, the elements are numbers representing the performance of five professional football teams in the 1985 season. The titles of the rows and columns are not part of the matrix.

National Football Conference
Central Division Results
1985

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>L</th>
<th>Pts. Earned</th>
<th>Opponents' Pts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chicago</td>
<td>15</td>
<td>1</td>
<td>456</td>
<td>198</td>
</tr>
<tr>
<td>Green Bay</td>
<td>8</td>
<td>8</td>
<td>337</td>
<td>355</td>
</tr>
<tr>
<td>Minnesota</td>
<td>7</td>
<td>9</td>
<td>346</td>
<td>359</td>
</tr>
<tr>
<td>Detroit</td>
<td>7</td>
<td>9</td>
<td>307</td>
<td>366</td>
</tr>
<tr>
<td>Tampa Bay</td>
<td>2</td>
<td>14</td>
<td>294</td>
<td>448</td>
</tr>
</tbody>
</table>

A second use of matrices is to describe transformations of various geometric figures. On the graph, \( QUAD \) has been reflected over the \( x \)-axis. The vertices of \( QUAD \) may be described by a matrix. You will learn that the image \( Q'U'A'D' \) can be calculated by multiplying two matrices.

\[
\begin{bmatrix}
Q & U & A & D \\
-4 & 0 & 5 & 4 \\
3 & 6 & 5 & 1
\end{bmatrix}
\]

A third use of matrices, which is even more geometric, is to describe the pictures you see on television screens and computer monitors. These matrices are rectangular arrays of square dots.

In this chapter you will study various operations using matrices and how those operations can be applied to geometric transformations and real life situations.
Information is often stored in matrices. The inventory of athletic clothing owned by a high-school cross-country team is shown in the matrix below.

\[
\begin{bmatrix}
\text{sweat pants} & \text{sweat shirts} & \text{shorts} \\
\text{small} & 9 & 10 & 8 & \text{row 1} \\
\text{medium} & 18 & 20 & 19 & \text{row 2} \\
\text{large} & 20 & 24 & 23 & \text{row 3} \\
\text{x-large} & 11 & 11 & 12 & \text{row 4} \\
\end{bmatrix}
\]

The elements of this matrix are enclosed by large square brackets. (Sometimes large parentheses are used in place of brackets.) This matrix has 4 rows and 3 columns. It is said to have dimensions 4 by 3, written $4 \times 3$. In general, a matrix with $m$ rows and $n$ columns has dimensions $m \times n$. 
Example 1  The Matterhorn Company produced 1500 trumpets and 1200 French horns in September; 2000 trumpets and 1400 French horns in October; 900 trumpets and 700 French horns in November.

a. Store the company’s production in a matrix.
b. What are the dimensions of the matrix?

Solution

a. There are two matrices that can be written. Matrix $M_1$ has the months as rows and matrix $M_2$ has the months as columns. Either matrix is an acceptable way to store the data.

<table>
<thead>
<tr>
<th></th>
<th>Matrix $M_1$</th>
<th>Matrix $M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>trumpets</td>
<td>French horns</td>
</tr>
<tr>
<td>Sept.</td>
<td>1500</td>
<td>1200</td>
</tr>
<tr>
<td>Nov.</td>
<td>900</td>
<td>700</td>
</tr>
</tbody>
</table>

b. Matrix $M_1$ has 3 rows and 2 columns, so its dimensions are $3 \times 2$. Matrix $M_2$ has 2 rows and 3 columns, so its dimensions are $2 \times 3$.

Although matrices $M_1$ and $M_2$ are equivalent ways to store the data, the two matrices are not considered equal. Two matrices are equal if and only if they have the same dimensions and corresponding elements are equal.

Points and polygons can also be represented by matrices. The ordered pair $(x, y)$ is generally represented by the matrix

$$
\begin{bmatrix}
x \\
y
\end{bmatrix}
$$

This $2 \times 1$ matrix is called a point matrix. Notice that the element in the first row is the $x$-coordinate and the element in the second row is the $y$-coordinate. Thus the point $(5, -1)$ is represented by the matrix

$$
\begin{bmatrix}
5 \\
-1
\end{bmatrix}
$$

Similarly, polygons can be written as matrices. The first row of the matrix contains the $x$-coordinates of the vertices in the order in which the polygon is named. The second row contains the corresponding $y$-coordinates. Example 2 on page 184 illustrates this.
Example 2

a. Write pentagon
   PENTA as a matrix.

b. Write pentagon
   NEPAT as a matrix.

c. Are the two matrices
   equal?

Solution

a. Starting with the coordinates
   of P, write the x-coordinates
   in the first row and the
   y-coordinates in the second
   row.

b. Start with the coordinates
   of N.

c. The two matrices are not equal because all corresponding elements
   are not equal. However, both matrices are valid ways to represent
   the polygon.

Questions

1. What is a matrix?

2. True or false Each element in a matrix must be a number.

In 3 and 4, for each matrix state: (a) the number of rows;
(b) the number of columns; and (c) the dimensions.

3. the National Football League matrix on page 181.

4. the matrix for pentagon PENTA above.

In 5–7, refer to the clothing matrix at the start of this lesson.

5. How many large sweatshirts did the cross-country team have?

6. What type of clothing does the element in the 2nd row, 3rd column
   represent?

7. What does the sum of the elements in the 3rd column represent?

8. Refer to Example 1. Suppose the Matterhorn Company produces 2500
   trumpets and 3800 French horns in December. Construct a 4 \times 2 ma-
   trix that gives the company’s production through December.
9. The ordered pair \((a, b)\) can be represented by the matrix \(?\). This matrix is called a \(?\) matrix.

10. \textit{Multiple choice} Which represents the point \((-1, 4)\)?
   
   - \((a) \begin{bmatrix} -1 & 4 \end{bmatrix}\)
   - \((b) \begin{bmatrix} 4 & -1 \end{bmatrix}\)
   - \((c) \begin{bmatrix} -1 \\ 4 \end{bmatrix}\)
   - \((d) \begin{bmatrix} 4 \\ -1 \end{bmatrix}\)

11. Refer to Example 2.
   \(a\). Write pentagon EPATN as a matrix.
   \(b\). Are the matrices for EPATN and PENTA equal? Why or why not?

12. Write \(\triangle TRI\) as a matrix.

\[
\begin{align*}
I &= (-2, 4) \\
R &= (2, 3) \\
T &= (-1, -2)
\end{align*}
\]

13. The matrix below gives, to the nearest thousand, the numbers of active duty U.S. military personnel as of 1988.

\[
\begin{bmatrix}
\text{Commissioned} & \text{Enlisted} \\
\text{Officers} & \text{Personnel}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th></th>
<th>Commissioned</th>
<th>Enlisted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army</td>
<td>107,000</td>
<td>660,000</td>
</tr>
<tr>
<td>Navy</td>
<td>72,000</td>
<td>514,000</td>
</tr>
<tr>
<td>Marines</td>
<td>20,000</td>
<td>177,000</td>
</tr>
<tr>
<td>Air Force</td>
<td>105,000</td>
<td>467,000</td>
</tr>
</tbody>
</table>

\(a\). What are the dimensions of this matrix?
\(b\). What does the sum of the elements in row 2 represent?
\(c\). What does the sum of the elements in column 1 represent?

14. If \(\begin{bmatrix} 7 & 4 \\ y & 2 \end{bmatrix} = \begin{bmatrix} x & 4 \\ 8 & 2 \end{bmatrix}\), then \(x = \) \(\) and \(y = \) \(\).

15. If \(\begin{bmatrix} 3a + 1 \\ b + 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}\), then \(a = \) \(\) and \(b = \) \(\).
16. Three cousins, Adam, Barbara, and Clem, write to each other from time to time. Last year Adam received 2 letters from Barbara and 5 from Clem; Barbara received 3 from Adam and 3 from Clem. Clem received 1 from Adam and 4 from Barbara.

a. Organize this information in a $3 \times 3$ matrix. (Hint: There are three 0s in the matrix.)

b. How many letters did each cousin write?

17. The matrix $\begin{bmatrix} 7 & 8 & 4 & 5 & 1 & 0 \\ 11 & -1 & 3 & 9 & 2 & 6 \end{bmatrix}$ describes a hexagon.

Graph this hexagon.

18. Consider the line $y = kx$ graphed on a coordinate plane in which the scales on the $x$- and $y$-axes are the same. In one sentence, describe how the position of the line $y = kx$ changes as $k$ changes from the number 1 to:

a. a large positive real number,

b. a very small positive fraction,

c. a negative real number. (Lesson 2-4)

19. Which four of the following describe the graph of $y = -4x^2$? (Lesson 2-5)

a. hyperbola

b. direct variation

c. parabola

d. inverse variation

e. continuous

f. symmetric to $y$-axis

20. Find the next two terms in the following inverse-variation sequence:

(Lessons 1-3, 2-7)

$(1, 36), (2, 18), (3, 12), (4, 9), (5, 7.2)$. 

21. Sports results in newspapers are often tabulated as matrices. Look in a newspaper to find an example of a matrix different from that in this lesson.
Matrix Multiplication

In linear-combination applications, it is quite useful to store data in matrices which can then be multiplied. For instance, a movie theater charges $5 for adults over 17, $2 for students 13–17 years old, and $1 for children 12 or under. What is the cost for 7 adults, 4 students, and 3 children under 12 to enter this theater?

The answer is \(7 \cdot 5 + 4 \cdot 2 + 3 \cdot 1 = 46\). This is the same arithmetic needed to calculate the product of these two matrices.

\[
\begin{bmatrix} 7 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}
\]

number of people
in each category

<table>
<thead>
<tr>
<th>cost per category</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5</td>
</tr>
<tr>
<td>$2</td>
</tr>
<tr>
<td>$1</td>
</tr>
</tbody>
</table>

Matrix multiplication is done by multiplying a row by a column. Multiply the first element in the row by the first element in the column, the second element in the row by the second element in the column, and so on. Finally, add the resulting products. The product is the \(1 \times 1\) matrix \([46]\), corresponding to the $46 total cost for the movie.

\[
\begin{bmatrix} 7 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \cdot 5 + 4 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 46 \end{bmatrix}
\]

If the left matrix has 2 rows and the right matrix 3 columns, then there are 6 ways to multiply a row by a column. That is exactly what is done to multiply two larger matrices. The 6 numbers are put in 2 rows and 3 columns in a natural way.

**Example 1** Let \(A = \begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix}\). Find \(AB\).
Solution  Find the dimensions of $AB$. The product has the same number of rows as the first matrix and the same number of columns as the second matrix. So $AB$ has 2 rows and 3 columns. Now fill in $AB$.

The product of row 1 of $A$ and column 1 of $B$ is

$$8 \cdot 1 + (-2) \cdot 0 = 8.$$ 

This is put in the 1st row and 1st column of the answer. Thus far we have

$$\begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 8 & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}.$$ 

The product of row 1 of $A$ and column 2 of $B$ is $8 \cdot 3 + (-2) \cdot 4 = 16$. Now you know the element in the 1st row and 2nd column of the answer.

$$\begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 & \_ \\ \_ & \_ & \_ \end{bmatrix}$$

The other four elements of $P$ are found using this row by column pattern.

For instance, the element in the 2nd row, 3rd column of $AB$ is found by multiplying the 2nd row of $A$ by the 3rd column of $B$, shown here along with the final result.

$$\begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 & 36 \\ 4 & 16 & 22 \end{bmatrix}$$

**Definition of matrix multiplication:**

Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Then the product $A \cdot B$ or $AB$ is the $m \times p$ matrix whose element in row $i$ and column $j$ is the product of row $i$ of $A$ and column $j$ of $B$.

Notice that the product of two matrices exists only if the rows of the left matrix can be multiplied by the columns of the right matrix. Thus the product of two matrices $A$ and $B$ exists only when the number of columns of $A$ equals the number of rows of $B$. So if $A$ is $m \times n$, $B$ must be $n \times p$ in order for $AB$ to exist.

These matrices can be multiplied. These matrices cannot be multiplied.

$$\begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ 4 & 1 \end{bmatrix}$$

$$2 \times 2 \quad 2 \times 2$$

$$2 \times 3 \quad 2 \times 2$$

equal

not equal
These two cases indicate that in general, multiplication of matrices is not commutative.

The following example illustrates a situation requiring multiplication of more than two matrices.

**Example 2**
Costumes have been designed for the school play. Each boy's costume requires 5 yards of fabric, 4 yards of ribbon, and 3 packets of sequins. Each girl's costume requires 6 yards of fabric, 5 yards of ribbon, and 2 packets of sequins. Fabric costs $4 per yard; ribbon costs $2 per yard; and sequins cost $0.50 per packet. Costumes are needed for 8 boys and 10 girls. Find the total cost of making these costumes.

**Solution**
The information can be stored in three matrices.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

Number of costumes

<table>
<thead>
<tr>
<th>Fabric</th>
<th>Ribbon</th>
<th>Sequins</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Materials for one costume

<table>
<thead>
<tr>
<th>cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>0.50</td>
</tr>
</tbody>
</table>

Unit cost of material

The total cost of making the costumes is given by the product of these three matrices. To multiply more than two matrices, multiply two at a time using the definition given earlier.

\[
\begin{bmatrix}
8 & 10 \\
\end{bmatrix}
\begin{bmatrix}
5 & 4 & 3 \\
6 & 5 & 2 \\
\end{bmatrix}
\begin{bmatrix}
4 \\
2 \\
0.50 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
8 & 10 \\
\end{bmatrix}
\begin{bmatrix}
5 \cdot 4 + 4 \cdot 2 + 3 \cdot 0.50 \\
6 \cdot 4 + 5 \cdot 2 + 2 \cdot 0.50 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
8 & 10 \\
\end{bmatrix}
\begin{bmatrix}
29.50 \\
35 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
8 \cdot 29.50 + 10 \cdot 35 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
586 \\
\end{bmatrix}
\]

The total cost is $586.

Notice that in Example 2 we calculated the cost of making one boy’s costume ($29.50) and one girl’s costume ($35) and then multiplied this matrix by the matrix representing the number of boys’ and girls’ costumes needed. In Question 8, you are asked to verify that the result is the same if you first find the total amount of material needed for all costumes and then multiply by the matrix giving the unit cost of each item. You will be verifying an instance that, in general, matrix multiplication is associative.
Questions

Covering the Reading

In 1 and 2, multiply the column by the row.

1. \[
\begin{bmatrix}
3 & 5 & 7
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
-2
\end{bmatrix}
= \begin{bmatrix}
\ ? 
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
1 & -1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
10 \\
9 \\
8 \\
7
\end{bmatrix}
= \begin{bmatrix}
\ ? 
\end{bmatrix}
\]

In 3 and 4, (a) determine the dimensions of each matrix; (b) decide if the product can or cannot be found; and (c) if so, find the product; if not, tell why not.

3. \[
\begin{bmatrix}
8 & 1 & 0 \\
6 & 3 & -4
\end{bmatrix}
\begin{bmatrix}
2 & 8 \\
5 & 4
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
9 & 4 & 8 & 6 \\
2 & 0 & 3 & 1 \\
1 & -2 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
12 & 2 \\
15 & 1 \\
3 & 9 \\
8 & 11
\end{bmatrix}
\]

5. Let \( M = \begin{bmatrix}
6 & 2 \\
0 & 3
\end{bmatrix} \) and \( N = \begin{bmatrix}
5 & 8 & -2 \\
-4 & 1 & 0
\end{bmatrix} \). Find the elements in \( MN = \begin{bmatrix}
\ ? \\
\ ? \\
\ ? \\
\ ?
\end{bmatrix} \).

6. If \( A \) has dimensions \( 11 \times 15 \) and \( B \) has dimensions \( 15 \times 19 \), what are the dimensions of \( AB \)?

7. If \( A \) is \( m \times n \) and \( B \) is \( p \times q \), when does \( AB \) exist?

In 8 and 9, refer to Example 2.

8. Verify that \( \begin{bmatrix}
7 & 8 \\
6 & 5
\end{bmatrix}
\begin{bmatrix}
5 & 4 & 3 \\
6 & 5 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
.50
\end{bmatrix}
= \begin{bmatrix}
586
\end{bmatrix} \).

9. What property is verified using the results from Example 2 and Question 8?

In 10 and 11, suppose \( X = \begin{bmatrix}
3 & 0 & 5 \\
-1 & 4 & 2
\end{bmatrix} \) and \( Y = \begin{bmatrix}
2 & -2 \\
0 & 1 \\
-3 & 4
\end{bmatrix} \).

Calculate each product.

10. \( XY \)

11. \( YX \)

12. \( True \ or \ false \) Matrix multiplication is commutative.
Applying the Mathematics

13. The matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] is called the \(2 \times 2\) identity matrix for multiplication. To see why, calculate the products in parts a and b.

a. \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

c. True or false Matrix multiplication with the identity matrix is commutative.

In 14 and 15, the matrix \(D\) below gives the daily delivery of cases of bakery products to two restaurants.

\[
\begin{array}{c|ccc}
& \text{whole wheat} & \text{white} & \text{rye} & \text{muffins} \\
\hline
\text{Shorty's} & 5 & 10 & 3 & 5 \\
\text{Slim's} & 0 & 15 & 8 & 10 \\
\end{array}
\]

The matrix \(C\) below gives the unit cost for each item in the bakery.

\[
\begin{array}{c|c}
& \text{whole wheat} & \text{white} & \text{rye} & \text{English muffin} \\
\hline
\text{whole wheat} & .70 & \text{white} & .70 & \text{rye} \\
\text{rye} & .65 & \text{English muffin} & .80 & \\
\end{array}
\]

14. a. Find \(DC\).

b. What is the daily cost of bakery products at Shorty’s?

15. Shorty’s restaurant is open 20 days this month and Slim’s is open 25 days. Let \(M = \begin{bmatrix} 20 & 25 \end{bmatrix}\). Find the total cost of bakery items for the month at these two restaurants.

16. Solve for \(x\). \[
\begin{bmatrix}
3 & 1 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
9
\end{bmatrix} = \begin{bmatrix}
10 \\
18
\end{bmatrix}
\]
17. In June, the Faucets & Fixtures Company produced 50 porcelain sinks, 40 stainless steel sinks, and 17 molded plastic sinks. In July, they produced 100 porcelain, 80 stainless steel, and 3 molded plastic sinks. In August they produced 42 porcelain, 58 stainless steel, and 5 molded plastic sinks. Write two different $3 \times 3$ matrices to store these data. (Lesson 4-1)

18. Use the matrix $\begin{bmatrix} 0 & -1 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{bmatrix}$.
   
   a. Graph the polygon represented by the matrix. (Lesson 4-1)
   b. What kind of polygon is it? (Previous course)
   c. Graph the image of this polygon under a size change of magnitude 2. (Previous course)

19. The matrix $\begin{bmatrix} 1 & 7 & 2 \\ 5 & 6 & -1 \end{bmatrix}$ can represent a triangle. Use the distance formula to show that this triangle is isosceles. (Lesson 4-1, previous course)

20. In this figure, $\triangle ADB \sim \triangle AEC$. If $DE = 12$, find $BC$ and $CE$. (Previous course)

   ![Diagram]

21. In Question 13 you worked with the $2 \times 2$ identity matrix. Make a conjecture regarding the $3 \times 3$ identity matrix for multiplication. Test your conjecture on the general $3 \times 3$ matrix.

   $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$
In the drawing above, figure I and point \( O \) were given. We call figure I the **preimage**. The points on figure II are located in the following way. For each point \( A \) on figure I, a corresponding point \( A' \) (read "‘A prime’") is found for figure II. The position of \( A' \) is determined by the following rules:

\[
A' \text{ is on } \overrightarrow{OA} \\
\frac{OA'}{OA} = 2
\]

This procedure was repeated with point \( B \) to find \( B' \) and with all other key points on the smaller car.

The resulting figure II is called a **size change image** of figure I. Specifically, the above size change has center \( O \) and magnitude 2. The two figures are similar with the ratio of similitude being 2.

Size changes with centers at \((0, 0)\) are easy to do using matrices.

As you have seen, matrices can represent geometric figures. For example, if \( P = (3, 1) \), \( Q = (-4, 0) \), and \( R = (-3, -2) \), then \( \triangle PQR \) can be represented by the matrix

\[
\begin{bmatrix}
3 & -4 & -3 \\
1 & 0 & -2
\end{bmatrix}.
\]

Notice what happens when this matrix is multiplied by \[
\begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix}.
\]

\[
\begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix} \begin{bmatrix}
3 & -4 & -3 \\
1 & 0 & -2
\end{bmatrix} = \begin{bmatrix}
9 & -12 & -9 \\
3 & 0 & -6
\end{bmatrix}
\]

\(P', Q', R'\)
The product can also be considered as a triangle. We call it \( \triangle P'Q'R' \), (read “triangle \( P' \), \( Q' \), \( R' \)”). Multiplying by

\[
\begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix}
\]

has transformed the original triangle. \( \triangle PQR \) is the preimage; \( \triangle P'Q'R' \) is its image.

Recall from geometry that this transformation is called a **size change of magnitude** or **scale factor** 3. We denote this size change by \( S_3 \). We write

\[
S_3 (3, 1) = (9, 3) \\
S_3 (-4, 0) = (-12, 0) \\
S_3 (-3, -2) = (-9, -6)
\]

We read the first one as “A size change of magnitude 3 maps (3, 1) onto (9, 3).” Since

\[
\begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
3x \\
3y
\end{bmatrix},
\]

in general

\[
S_3 (x, y) = (3x, 3y).
\]

We read this as “A size change of magnitude 3 maps any point \((x, y)\) onto the point \((3x, 3y)\).”

In general, \( S_k \) represents the size change with center \((0, 0)\) and magnitude \( k \neq 0 \). In earlier courses, you may have learned that \( S_k(x, y) = (kx, ky) \). Now, with matrix multiplication,

\[
\begin{bmatrix}
k & 0 \\
0 & k
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
kx + 0y \\
0x + ky
\end{bmatrix} = \begin{bmatrix}
kx \\
ky
\end{bmatrix}.
\]

This proves the following theorem.
Theorem: 
\[
\begin{bmatrix}
k & 0 \\
0 & k \\
k & 0
\end{bmatrix}
\] is the matrix for $S_k$.

Example: Given $ABCD$ with $A = (0, 3), B = (-2, -4), C = (-6, -4)$, and $D = (-6, 4)$, find the image $A'B'C'D'$ under $S_5$.

Solution: Write $ABCD$ and $S_5$ in matrix form and multiply.

\[
S_5 \begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix} \begin{bmatrix}
0 & -2 & -6 & -6 \\
3 & -4 & -4 & 4
\end{bmatrix} = \begin{bmatrix}
0 & -10 & -30 & -30 \\
15 & -20 & -20 & 20
\end{bmatrix}
\]

Thus $A'B'C'D'$ has vertices $A' = (0, 15), B' = (-10, -20), C' = (-30, -20)$, and $D' = (-30, 20)$.

Check: Graph the preimage and image. They should look similar, and they do.

In similar figures, (1) corresponding angles are congruent and (2) ratios of corresponding segments equal the ratio of similitude. Referring back to the Example, (1) $\angle D \cong \angle D'$ and (2) $\frac{D'C'}{DC} = \frac{40}{8} = 5$.

The ratio of similitude is thus the same as the magnitude of the size change. Any two other corresponding distances will have the same ratio. For instance,

\[
AB = \sqrt{(-2 - 0)^2 + (-4 - 3)^2} = \sqrt{53}
\]

and $A'B' = \sqrt{(-10 - 0)^2 + (-20 - 15)^2} = \sqrt{1325}$.

Thus $\frac{A'B'}{AB} = \frac{\sqrt{1325}}{\sqrt{53}} = 5$,

which is the scale factor.

In general, a size change of magnitude $k$ multiplies distances by $k$ so that the ratio of image lengths to preimage lengths is $k$. 

LESSON 4-3 Size Changes 195
Questions

Covering the Reading

In 1 and 2, how is the expression read?
1. $S_3: (3, 1) = (9, 3)$
2. $\triangle P'Q'R'$

3. Refer to $\triangle PQR$ and $\triangle P'Q'R'$ in this lesson.
   a. $\triangle P'Q'R'$ is the image of $\triangle PQR$ under what transformation?
   b. What is the matrix for this transformation?

4. The matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ is associated with a(n) __ change with center ___ of magnitude ___.

In 5 and 6, refer to the Example in the lesson.
5. What are the coordinates of the image of point $B$?
6. What is the scale factor?

True or false In 7–9, under a size change,
7. an angle and its image are congruent.
8. a segment and its image are congruent.
9. a figure and its image are similar.

10. Let the quadrilateral $MATH$ be represented by the
    matrix $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$.
    a. Calculate the image $M'A'T'H'$ of this quadrilateral under $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.
    b. Graph the preimage and image.
    c. What type of quadrilateral is $MATH$?
    d. What type of quadrilateral is $M'A'T'H'$?

11. Suppose $P = (3, 4)$ and $P'$ is the image of $P$ under a size change with center $O = (0, 0)$ and magnitude 2.5.
    a. Verify that $\frac{OP'}{OP} = 2.5$.
    b. Give an equation for the line containing $O$, $P$, and $P'$.

12. $\triangle ABC$ has matrix $\begin{bmatrix} 6 & -4 & 2 \\ 8 & 2 & -2 \end{bmatrix}$. Graph $\triangle ABC$ and its image $\triangle A'B'C'$ under $S_{1/2}$. 
13. Refer to the Example in the lesson.
   a. Find the slope of $AB$.
   b. Find the slope of $A'B'$.
   c. Is $AB$ parallel to $A'B'$? Why or why not?
   d. Is $AD \parallel A'D'$? Justify your answer.

14. In the example, verify that $\frac{A'D'}{AD} = 5$.

15. A $4 \times 5$ drawing is enlarged to $8 \times 10$ by using a size change.

16. A matrix lists the vertices of an $n$-gon. What are the dimensions of the matrix? (Lesson 4-1)

17. A clothing manufacturer has factories in Chicago, Minneapolis, and Syracuse. Sales (in thousands) can be summarized by the following matrix $S$.

<table>
<thead>
<tr>
<th></th>
<th>Blouses</th>
<th>Dresses</th>
<th>Skirts</th>
<th>Slacks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chicago</td>
<td>9</td>
<td>14</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>Minneapolis</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Syracuse</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

   a. What are the dimensions of $S$?
   b. The selling price of a blouse is $25, of a dress is $70, of a skirt is $30, and of a pair of slacks is $30. Write a $4 \times 1$ matrix representing the selling prices of the items.
   c. Use matrix multiplication to determine the total revenue of each factory. (Lessons 4-1, 4-2)

18. Let $M = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ and $N = \begin{bmatrix} -8 & -6 \\ -4 & -2 \end{bmatrix}$. Calculate:

   a. $MN$
   b. $NM$ (Lesson 4-2)

19. Solve for $a$ and $b$: $\begin{bmatrix} 2 & a \\ 3 & b \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. (Lesson 4-2)
20. Two figures, $F$ and $G$, are similar. The perimeter of $F$ is 20 cm and of $G$ is 15 cm. If the area of $F$ is 100 cm$^2$, what is the area of $G$? (Previous course)

21. Graph $2x + 4y = 12$. (Lesson 3-9)

22. Refer to the graph below. The graph shows the location of an elevator over a one minute period. (Lessons 2-4, 3-8)

![Graph showing elevator location over time]

a. When is the elevator on the sixth floor?

b. At what rate does the elevator ascend?

c. Does the elevator descend at the same rate it ascends?

23. $ABCD$ is the square defined by the matrix $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

Transform $ABCD$ by multiplying its matrix by each of the following size-change matrices (and by some others of your own choice).

a. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

e. Find the area of each new shape. Enter your results in a table like this one:

<table>
<thead>
<tr>
<th>Original Area</th>
<th>Matrix</th>
<th>New Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 units$^2$</td>
<td>$\begin{bmatrix} 2 &amp; 0 \ 0 &amp; 2 \end{bmatrix}$</td>
<td>? units$^2$</td>
</tr>
<tr>
<td>4 units$^2$</td>
<td>$\begin{bmatrix} 3 &amp; 0 \ 0 &amp; 3 \end{bmatrix}$</td>
<td>? units$^2$</td>
</tr>
<tr>
<td>?</td>
<td>$\begin{bmatrix} 4 &amp; 0 \ 0 &amp; 4 \end{bmatrix}$</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>$\begin{bmatrix} 5 &amp; 0 \ 0 &amp; 5 \end{bmatrix}$</td>
<td>?</td>
</tr>
</tbody>
</table>

f. There is a connection between the entries of a matrix associated with a size change and the effect the matrix has on the area of a shape. What is this connection?
In contrast to a size change, which you studied in the previous lesson, a **scale change** can change a figure by stretching or shrinking it in either a horizontal direction only, in a vertical direction only, or in both directions.

The scale change $S_{a,b}$ combines a horizontal scale change of magnitude $a$ with a vertical scale change of magnitude $b$. Here we require the magnitudes $a$ and $b$ to be positive. If a magnitude is larger than 1, the scale change is a **stretch** in that direction. When a magnitude is less than 1, the scale change is a **shrink** in that direction.

**Definition:**

The scale change $S_{a,b}$ is the transformation that maps $(x, y)$ onto $(ax, by)$.

**Example 1**

Given $\triangle ABC$ with $A = (0, 3), B = (-2, -4), and C = (4, 0)$. Find the image $\triangle A'B'C'$ of $\triangle ABC$ under $S_{2,5}$.

**Solution**

$S_{2,5} (x, y) = (2x, 5y)$. That is, each image point is found by multiplying the $x$-coordinate of the preimage by 2 and the $y$-coordinate by 5.

$S_{2,5} (0, 3) = (0, 15)$
$S_{2,5} (-2, -4) = (-4, -20)$
$S_{2,5} (4, 0) = (8, 0)$

$\triangle A'B'C'$ has vertices $A' = (0, 15)$,
$B' = (-4, -20)$, and $C' = (8, 0)$. 
Example 1 shows that a scale change is not necessarily a similarity transformation. The ratios of the lengths of corresponding sides in \( \Delta ABC \) and \( \Delta A'B'C' \) are not equal.

\[
\frac{A'B'}{AB} = \frac{\sqrt{(-20 - 15)^2 + (-4 - 0)^2}}{\sqrt{(-4 - 3)^2 + (-2 - 0)^2}} = \frac{\sqrt{1241}}{\sqrt{53}} \approx 4.84
\]

and

\[
\frac{B'C'}{BC} = \frac{\sqrt{(0 - 20)^2 + (8 - 4)^2}}{\sqrt{(0 - 4)^2 + (4 - 2)^2}} = \frac{\sqrt{544}}{\sqrt{52}} \approx 3.23
\]

Because the ratios are different, the two triangles are not similar.

Because a size change has a \( 2 \times 2 \) matrix, it is reasonable to expect that a scale change also has one. Suppose that \( S_{a,b} \) has the matrix

\[
\begin{bmatrix}
e & f \\
g & h
\end{bmatrix},
\]

where \( e, f, g, \) and \( h \) are real numbers. Because \( (x, y) \rightarrow (ax, by) \) under \( S_{a,b} \), we want to find \( e, f, g, \) and \( h \) such that

\[
\begin{bmatrix}
e & f \\
g & h
\end{bmatrix}
\begin{bmatrix}
x \\y
\end{bmatrix} = \begin{bmatrix}
ax \\by
\end{bmatrix}.
\]
Theorem:
\[
\begin{bmatrix}
  a & 0 \\
  0 & b
\end{bmatrix}
\]

is the matrix for \( S_{a,b} \).

Proof:
By matrix multiplication,
\[
\begin{bmatrix}
  a & 0 \\
  0 & b
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \begin{bmatrix}
  ax \\
  by
\end{bmatrix},
\]
which proves the theorem.

Example 2
Refer to \( \triangle ABC \) from Example 1. Use matrix multiplication to find the image \( \triangle A'B'C' \) under \( S_{2,5} \).

Solution
Write \( S_{2,5} \) and \( \triangle ABC \) in matrix form.
\[
S_{2,5} = \begin{bmatrix}
  2 & 0 \\
  0 & 5
\end{bmatrix}, \quad \triangle ABC = \begin{bmatrix}
  0 & -2 & 4 \\
  3 & -4 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 & -4 & 8 \\
  15 & -20 & 0
\end{bmatrix}
\]

Check
The product matrix gives the same result for \( \triangle A'B'C' \) that was found in Example 1.

Notice that a scale change may stretch or shrink by different amounts in the horizontal and vertical directions. If the amounts are the same in both directions, then the scale change matrix has the form
\[
\begin{bmatrix}
  a & 0 \\
  0 & a
\end{bmatrix}
\]
and is really just a size change. Thus a size change is a special type of scale change. This can be stated in symbols as \( S_{k,k} = S_k \).

Questions
1. \( S_{a,b} \) maps \((x, y)\) onto \(?\).
2. What is the image of \((1, -2)\) under \( S_{2,5} \)?
3. If the horizontal and vertical scale change of the right-most drawing of the person at the beginning of this lesson were to be done by applying \( S_{a,b} \), what are the values of \( a \) and \( b \)?
4. a. Multiply \( \begin{bmatrix}
  100 & 0 \\
  0 & 200
\end{bmatrix} \begin{bmatrix}
  7 \\
  9
\end{bmatrix} \)
   b. You have found the image of \(?\) under \(?\).
Applying the Mathematics

5. The scale change with matrix \[
\begin{bmatrix}
0.5 & 0 \\
0 & 1.5
\end{bmatrix}
\] is a horizontal \_\_ and a vertical \_\_.

6. True or false A size change is a special type of scale change.

7. Refer to Example 1.
   a. Find the slope of \(AB\).
   b. Find the slope of \(A'B'\).
   c. Is \(AB\) parallel to \(A'B'\)? Justify your answer.
   d. Under a scale change, is a line necessarily parallel to its image?

In 8–10, determine whether a size change or a scale change is needed. Write the matrix that could be associated with each change.

8. A 4 \times 6\) photograph is to be enlarged to 10 \times 12.

9. A cabinet maker wants to make a new bookcase twice as tall and twice as wide as the original.

10. A drawing that fills an 8" \times 10\(\frac{1}{2}\)" regular notebook paper is enlarged for 8\(\frac{1}{2}\)" \times 11" college-size notebook paper.

11. a. Quadrilateral \(TOPS\) is represented by the matrix
   \[
   \begin{bmatrix}
   0 & 4 & 4 & 0 \\
   0 & 0 & 6 & 6
   \end{bmatrix}
   \]
   What type of quadrilateral is \(TOPS\)?
   b. Find the matrix of the image of quadrilateral \(TOPS\) under
   \[
   \begin{bmatrix}
   3 & 0 \\
   0 & 2
   \end{bmatrix}
   \]
   c. What type of quadrilateral is \(T'O'P'S'\)?
   d. Graph the preimage and image of the quadrilateral.

12. Consider the matrix equation below.
   \[
   S_{a,b} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \triangle TRY \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 16 & -4 & 12 \end{bmatrix}
   \]
   a. What scale change is represented by this equation?
   b. Draw the preimage and the image of \(\triangle TRY\).
   c. Find \(\frac{T'R'}{TR}\) and \(\frac{T'Y'}{TY}\).
   d. Should the ratios be the same? Why or why not?

13. Consider the line with equation \(y = x\).
   a. Graph this line using the same scales for the \(x\)- and \(y\)-axes.
   b. What is the measure of the angle that the line makes with the positive \(x\)-axis? (Previous course, Lesson 2-4)

14. Evaluate \(P_n = 500(2)^{n-1}\) when \(n = 1\). (Lesson 1-2)
15. Find $AX$ when $X = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & -3 \\ 3 & 4 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \\ -3 & -2 & 0 \end{bmatrix}$  

(Lesson 4-2)

16. The matrix below gives the daily delivery of boxes of apples and pears to two markets.

\[
\begin{array}{ccc}
\text{apples} & \text{Troy's} & \text{Abby's} \\
5 & 4 \\
1 & 2
\end{array}
\]

During peak season the markets triple their demand for fruit.

a. What size change is needed to meet the increased demand? Represent the size change by a matrix.

b. Multiply the original matrix by the size-change matrix to find the new matrix which meets the increased demand. (Lesson 4-3)

17. a. Solve $\frac{x}{5} = 15$.  
b. Solve $\frac{y}{9} = 15$. (Lesson 1-5)

18. $ABCD$ is the square defined by the matrix $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

Transform $ABCD$ by multiplying its matrix by each of the following matrices (and by some others of your own choice).

a. $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$  
b. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  
d. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

e. Find the area of each new shape. Enter your results in a table like this one:

<table>
<thead>
<tr>
<th>Original Area</th>
<th>Matrix</th>
<th>New Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 units$^2$</td>
<td>$\begin{bmatrix} 3 &amp; 0 \ 0 &amp; 4 \end{bmatrix}$</td>
<td>units$^2$</td>
</tr>
<tr>
<td>4 units$^2$</td>
<td>$\begin{bmatrix} 3 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>units$^2$</td>
</tr>
<tr>
<td>?</td>
<td>$\begin{bmatrix} 3 &amp; 0 \ 0 &amp; 2 \end{bmatrix}$</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>$\begin{bmatrix} 2 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>?</td>
</tr>
</tbody>
</table>

f. There is a connection between the elements $a$ and $b$ of the scale-change matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and the effect the scale change has on the area of a shape. What is this connection?
Recall from geometry that the reflection image of a point $A$ over a line $m$ is:

1. the point $A$, if $A$ is on $m$;
2. the point $A'$ such that $m$ is the perpendicular bisector of $AA'$, if $A$ is not on $m$.

The line $m$ is called the reflecting line or line of reflection.

Suppose the reflecting line is the $y$-axis, as shown below. If $A = (x, y)$, its reflection image has the opposite first coordinate but the same second coordinate. So $A' = (-x, y)$.

![Reflection over the y-axis](image)

You can verify that the $y$-axis is the perpendicular bisector of $AA'$. Reflection over the $y$-axis can be denoted $r_{y\text{-axis}}$ or $r_y$. In this book we use $r_y$. So we can write

$$r_y(x, y) = (-x, y).$$

We read this as “the reflection over the $y$-axis maps point $(x, y)$ onto point $(-x, y)$.”
Can a matrix associated with \( r_y \) be found? Notice that
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
-1 \cdot x + 0 \cdot y \\
0 \cdot x + 1 \cdot y
\end{bmatrix}
= \begin{bmatrix}
-x \\
y
\end{bmatrix}.
\]
This proves the following theorem

**Theorem:**
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\text{ is the matrix for } r_y.
\]

**Example 1** If \( A = (1, 2), B = (1, 4), \) and \( C = (2, 4) \), find the image of \( \triangle ABC \) under the transformation \( r_y \).

**Solution** Represent \( r_y \) and \( \triangle ABC \) as matrices and multiply.
\[
r_y \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 2 \\
2 & 4 & 4
\end{bmatrix}
= \begin{bmatrix}
-1 & -1 & -2 \\
2 & 4 & 4
\end{bmatrix}
\]
The image \( \triangle A'B'C' \) has \( A' = (-1, 2), B' = (-1, 4), \) and \( C' = (-2, 4) \).

**Check** The preimage and image are graphed here.

Two other important reflecting lines are the \( x \)-axis and the line with equation \( y = x \). Reflection over the \( x \)-axis is denoted by \( r_x \); and reflection over the line \( y = x \), by \( r_{y=x} \). You can verify the following results:
\[
\begin{align*}
r_x: (x, y) & \rightarrow (x, -y) \\
r_{y=x}: (x, y) & \rightarrow (y, x)
\end{align*}
\]
The graphs on the top of page 206 show the effects of \( r_x \) and \( r_{y=x} \) on \( \triangle ABC \) of Example 1.
The matrices for \( r_x \) and \( r_{y=x} \) also involve only 0s, 1s, and -1s.

**Theorem:**
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
is the matrix for \( r_x \).

**Proof:**
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
1 \cdot x + 0 \cdot y \\
0 \cdot x + -1 \cdot y
\end{bmatrix}
= \begin{bmatrix}
x \\
-y
\end{bmatrix}
\]

**Theorem:**
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
is the matrix for \( r_{y=x} \).

**Proof:**
You are asked to do this proof in Question 13.

**Example 2**

Find the reflection image of pentagon \( WEIRD \) over the line \( y = x \) if \( W = (-1, -1), E = (3, -2), I = (6, 0), R = (6, 5), \) and \( D = (-1, 7) \).

**Solution**
Represent \( r_{y=x} \) and \( WEIRD \) by matrices and multiply.
\[
\begin{align*}
s_{y=x} & = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \\
WEIRD & = \begin{bmatrix}
-1 & 3 & 6 & 6 & -1 \\
-1 & -2 & 0 & 5 & 7
\end{bmatrix}
\end{align*}
\]
\[
s_{y=x} \cdot WEIRD = \begin{bmatrix}
-1 & -2 & 0 & 5 & 7
\end{bmatrix}
\]
\( W'E'I'R'D' \) is represented by the product matrix.
Check *WEIRD* and *W’E’I’R’D’* are graphed below.

It is important to note one significant way that reflections differ from size and scale changes. Reflection images are *congruent* to their preimages. For instance, in Example 1 \( \triangle ABC \cong \triangle A'B'C' \). Under size changes, preimages and images are similar but not necessarily congruent.

At this point you have learned matrices for some size changes, some scale changes, and three reflections. You may wonder: How do I remember them? Question 11 will help you.

### Questions

#### Covering the Reading

1. Suppose that \( A \) is not on line \( m \) and that \( A' \) is the reflection image of \( A \) over \( m \). Then \( m \) is the ___ of \( AA' \).

2. **a.** What is the reflection image of a point \( A \) over a line \( m \) if \( A \) is on \( m \)?  
   **b.** Which vertex on pentagon *WEIRD* shows this?

3. Refer to Example 1. Use matrices to find the image of \( \triangle ABC \) under \( r_{y=x} \). Graph \( \triangle ABC \) and this image.

4. How can the following sentence be read?  
   \[ r_x (x, y) = (x, -y) \]

   **Multiple choice** In 5–7, choose the matrix that corresponds to the given reflection.

   (a) \[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \] (b) \[ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \] (c) \[ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

   (d) \[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] (e) \[ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \]

5. \( r_x \)  
6. \( r_y \)  
7. \( r_{y=x} \)

8. Refer to Example 2. Find the matrix for the reflection image of *WEIRD* over the \( y \)-axis. Graph *WEIRD* and its image *W"E"I"R"D"*. 

---

**LESSON 4-5 Reflections** 207
9. True or false Reflection images are congruent to their preimages.

10. Translate the matrix equation below by filling in the blanks.

\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
3
\end{bmatrix}
= \begin{bmatrix}
-2 \\
3
\end{bmatrix}
\]

The reflection image of the point \(\ldots\) over the line \(\ldots\) is the point \(\ldots\).

11. Let \(F = (1, 0)\) and \(S = (0, 1)\).
   a. Find their images under the transformation with matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

b. The image of \((1, 0)\) is the first \(\ldots\) of this matrix.

c. The image of \((0, 1)\) is the second \(\ldots\) of this matrix.

d. Explain how this idea leads to a way of remembering the \(2 \times 2\) matrix for any transformation that has one.

12. Write the matrices for \(r_x\), \(r_y\), and \(r_{y=x}\). Multiply each matrix by itself and explain your results.

13. Prove that \(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) is the matrix for reflection over the line \(y = x\).

14. Let \(P = (x, y)\), \(Q = (y, x)\), and let \(R\) be any point on the line \(y = x\). Then \(R = (a, a)\).
   a. Verify that \(PR = QR\).
   b. From part a, what theorem from geometry allows you to conclude that the line \(y = x\) is the perpendicular bisector of \(PQ\)?

15. Recall from geometry that the area of a parallelogram varies jointly as the length of the base and the height. What happens to the area of the parallelogram when both the length and height are tripled? Prove your conjecture. (Lesson 2-10)

16. Find the change in the area of the polygon represented by

\[
\begin{bmatrix}
3 & 2 & 5 & -3 \\
4 & -1 & 0 & 1
\end{bmatrix}
\]

when it is multiplied on the left by

\[
\begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix}
\]

(Lesson 4-3, previous course)
17. Refer to the diagram below. Find $y$ given $\ell \parallel m$. *(Previous courses)*

\[ (2x + 10)^\circ \]
\[ (3x - 40)^\circ \]

18. Find an equation for the line parallel to $y = 3x - 4$ and containing the point $(2, -5)$. *(Lesson 3-2)*

19. Trace the figure below. Point $H$ on pentagon HOUSE has been rotated $90^\circ$ about point $C$ to the position of $H'$. Rotate the other vertices of HOUSE $90^\circ$ about $C$ and draw $H'O'U'S'E$. *(Previous course)*

20. Let $r_{y=-x}$ denote reflection over the line $y = -x$.
   a. By graphing and testing points, complete this sentence.
      \[ r_{y=-x}(x, y) = \_\_\_\_. \]
   b. Find the $2 \times 2$ matrix associated with $r_{y=-x}$.
A **transformation** is a one-to-one correspondence between sets of points. Transformations are described by rules. These rules may be algebraic (by a formula), geometric (by giving the location of image points), or arithmetic (by a matrix). In the past three lessons, you have encountered the $2 \times 2$ matrices for transformations known as size changes, scale changes, and reflections. Here we summarize some of the properties of multiplication of $2 \times 2$ matrices, and discuss how they are related to transformations.

It is reasonable to compare multiplication of $2 \times 2$ matrices with multiplication of real numbers.

1. **(closure)** The set of $2 \times 2$ matrices is closed under multiplication.
   Closure means: If you multiply two $2 \times 2$ matrices, the result is a $2 \times 2$ matrix. This property follows from the definition of multiplication of matrices.

2. **(noncommutativity)** In general, multiplication of $2 \times 2$ matrices is not commutative.
   As you will learn in the questions, multiplication with some $2 \times 2$ matrices is commutative. But in general, you cannot assume $AB = BA$.

3. **(associativity)** Multiplication of $2 \times 2$ matrices is associative.
Proof:

Remember that, for real numbers, associativity of multiplication means

\[(ab)c = a(bc)\].

For matrices, therefore, it must be shown that

\[(AB)C = A(BC)\].

The calculation must work for all \(2 \times 2\) matrices, so let

\[A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad C = \begin{bmatrix} i & j \\ k & l \end{bmatrix}.\]

Now it is only a matter of some manipulation, which is left for you in Question 5.

4. (identity) The matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) is the identity for multiplication of \(2 \times 2\) matrices.

Proof:

For the real numbers, 1 is the identity for multiplication: For all \(a\), \(1 \cdot a = a \cdot 1 = a\). Therefore, we need to find a \(2 \times 2\) matrix that we will call \(I\), such that for all \(2 \times 2\) matrices \(A\),

\[I \cdot A = A \cdot I = A.\]

The matrix for a size change of magnitude 1, \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), serves this purpose.

\[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\]

(Both multiplications are needed because multiplication is not always commutative.)

The proof of the identity property illustrates the close relationship between transformations and matrices. When the matrix \( \begin{bmatrix} x \\ y \end{bmatrix} \) is multiplied on the left by \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), each point \((x, y)\) coincides
with its image. Thus \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
1 \cdot x + 0 \cdot y \\
0 \cdot x + 1 \cdot y
\end{bmatrix} =
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Properties of matrices correspond to properties of transformations. The example shows how the product of two matrices is related to the corresponding transformations.

**Example**  
The flag F has key points at \( A = (1, 2) \), \( B = (1, 6) \), and \( C = (3, 6) \). First, reflect this flag over the x-axis. Then reflect the image about the line \( y = x \).

**Solution 1**  \( A' \), \( B' \), and \( C' \) are key points on the first image over the x-axis. \( A'', B'', \) and \( C'' \) are on the final image found by reflecting \( A', B', \) and \( C' \) over the line \( y = x \).

**Solution 2**  Represent \( A \), \( B \), and \( C \) by the matrix \[
\begin{bmatrix}
1 & 1 & 3 \\
2 & 6 & 6
\end{bmatrix}
\]
To find the first image, multiply this matrix on the left by the matrix for \( r_x \).

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 3 \\
2 & 6 & 6
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 3 \\
-2 & -6 & -6
\end{bmatrix}
\]
To find the second image, multiply the matrix for \( A' \), \( B' \), and \( C' \) by the matrix for \( r_{y=x} \).

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 3 \\
-2 & -6 & -6
\end{bmatrix} =
\begin{bmatrix}
-2 & -6 & -6 \\
1 & 1 & 3
\end{bmatrix}
\]
The points \( A'' = (-2, 1) \), \( B'' = (-6, 1) \), and \( C'' = (-6, 3) \) enable the final flag to be drawn.

**Check**  The coordinates for the flag determined by using matrix methods are the same as the coordinates in the graph of the final image.
We call the final flag the image of the original flag under the **composite** of the reflections \( r_x \) and \( r_{y=x} \).

**Definition:**

Suppose transformation \( T_1 \) maps figure \( F \) onto figure \( F' \), and transformation \( T_2 \) maps figure \( F' \) onto figure \( F'' \). The transformation that maps \( F \) onto \( F'' \) is called the composite of \( T_1 \) and \( T_2 \), written \( T_2 \circ T_1 \).

The symbol \( \circ \) means "following." Thus in the example, the composite is

\[ r_{y=x} \circ r_x. \]

To describe the composite, ignore the first image and look only at the preimage and the final image.

How are the two flags related?

![Graph showing the composite transformation](image)

The graph shows that the composite is neither a reflection nor a size or scale change. Recall from geometry that the preimage and image are related by a different type of transformation called a **rotation**. This rotation has center \((0, 0)\). If the preimage flag is turned \(90^\circ\) about \((0, 0)\) in a counterclockwise direction, the image flag results. We denote this rotation by \( R_{90} \). This rotation is the composite of the two reflections.

\[ R_{90} = r_{y=x} \circ r_x \]

How do we find the matrix associated with \( R_{90} \)? Notice that the composition of transformations of the example led to the following matrix multiplication.

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  1 & 0 \\
  0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
  1 & 1 & 3 \\
  2 & 6 & 6 \\
\end{bmatrix}
\]
Because matrix multiplication is associative, this product could be computed as follows:

\[
\begin{align*}
\Delta ABC &= \left( \begin{bmatrix} r_{y=x} & r_x \\ 0 & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 6 \end{bmatrix} \right) \\
\Delta ABC &= \left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right] \left[ \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 6 \end{bmatrix} \right] \\
\Delta ABC &= \left[ \begin{bmatrix} -2 & -6 \\ -6 & 1 \\ 1 & 3 \end{bmatrix} \right]
\end{align*}
\]

Multiplying the two reflection matrices gives the single matrix
\[
\left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right].
\]

Applying this matrix to \( A, B, \) and \( C \) results in the final images \( A'', B'', \) and \( C''. \) Thus the single matrix \( \left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right] \)
can be used to do the composition \( r_{y=x} \circ r_x. \) The general idea is summarized in the following theorem.

**Theorem:**

If transformation \( T_1 \) has matrix \( M_1 \) and transformation \( T_2 \) has matrix \( M_2, \) then \( T_2 \circ T_1 \) has matrix \( M_2M_1. \)

### Questions

1. When a \( 2 \times 2 \) matrix is multiplied by a \( 2 \times 2 \) matrix, what are the dimensions of the product matrix?

2. Give an example to show that multiplication of \( 2 \times 2 \) matrices is sometimes commutative.

3. Find two \( 2 \times 2 \) matrices \( A \) and \( B \) such that \( AB \neq BA. \)

4. Let \( X = \left[ \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \right], \ Y = \left[ \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} \right], \) and \( Z = \left[ \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} \right]. \)
   
   a. Show that \( (XY)Z = X(YZ). \)
   
   b. The answer to part a is an instance of what property?

5. Finish the proof (on page 211) that multiplication of \( 2 \times 2 \) matrices is associative, by calculating the following:
   
   a. \( (AB)C \)
   
   b. \( A(BC). \)
6. Multiply \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] by \[
\begin{bmatrix}
\pi & \sqrt{2} \\
-3 & 3/4
\end{bmatrix}
\].

7. The identity transformation maps each point onto \_\_\_.

8. The symbol \( \circ \) means \_\_\_.

9. In the rotation \( r_y = x \circ r_x \), which reflection is done first, \( r_y = x \) or \( r_x \)?

10. \( R_{90} \) represents a rotation of \_\_\_ degrees around \_\_\_ in a(n) \_\_\_ direction.

11. What property of matrix multiplication justifies that
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & 6
\end{bmatrix}
\]

12. If \( T_1 \) has matrix \[
\begin{bmatrix}
-2 & 0 \\
0 & 2
\end{bmatrix}
\] and \( T_2 \) has matrix \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\], what is a matrix for \( T_1 \circ T_2 \)?

13. a. Find the matrix for \( r_x \circ r_y = x \).
    b. To what single transformation is \( r_x \circ r_y = x \) equivalent?
    c. How does your answer to part b compare with \( r_y = x \circ r_x \)?

14. Graph the image of the flag of this lesson under the transformation \( S_2 \circ r_y \).

15. a. Prove that if \( C \) is any \( 2 \times 2 \) matrix, then \( S_k \cdot C = C \cdot S_k \).
    b. What does part a imply is true about size-change transformations?
16. \( \triangle BAT \) can be represented by \[
\begin{bmatrix}
0 & 3 & -1 \\
5 & -2 & -1
\end{bmatrix}.
\]
   a. Find a matrix to represent the image \( \triangle B'A'T' \) under \( r_x \).
   b. True or false \( \triangle BAT \equiv \triangle B'A'T' \). (Lesson 4-5)

17. a. What kind of triangle is represented by \[
\begin{bmatrix}
-7 & 7 & 0 \\
0 & 0 & 7
\end{bmatrix}?
\]
   b. What matrix describes the image of the triangle in part a under \[
\begin{bmatrix}
4 & 0 \\
0 & 1
\end{bmatrix}?
\]
   c. What special kind of triangle is the image?
   d. Find the areas of the two triangles. (Previous course: Lessons 4-1, 4-4)

18. Each month the population of Boomtown increases by 50 people. In contrast, the population of Bustville has been decreasing by 80 people/month. Suppose Boomtown now has 25,620 people and Bustville 31,250, and these growth rates continue.
   a. In how many months will Boomtown have more people?
   b. What will the population be then? (Lesson 1-7)

19. Multiple choice Which expression equals \(-(x_1 - x_2)\)? (Previous course)
   a. \(x_2 - x_1\)  \hspace{1cm} b. \(x_1 - x_2\)  \hspace{1cm} c. \(x_1 + x_2\)
   d. \(-x_1\)  \hspace{1cm} e. none of these

20. Explore whether multiplication of \(3 \times 3\) matrices has properties identical or similar to the properties for multiplication of \(2 \times 2\) matrices given in this lesson.
Rotations are closely related to angles. The arcs used to denote angles suggest turns. Angles with larger measure require more turn.

Rotations often occur one after the other, as when going from one frame to another in animated cartoons or in computer generated images.

In these frames, the monkey undergoes a series of 45° counterclockwise rotations. Notice that monkeys two frames apart are turned 90°. This is a result of a fundamental property of rotations which itself is derived from the Angle Addition Postulate in geometry.

**Theorem:**

A rotation of $x^\circ$ following one of $y^\circ$ results in a rotation of $(x + y)^\circ$. In symbols: $R_x \circ R_y = R_{x+y}$.

Notice that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $a$, $b$, $c$, and $d$.

So any transformations with a $2 \times 2$ matrix must map $(0, 0)$ onto itself. Thus the only rotations that can have $2 \times 2$ matrices are those with center $(0, 0)$. 
In Lesson 4-6, you learned the following theorem.

**Theorem:**

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\] is the matrix for \( R_{90} \).

By composing two 90° rotations, a matrix for \( R_{180} \) can be found.

---

**Example**  
Find the matrix for \( R_{180} \).

**Solution**  
A rotation of 180° can be considered as a 90° rotation followed by another 90° rotation. That is,

\[ R_{90} \circ R_{90} = R_{180} \]

In matrix form,

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} = 
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}.
\]

The matrix for \( R_{180} \) is

\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}.
\]

**Check**  
Apply this matrix to a figure. We use \( A, B, \) and \( C \) from the last lesson.

\[
R_{180} \quad \Delta ABC \quad \Delta A'B'C' 
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 3 \\
2 & 6 & 6
\end{bmatrix} = 
\begin{bmatrix}
-1 & -1 & -3 \\
-2 & -6 & -6
\end{bmatrix}.
\]

The graph verifies that each point of \( \Delta ABC \) has been rotated 180° to the corresponding image point of \( \Delta A'B'C' \).

In Question 8 of this lesson you will be asked to show, in a similar way, that the matrix for \( R_{270} \) is

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]
Rotations in the clockwise direction have negative magnitudes. So $R_{-90}$ represents a 90° turn clockwise. Because a rotation of $-90°$ has the same image as a rotation of 270°, $R_{-90}$ equals $R_{270}$.

Here is a summary of the rotations of this lesson:

Rotation of 90° or $-270°$  
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

Rotation of 180° or $-180°$  
\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

Rotation of 270° or $-90°$  
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

These matrices make it possible to get algebraic formulas for rotation images. For instance, for $R_{90}$,
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \cdot x + -1 \cdot y \\
1 \cdot x + 0 \cdot y
\end{bmatrix}
= 
\begin{bmatrix}
-y \\
x
\end{bmatrix}.
\]

Thus $R_{90}(x, y) = (-y, x)$.

You have learned matrices for many transformations in this chapter. To remember the $2 \times 2$ matrix for a particular transformation $T$, use this rule: The first column is the image of $(1, 0)$ under $T$. The second column is the image of $(0, 1)$. So, for example, to remember the matrix for $R_{90}$, use the picture below and record the images as shown in the matrix.

Questions

1. The composite of a rotation of 45° and a rotation of 90° is a rotation of __?__.

2. In general, $R_x \circ R_y = __?__.$

3. How much of a turn did the monkey pictured in the lesson undergo from the frame $A$ to the frame $E$?
4. A rotation of negative magnitude is in what direction?

In 5–7, identify the matrix for the given rotation.

\[
\begin{array}{ccc}
\text{a.} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{b.} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \text{c.} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
\text{d.} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{e.} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\end{array}
\]

5. \( R_{90} \)  
6. \( R_{-90} \)  
7. \( R_{180} \)

8. Consider a rotation of 270° as being a rotation of 90° followed by a rotation of 180°. Using the method of the example, show that

\[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

is the matrix for \( R_{270} \).

In 9–12, find each image.

9. \( R_{90}(3, 5) \)  
10. \( R_{180}(3, 5) \)  
11. \( R_{270}(3, 5) \)  
12. \( R_{90}(-2, -1) \)

13. What is the matrix for \( R_0 \)?

14. Quadrilateral \( MATH \) has coordinates \( M = (0, 0), A = (5, 0), T = (5, 7), \) and \( H = (-1, 3) \). Graph \( MATH \) and its image under \( R_{180} \).

15. a. Calculate a matrix for \( R_{180} \circ r_y \).
   b. To what transformation does the matrix in part a correspond?

16. The point \( (3, 4) \) lies on the circle with center \( (0, 0) \) and radius 5. 
   a. Rotate this point 90°, 180°, and 270° around \( (0, 0) \) to find the coordinates of 3 other points on this circle.
   b. Graph all 4 points.

17. a. Graph triangle \( ABC \) with vertices \( A = (-5, 0), B = (-1, 2), \) and \( C = (-1, 4) \).
   b. Graph the image of \( \triangle ABC \) under \( r_y \circ r_x \).
   c. The triangle in part b is the image of \( \triangle ABC \) under what rotation?

In 18–21, write the matrix for each transformation. (Lessons 4-3, 4-4, 4-5, 4-6)

18. the size change of magnitude 3, center \( (0, 0) \)

19. the identity transformation

20. \( r_y = x \)

21. \((x, y) \rightarrow (x, 3y)\)
22. a. Let \( T = (-2, 3), R = (5, 3), A = (5, 0), \) and \( P = (3, 0). \) Graph polygon TRAP.

b. Graph the image of TRAP under the transformation with matrix

\[
\begin{bmatrix}
2 & 0 \\
0 & -2
\end{bmatrix}
\]

c. Describe the transformation. (Lessons 4-1, 4-4, 4-6)

23. If \( y \) varies inversely as the square of \( x \) and \( y = 10 \) when \( x = 5, \) what is \( y \) when \( x = 6? \) (Lesson 2-2)

24. Calculate the coordinates of the midpoint of the segment joining (-4, 6) and (2, 3). (Previous course)

25. What is the definition of “perpendicular lines”? (Previous course)

26. Show that the matrix \[
\begin{bmatrix}
5 & -3 & 0 \\
3 & 5 & 0
\end{bmatrix}
\]
represents a right triangle by using the Pythagorean theorem. (Lesson 4-1, Previous course)

27. Use the figure below. Describe the transformation that maps ABCDE onto A'B'C'D'E'.

a. in words.
b. by a matrix.
c. with an algebraic formula.
(Lesson 4-5)

![Diagram of ABCDE and its transformation to A'B'C'D'E']

28. The matrix for \( R_{30} \) is \[
\begin{bmatrix}
\frac{\sqrt{3}}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix}
\]
Use this information to determine matrices for some other rotations.

29. The matrix \[
\begin{bmatrix}
0.6 & -0.8 \\
0.8 & 0.6
\end{bmatrix}
\]
is a matrix for a rotation \( R_x. \) By carefully plotting points and their images, estimate \( x, \) the magnitude of the rotation.
In this lesson, ideas about rotations help to deduce an important theorem about the slopes of perpendicular lines. Recall from geometry that two lines are perpendicular if and only if they form a 90° angle. In the last section you learned that \( R_{90}(x, y) = (-y, x) \). We can use \( R_{90} \) to rotate a line 90° by taking any two points \( P \) and \( Q \) on the line and rotating them 90° about the origin. The line drawn through the two image points \( P' \) and \( Q' \) will then be perpendicular to the original line.
Example 1 Given \( A = (4, 1) \) and \( B = (-5, -3) \):

a. Rotate \( \overrightarrow{AB} \) 90°.

b. Graph \( \overrightarrow{AB} \) and its image \( \overrightarrow{A'B'} \).

c. Find the slopes of \( \overrightarrow{AB} \) and \( \overrightarrow{A'B'} \).

d. What relationship exists between the two slopes?

Solution

a. Use \( R_{90}(x, y) = (-y, x) \).

\[ A' = R_{90}(A) = R_{90}(4, 1) = (-1, 4) \]

\[ B' = R_{90}(B) = R_{90}(-5, -3) = (3, -5) \]

Thus two points on the image line are \( A' = (-1, 4) \) and \( B' = (3, -5) \).

b. The preimage \( \overrightarrow{AB} \) and image \( \overrightarrow{A'B'} \) are graphed below.

\[ \text{c. The slope of } \overrightarrow{AB} : \frac{3 - 1}{-5 - 4} = \frac{4}{9}; \text{ the slope of } \overrightarrow{A'B'}: \frac{5 - 4}{3 - (-1)} = \frac{1}{4}. \]

d. The slopes are negative reciprocals of each other. Another way to say that is to say that the product of the slopes is -1.

Example 1 is an instance of the following theorem.

**Theorem:**

If two lines with slopes \( m_1 \) and \( m_2 \) are perpendicular, then \( m_1 \cdot m_2 = -1 \).

The following argument proves that if the two lines are perpendicular, then the product of their slopes is -1. We are given lines with slopes \( m_1 \) and \( m_2 \). We must find values for \( m_1 \) and \( m_2 \) and multiply those values. The given lines are parallel to lines with the same slopes that intersect at the origin. We prove the theorem for two lines through the origin; this proves the property for perpendicular lines elsewhere.
Proof

Let \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) be two points on a line \( P\bar{Q} \) that contains the origin.

Since \( R_{90}(x, y) = (-y, x) \) for all \((x, y)\),
\[ R_{90}(x_1, y_1) = (-y_1, x_1) \]
and \( R_{90}(x_2, y_2) = (-y_2, x_2) \).

The image line contains \( P' = (-y_1, x_1) \) and \( Q' = (-y_2, x_2) \).

Let the slopes of the lines be \( m_1 \) and \( m_2 \):

\[
m_1 = \text{slope of } P\bar{Q} = \frac{y_2 - y_1}{x_2 - x_1}
\]
\[
m_2 = \text{slope of } P'\bar{Q}' = \frac{x_2 - x_1}{-y_2 - (-y_1)} = \frac{x_2 - x_1}{-y_2 + y_1} = \frac{x_2 - x_1}{y_2 - y_1}
\]

The product of the slopes is

\[
m_1 \cdot m_2 = \frac{y_2 - y_1}{x_2 - x_1} \cdot \left( \frac{x_2 - x_1}{y_2 - y_1} \right) = -1.
\]

Example 2

Line \( n \) goes through \((-4, 1)\) and is perpendicular to line \( l \) whose equation is \( y = -\frac{3}{2}x + 2 \). Find an equation for line \( n \).

Solution  
The situation is graphed at the right. Line \( l \) is in slope-intercept form; its slope is \(-\frac{3}{2}\). Any line perpendicular to it has slope \( \frac{2}{3} \), since \(-\frac{3}{2} \cdot \frac{2}{3} = -1 \). An equation for line \( n \) having slope \( \frac{2}{3} \) and going through \((-4, 1)\) is

\[
y - 1 = \frac{2}{3}(x + 4).
\]
Recall from geometry that all points on the perpendicular bisector of a segment \( AB \) are equidistant from the endpoints \( A \) and \( B \). This idea is used in the next example.

**Example 3** Find an equation for the set of points equidistant from \( Y = (-2, -5) \) and \( C = (6, 5) \).

**Solution** The set of points equidistant from \( Y \) and \( C \) is the perpendicular bisector of \( YC \). To find the equation, first find the coordinate of \( M \), the midpoint of \( YC \).

\[
M = \left( \frac{-2 + 6}{2}, \frac{-5 + 5}{2} \right) = (2, 0)
\]

Then find the slope \( m \) of \( YC \).

\[
m = \frac{5 - (-5)}{6 - (-2)} = \frac{10}{8} = \frac{5}{4}
\]

The slope of any perpendicular to \( YC \) is \( -\frac{4}{5} \). Now use the point-slope formula to write an equation for the line through \((2, 0)\) with slope \(-\frac{4}{5}\).

\[
y - 0 = -\frac{4}{5} (x - 2)
\]

**Check** Let \( x = -3 \). Then \( y = 4 \). Does the line seem to go through \((-3, 4)\)? Yes.

Suppose line \( l_1 \) has slope \( m_1 \), line \( l_2 \) has slope \( m_2 \), and \( m_1m_2 = -1 \). Are \( l_1 \) and \( l_2 \) perpendicular? The answer is yes, by the following argument. Any line \( l_3 \) perpendicular to \( l_1 \) has slope \( m_3 \), where \( m_1m_3 = -1 \). Thus \( m_1m_3 = m_1m_2 \), which means that \( m_3 = m_2 \). Therefore \( l_3 \) and \( l_2 \) have the same slope; so \( l_3 \parallel l_2 \). But we know that \( l_1 \perp l_3 \). We also know that if a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. Thus, \( l_1 \perp l_2 \).

We have proved the converse of the previous theorem:

**Theorem:**

If two lines have slopes \( m_1 \) and \( m_2 \) and \( m_1m_2 = -1 \), then the lines are perpendicular.
Questions

Covering the Reading

1. Under $R_{90}$, the image of $(x, y)$ is $^2_0$.

2. Let $\overline{AB}$ contain points $A = (3, 5)$ and $B = (-1, -6)$.
   a. Find two points on the image of $\overline{AB}$ under $R_{90}$.
   b. Graph $\overline{AB}$ and its image $\overline{A'B'}$.
   c. Find the slopes of $\overline{AB}$ and of $\overline{A'B'}$.
   d. The product of the slopes is $^2_0$.

3. Two lines with nonzero slopes $m_1$ and $m_2$ are perpendicular if and only if $\overline{m_1m_2}$.

4. Refer to the proof of the first theorem in this lesson.
   a. What rotation maps points on $\overline{PQ}$ onto corresponding points on $\overline{PQ'}$?
   b. The slope of $\overline{PQ}$ is $^2_0$.
   c. The slope of $\overline{PQ'}$ is $^2_0$.

5. Given the line with equation $2x + 6y = 1$ and the point $P = (7, -2)$:
   a. Find the slope of a line perpendicular to the given line.
   b. Find an equation for the line through $P$ and perpendicular to the given line.

6. Find an equation of the line through $(6, 1)$ and perpendicular to the line $y = \frac{4}{3}x - 2$.

7. Suppose $Y$ is $(1, -4)$ and $C$ is $(-3, 10)$. What is an equation for the perpendicular bisector of $YC$?

8. $\overline{CD}$ has endpoints $C = (9, 5)$ and $D = (-7, 11)$. Find an equation for the perpendicular bisector of $\overline{CD}$.

9. Multiple choice A line perpendicular to the line with equation $x = 7$ has
   (a) slope 0.   (b) undefined slope.   (c) slope $\frac{1}{7}$.   (d) slope 7.

10. Why do the statements of the theorems in this lesson apply only to lines with nonzero slopes?

11. Find an equation for the line through $(6, 2)$ and perpendicular to $y = 4$.

12. Given the line containing points $A = (7, 3)$ and $B = (-4, 1)$:
   a. Find the coordinates of $A'$ and $B'$ under $R_{270}$.
   b. Graph both the preimage and the image.
   c. Find the slopes of $\overline{AB}$ and $\overline{A'B'}$.
   d. What relationship exists between the slopes? What does this tell you about the lines?
   e. A counterclockwise rotation of $270^\circ$ is the same as a clockwise rotation of $^0_{180}$.
13. Fill each blank with $\parallel$ or $\perp$. Assume all lines lie in the same plane.
   a. If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.
   b. If $l \parallel m$ and $m \perp n$, then $l \perp n$.
   c. If $l \perp m$ and $m \parallel n$, then $l \perp n$.
   d. If $l \perp m$ and $m \perp n$, then $l \parallel n$.

14. a. Find the matrix for the image of the triangle defined by
   \[
   \begin{bmatrix}
   1 & 1 & 3 \\
   2 & 6 & 7 
   \end{bmatrix}
   \] under $R_{270}$. \\
   b. Graph the preimage and image. (Lesson 4-7)

15. a. Calculate a matrix for $r_y \circ R_{180}$.
   b. To what transformation does this matrix correspond? (Lesson 4-7)

16. Matrix $D$ gives the daily delivery of fish to two markets. Matrix $C$ gives the unit cost for each item in the market.

   \[
   \begin{bmatrix}
   \text{cod} & \text{perch} & \text{grouper} \\
   \end{bmatrix}
   \begin{bmatrix}
   12 & 6 & 20 \\
   8 & 5 & 32 
   \end{bmatrix} = D
   \]

   \[
   \begin{bmatrix}
   \text{unit cost} \\
   \end{bmatrix}
   \begin{bmatrix}
   2.89 \\
   2.59 \\
   1.98 
   \end{bmatrix} = C
   \]

   a. Find $DC$.
   b. What is the daily cost of fish at Carlita's? (Lesson 4-2)

17. Find the first five terms of the sequence $\left\{ S_n \right\}$
   \[
   \begin{align*}
   S_1 &= 1 \\
   S_n &= S_{n-1} + n^2 \text{ for } n > 1.
   \end{align*}
   \]

18. a. Solve for $x$: $u + vx = w + xy$.
   b. When does the equation in part a have no solution? (Lesson 1-8)

19. Wee Willie Winkle determined that he gets about $\frac{1}{4}$ of his daily calories from breakfast, $\frac{1}{3}$ from lunch, $\frac{1}{4}$ from dinner, and 500 calories from snacks. About how many calories does he consume daily? (Lesson 1-8)

20. Begin with the line $y = 2x + 7$. Choose five different transformations. Find an equation for the image of this line under each transformation you have chosen.
There are many situations which require adding the information stored in matrices. For instance, suppose matrix $C$ represents the current inventory of Elizabeth’s Boutique Department.

The boutique receives a delivery of new items represented by matrix $D$.

The new inventory is found by taking the sum of matrices $C$ and $D$. This **matrix addition** is performed according to the following rule.

**Definition:**

If two matrices $A$ and $B$ have the same dimensions, their sum $A + B$ is the matrix in which each element is the sum of the corresponding elements in $A$ and $B$. 
For the matrices above, the sum $C + D$ is a $4 \times 5$ matrix. Add corresponding elements of $C$ and $D$ to find the elements of $C + D$. We have circled one set of corresponding elements.

$$C + D = \begin{bmatrix} 8 & 9 & 12 & 13 & 10 \\ 4 & 6 & 9 & 6 & 4 \\ 20 & 26 & 22 & 26 & 12 \\ 16 & 21 & 19 & 28 & 17 \end{bmatrix}$$

Because addition of real numbers is commutative, *addition of matrices is commutative*. Thus for any two matrices $A$ and $B$ with the same dimensions, $A + B = B + A$. Also, for all matrices $A$, $B$, and $C$, $(A + B) + C = A + (B + C)$; *addition of matrices is associative*.

Subtraction of matrices is defined in a similar manner: Given two matrices $A$ and $B$, their difference $A - B$ is the matrix whose element in each position is the difference of the corresponding elements in $A$ and $B$.

**Example 1** The matrix $W1$ below represents the costs of 1 dozen each of eggs and oranges in three different markets during one week. The matrix $W2$ represents the cost of these same items in the same stores during another week.

<table>
<thead>
<tr>
<th>market</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.97</td>
<td>.90</td>
<td>.95</td>
<td>eggs</td>
</tr>
<tr>
<td></td>
<td>1.99</td>
<td>1.79</td>
<td>1.59</td>
<td>oranges</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>market</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.97</td>
<td>.85</td>
<td>1.05</td>
<td>eggs</td>
</tr>
<tr>
<td></td>
<td>1.49</td>
<td>1.79</td>
<td>1.89</td>
<td>oranges</td>
</tr>
</tbody>
</table>

**a.** Find $W2 - W1$.

**b.** Which of the markets had the greatest change in the price of oranges from Week 1 to Week 2?

**Solution**

**a.** $W2 - W1 = \begin{bmatrix} 0 & .05 & .10 \\ -.50 & 0 & .30 \end{bmatrix}$

**b.** The changes in prices of oranges are given in row 2 of the matrix $W2 - W1$. The greatest change in prices occurred with oranges in market 1. The price of oranges decreased $.50 per dozen in this period.
Matrix addition is related to a special type of matrix multiplication called *scalar multiplication*. Consider

\[
\begin{bmatrix}
7 & 8 \\
4 & 2 \\
\end{bmatrix} + \begin{bmatrix}
7 & 8 \\
4 & 2 \\
\end{bmatrix} + \begin{bmatrix}
7 & 8 \\
4 & 2 \\
\end{bmatrix} = \begin{bmatrix}
21 & 24 \\
12 & 6 \\
\end{bmatrix}.
\]

Notice that in the final result, every element of the original matrix has been multiplied by 3. We rewrite this as 3\[\begin{bmatrix}
7 & 8 \\
4 & 2 \\
\end{bmatrix}\]. The constant 3 is called a *scalar*. *Scalar multiplication* is defined as follows.

**Definition:**

The product of a scalar \(k\) and a matrix \(A\) is the matrix \(kA\) in which each element is \(k\) times the corresponding element in \(A\).

**Example 2** Find the product 5\[\begin{bmatrix}
7 & 2 & -1 \\
4 & 9 & 11 \\
\end{bmatrix}\].

**Solution** Every element in the matrix must be multiplied by 5.

\[
5\begin{bmatrix}
7 & 2 & -1 \\
4 & 9 & 11 \\
\end{bmatrix} = \begin{bmatrix}
5 \cdot 7 & 5 \cdot 2 & 5 \cdot (-1) \\
5 \cdot 4 & 5 \cdot 9 & 5 \cdot 11 \\
\end{bmatrix} = \begin{bmatrix}
35 & 10 & -5 \\
20 & 45 & 55 \\
\end{bmatrix}
\]

**Questions**

1. What must be true about the dimensions of two matrices in order for addition or subtraction to be possible?

In 2 and 3, refer to the clothing matrices \(C\) and \(D\) at the beginning of this lesson.

2. Does \(C + D = D + C\)?

3. Suppose the shop gets another delivery described by matrix \(P\) below. Find the new inventory \(P + C + D\).

\[
P = \begin{bmatrix}
8 & 10 & 12 & 14 & 16 \\
5 & 2 & 1 & 0 & 3 \\
4 & 1 & 1 & 1 & 2 \\
3 & 6 & 4 & 10 & 5 \\
4 & 2 & 5 & 11 & 12 \\
\end{bmatrix}
\]

sizes

- dresses
- suits
- skirts
- blouses
4. Refer to Example 1.
   a. In which market did the price of a dozen eggs change the most from Week 1 to Week 2?
   b. Was that change an increase or decrease?

In 5–7, let \( A = \begin{bmatrix} 3 & 5 \\ 0 & -3 \end{bmatrix} \), \( B = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix} \), and \( C = \begin{bmatrix} 1 & -1 \\ -6 & 3 \end{bmatrix} \). Find:

5. \( 6C \)
6. \( A - B \)
7. \( B - A \)

8. True or false Subtraction of matrices is commutative.

   a. Find \( (A + B) + C \).
   b. Find \( A + (B + C) \).
   c. What property is illustrated by the results of parts a and b?

10. Let \( M = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \), \( N = \begin{bmatrix} -2 & 3 \\ -5 & 0 \end{bmatrix} \), and \( P = \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} \).
   a. Compute \( M(N + P) \).
   b. Compute \( MN + MP \).
   c. Is matrix multiplication distributive over matrix addition in this case?

11. Solve for \( a, b, c, \) and \( d \).

\[
3 \begin{bmatrix} a & -1 \\ c & 4 \end{bmatrix} - 5 \begin{bmatrix} 3 & b \\ 11 & -2.5 \end{bmatrix} = \begin{bmatrix} 9 & 0 \end{bmatrix}
\]

12. The matrices \( N, C, \) and \( S \) give the enrollments by sex and grade at North, Central, and South high schools. In each matrix Row 1 gives the number of boys and Row 2 the number of girls. Columns 1 to 4 give the number of students in grades 9 through 12, respectively. Calculate entries in the matrix \( T \) that shows the total enrollment by sex and grade in the three schools.

### 1982–1983

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>L</th>
<th>T</th>
<th>Pts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston</td>
<td>50</td>
<td>20</td>
<td>10</td>
<td>110</td>
</tr>
<tr>
<td>Montreal</td>
<td>42</td>
<td>24</td>
<td>14</td>
<td>98</td>
</tr>
<tr>
<td>Buffalo</td>
<td>38</td>
<td>29</td>
<td>13</td>
<td>89</td>
</tr>
<tr>
<td>Quebec</td>
<td>34</td>
<td>34</td>
<td>12</td>
<td>80</td>
</tr>
<tr>
<td>Hartford</td>
<td>19</td>
<td>54</td>
<td>7</td>
<td>45</td>
</tr>
</tbody>
</table>

### 1983–1984

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>L</th>
<th>T</th>
<th>Pts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston</td>
<td>49</td>
<td>25</td>
<td>6</td>
<td>104</td>
</tr>
<tr>
<td>Montreal</td>
<td>35</td>
<td>40</td>
<td>5</td>
<td>75</td>
</tr>
<tr>
<td>Buffalo</td>
<td>48</td>
<td>25</td>
<td>7</td>
<td>103</td>
</tr>
<tr>
<td>Quebec</td>
<td>42</td>
<td>28</td>
<td>10</td>
<td>94</td>
</tr>
<tr>
<td>Hartford</td>
<td>28</td>
<td>42</td>
<td>10</td>
<td>66</td>
</tr>
</tbody>
</table>

a. Subtract the top matrix from the bottom matrix. Call the difference \( M \).

b. What is the meaning of the 4th column of \( M \)?

c. What is the meaning of the 1st row of \( M \)?

14. Mr. Toi makes handcrafted toys for children. His output last year is represented by the matrix at the right. He wants to increase his output by 30%. Find the matrix that describes the needed output. Round elements to the nearest whole number.

<table>
<thead>
<tr>
<th></th>
<th>sm</th>
<th>med</th>
<th>lg</th>
</tr>
</thead>
<tbody>
<tr>
<td>dolls</td>
<td>5</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>stuffed animals</td>
<td>12</td>
<td>22</td>
<td>9</td>
</tr>
</tbody>
</table>

15. Let \( l \) be the line \( x - 2y = -4 \) and let \( P = (3, 0) \). 
Find an equation for the line through \( P \):

a. parallel to \( l \).

b. perpendicular to \( l \).

(Lessons 3-5, 4-8)
16. Match each numbered item on the right with the best lettered choice on the left. (*Lessons 4-1, 4-6, 4-7*)

- a. \[
\begin{bmatrix}
2 \\
3
\end{bmatrix}
\] (i) identity

- b. \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] (ii) point

- c. \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\] (iii) \(R_{90}\)

- d. \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\] (iv) \(R_{-90}\)

In 17–20, give the \(2 \times 2\) matrix for the transformation. (*Lessons 4-3, 4-4, 4-5, 4-7*)

17. \(r_x\)  
18. \(S_2\)  
19. \(S_{3,4}\)  
20. \(R_{180}\)

21. Scenic City and Watertown are both along Raging River. A bridge is to be built across the river and be equidistant from the two cities. If Scenic City is at \((-12, 8)\) and Watertown is at \((22, 34)\), on what line should the bridge be built? (*Lesson 4-8*)

22. Approximate the solution to \(\sqrt{3}(5 - .7x) = 19.04\) to the nearest hundredth. (*Lesson 1-7*)

23. Yet another operation with matrices is powering. Consider the matrix

\[
M = \begin{bmatrix}
0.7 & 0.3 \\
0.6 & 0.4
\end{bmatrix}
\]

- a. Calculate \(M \cdot M\). Call it \(M^2\).

- b. Write a computer program to calculate \(M^3, M^4\), and so on, up to \(M^{20}\).

- c. What matrix does \(M^n\) seem to approach as \(n\) gets larger and larger?
In this chapter you have found transformation images by multiplying matrices. There is one transformation for which images can be found by adding matrices. Consider $\triangle ABC$ and $\triangle A'B'C'$ below. $\triangle A'B'C'$ is a slide or **translation** image of the preimage $\triangle ABC$. The matrices $M$ and $M'$ for these triangles are given at the left.

$$M = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 6 & 6 \end{bmatrix}$$

$$M' = \begin{bmatrix} 5 & 5 & 7 \\ -1 & 3 & 3 \end{bmatrix}$$

Now we calculate $M' - M$.

$$M' - M = \begin{bmatrix} 5 & 5 & 7 \\ -1 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 \\ 2 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ -3 & -3 & -3 \end{bmatrix}$$

In $M' - M$, all the elements in the first row are equal and all the elements in the second row are equal. Thus, to get the image $\triangle A'B'C'$, add 4 to every $x$-coordinate and -3 to every $y$-coordinate of the preimage.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 \\ -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 7 \\ -1 & 3 & 3 \end{bmatrix}$$

This leads to an algebraic definition of translation.

**Definition:**

The transformation that maps $(x, y)$ onto $(x + h, y + k)$ is a translation of $h$ units horizontally and $k$ units vertically and is denoted by $T_{h,k}$.

$\triangle A'B'C'$ was obtained from $\triangle ABC$ with the translation $T_{4,-3}$.

There is no single matrix for a translation because the dimensions of that matrix would depend on the figure being translated. Translations are easy to do using the formula $T_{h,k}(x, y) = (x + h, y + k)$. 
Example
A quadrilateral has vertices \( Q = (-4, 2), U = (-2, 6), A = (0, 5), \) and \( D = (0, 3). \)

a. Find its image under the transformation \( T_{3,5}. \)
b. Graph the image and preimage on the same graph.

Solution

a. \( T_{3,5}(x, y) = (x + 3, y + 5), \) so \( T(-4, 2) = (-1, 7). \)
   \( T(-2, 6) = (1, 11) \)
   \( T(0, 5) = (3, 10) \)
   \( T(0, 3) = (3, 8) \)

b. The image appears to be 3 units to the right and 5 units up, as it should.

Questions

In 1 and 2, refer to \( \triangle ABC \) and \( \triangle A'B'C' \) at the beginning of the lesson.

1. What translation maps \( \triangle ABC \) onto \( \triangle A'B'C'? \)
2. What translation maps \( \triangle A'B'C' \) onto \( \triangle ABC? \)
3. A translation is a transformation mapping \((x, y)\) to \(\_\_\_\). 
4. \( T_{h,k} \) is a translation \(\_\_\_\) units horizontally and \(\_\_\_\) units vertically.
5. Refer to the Example. Graph \( QUAD \) and its image \( Q'U'A'D' \) under \( T_{4,7}. \)

In 6–8, find the image of the point under \( T_{-2,6}. \)

6. \((0, 0)\) 
7. \((100, -98)\) 
8. \((a, b)\)

9. A transformation \( T \) has a \(2 \times 2\) matrix. If \( T(1, 0) = (7, 11) \) and \( T(0, 1) = (2, 3) \), what is the matrix for \( T? \)
10. Refer to the graph at the right.
   a. What translation maps $ABCDE$ onto $A'B'C'D'E'$?
   b. Verify that $CD \cong C'D'$. 
   c. Verify that $BC \parallel B'C'$.

11. $[ \begin{array}{rrrrr} 9 & 4 & 3 & 1 & 6 \\ -1 & 10 & 5 & 1 & -8 \end{array} ]$ represents pentagon $FAITH$.
   a. Apply the translation $T_{2,-7}$ to the pentagon.
   b. Graph the preimage and the image on the same set of axes.

12. $\triangle CUB$ is translated under $T_{4,9}$ to get $\triangle C'U'B'$. $\triangle C'U'B'$ is then translated under $T_{6,5}$ to get $\triangle C''U''B''$. What single translation will give the same result as $T_{6,5} \circ T_{4,9}$?

13. The matrices below represent U.S. foreign trade with other countries in millions of dollars.

<table>
<thead>
<tr>
<th></th>
<th>1982</th>
<th>1983</th>
</tr>
</thead>
<tbody>
<tr>
<td>Western</td>
<td>Exports</td>
<td>Imports</td>
</tr>
<tr>
<td>Hemisphere</td>
<td>67,312</td>
<td>84,467</td>
</tr>
<tr>
<td>Europe</td>
<td>63,664</td>
<td>53,413</td>
</tr>
<tr>
<td>Asia</td>
<td>64,822</td>
<td>85,170</td>
</tr>
<tr>
<td>Africa</td>
<td>10,271</td>
<td>8,768</td>
</tr>
<tr>
<td>Oceania</td>
<td>5,700</td>
<td>4,827</td>
</tr>
</tbody>
</table>

   a. Find the growth in the foreign trade from 1982 to 1983.
   b. From which area did the imports increase the most? (Lesson 4-8)
14. Find a single matrix equal to
\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 8 \\
-1 & 0 \\
1 & -2
\end{bmatrix}
- \begin{bmatrix}
5 & 5 \\
4 & -4 \\
1 & 3
\end{bmatrix}. 
\]
(Lessons 4-8, 4-2)

15. Find a matrix A so that \(A \cdot \begin{bmatrix}
5 & 0 & -1 \\
2 & 6 & -4 \\
1 & 3 & -2
\end{bmatrix} = \begin{bmatrix}
-10 & 0 & 2 \\
1 & 3 & -2
\end{bmatrix}.\) 
(Lesson 4-4)

16. \(H = (5, 1)\) and \(I = (-3, -1)\).
   a. Find the image \(H' I'\) under \(r_y\).
   b. Find \(HI\) and \(H'I'\) and compare the two lengths.
   c. Are \(HI\) and \(H'I'\) perpendicular? Justify your answer. (Lessons 4-5, 4-7)

In 17–22, (a) give the matrix for the transformation; (b) give the image of \((a, b)\). (Lessons 4-3, 4-4, 4-5, 4-6)

17. \(R_{270}\)  
18. \(r_x\)  
19. a size change of magnitude 4  
20. \(S_{1.6}\)  
21. \(R_{180}\)  
22. reflection over the line \(y = x\)

23. A transformation has the following rule: The image of \((x, y)\) is \((3x, y + 2)\). Find images of a figure of your own choosing. Geometrically describe what the transformation does to a figure.
A matrix is a rectangular array for storing data. The product of two matrices contains the sums of linear combinations of the rows and columns being multiplied. Not all matrices can be multiplied; the number of columns of the left matrix must equal the number of rows of the right matrix. Matrix multiplication is associative but not commutative.

Matrices can be added if they have the same dimensions. Any matrix can be multiplied by a number called a scalar.

Matrices with 2 rows can represent points and figures in the coordinate plane. Multiplying such a matrix by a $2 \times 2$ matrix on the left may yield a transformation image of the figure. Transformations for which matrices are found in this chapter include reflections, rotations, size changes, and scale changes. A summary is given below. The rotation of 90° about the origin is a particularly important transformation. From it, we proved that two nonvertical lines are ⊥ if and only if the product of their slopes is -1.

The set of $2 \times 2$ matrices under multiplication has many properties. It is closed; though not commutative, it is associative; there is an identity

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Transformations Yielding Images Congruent to Preimages

**Reflections:**

- over the x-axis
  $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
  $$r_x(x, y) = (x, -y)$$

- over the y-axis
  $$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
  $$r_y(x, y) = (-x, y)$$

- over the line $x = y$
  $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
  $$r_y(x, y) = (y, x)$$

**Rotations with center (0, 0):**

- magnitude 90°
  $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
  $$R_{90}(x, y) = (-y, x)$$

- magnitude 180°
  $$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
  $$R_{180}(x, y) = (-x, -y)$$

- magnitude 270°
  $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
  $$R_{270}(x, y) = (y, -x)$$
Translations:
No general matrix, $T_{h,k}(x, y) = (x + h, y + k)$.

Transformations Yielding Images
Similar to Preimages

Size changes with center $(0, 0)$, magnitude $k$:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$S_k(x, y) = (kx, ky)$

Other Transformations

Scale changes with horizontal magnitude $a$ and vertical magnitude $b$:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$S_{a,b}(x, y) = (ax, by)$

The identity transformation maps any figure onto itself. It can be considered as the size change $S_1$, the rotation $R_0$, or the translation $T_{0,0}$.

Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description or a specific example.

**Lesson 4-1**
*matrix, element of a matrix, dimensions $m \times n$, equal matrices, point matrix

**Lesson 4-2**
matrix multiplication

**Lesson 4-3**
*size change, image, preimage, scale factor, magnitude of size change

**Lesson 4-4**
*scale change, horizontal scale change, vertical scale change, stretch, shrink

**Lesson 4-5**
*reflection

**Lesson 4-6**
*transformation, *identity transformation, *composite of transformations, composition, rotation

**Lesson 4-9**
*matrix addition, matrix subtraction, scalar, scalar multiplication

**Lesson 4-10**
*translation
Progress Self-Test

Take this test as you would take a test in class. Use graph paper. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. Graph the polygon described by the matrix
   \[
   \begin{bmatrix}
   3 & -5 & -6 & -5 & -1 & 5 \\
   4 & 2 & 0 & -2 & -3 & -4
   \end{bmatrix}
   \]

2. One day on Fruhtair flying from Appleton there were 14 first-class and 120 economy passengers going to Peachport; 3 first-class and 190 economy passengers bound for Bananasville; and 8 first-class and 250 economy passengers flying to Grapetown. Write a \(2 \times 3\) matrix to store this information.

3. Which product exists, \(AB\) or \(BA\)?
4. Find \(BC\).
5. Find \(B - C\).
6. Find the image of \(B\) under \(r_y\).
7. Find the image of \(C\) under \(R_{90}\).
8. Calculate \(7B\).
9. Why is \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) called the identity matrix?
10. Find an equation for the line through \((3, -2.5)\) that is perpendicular to \(y = 5x - 3\).
11. Calculate the matrix for \(r_x \circ R_{270}\).
12. Refer to the graph below. What translation maps \(FIGURE\) onto \(F'IG'U'R'E'\)?

\[\begin{aligned}
F & \rightarrow R \\
E & \rightarrow G \\
R & \rightarrow U \\
\end{aligned}\]

13. A shoe manufacturer has factories in Los Angeles, Tucson, and Santa Fe. One year’s sales (in thousands) can be summarized by the following matrix.

\[
\begin{bmatrix}
\text{Deck shoes} & \text{Pumps} & \text{Sandals} & \text{Boots} \\
\text{Los Angeles} & 23 & 8 & 10 & 5 \\
\text{Tucson} & 11 & 5 & 10 & 15 \\
\text{Santa Fe} & 2 & 3 & 15 & 15
\end{bmatrix}
\]

The selling prices of deck shoes, pumps, sandals, and boots are $18, $58, $12, and $76, respectively. Use matrix multiplication to determine the total revenue of each factory.

14. Two pet stores merge and combine inventories. If the matrices below represent each store’s inventory before the merger, what will be their inventory after the merger?

\[
\begin{array}{c|c|c|c}
\text{Lois's Pet Shop} & \text{Doug's Pet Shop} \\
\text{Males} & \text{Females} & \text{Males} & \text{Females} \\
\hline
\text{Dogs} & 8 & 11 & 10 & 14 \\
\text{Cats} & 5 & 4 & 11 & 13 \\
\text{Birds} & 15 & 16 & 7 & 9 \\
\text{Monkeys} & 2 & 0 & 0 & 3 \\
\end{array}
\]

15. Solve for \(a\) and \(b\).
\[
\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} -9 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 14 \end{bmatrix}
\]

In 16 and 17, give the matrix you might use if you wanted to perform the given transformation.

16. a horizontal stretch of magnitude 2 and a vertical shrink of magnitude \(\frac{1}{2}\)
17. a reflection over the line \(y = x\)
18. What is the image of \((x, y)\) under a translation 4 units left and 12 units up?
19. Let \(A = (7, 6), B = (-1, 2)\), and \(C = (3, -4)\). Graph \(\triangle ABC\) and \(R_{90}(\triangle ABC)\).
20. The transformation with matrix \[\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\]
is applied to \(\triangle PQR\) with sides 3 cm, 4 cm, and 5 cm long. What are the lengths of the sides of \(\triangle P'Q'R'\)?
**SPUR** stands for **Skills**, **Properties**, **Uses**, and **Representations**. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deals with the procedures used to get answers.

- **Objective A:** *Perform matrix operations.* (Lessons 4-2, 4-8, 4-9)

  In 1–3, calculate the product.

  1. \[
  \begin{bmatrix}
  6 & -1 & -4 \\
  
  \end{bmatrix}
  \begin{bmatrix}
  8 \\
  -3 \\
  -2
  \end{bmatrix}
  \]

  2. \[
  \begin{bmatrix}
  2 & 3 \\
  -3 & 5
  \end{bmatrix}
  \begin{bmatrix}
  4 & 9 \\
  7 & 6
  \end{bmatrix}
  \]

  3. \[
  \left(\begin{bmatrix}
  1 & 2 & 3 \\
  4 & 7 & 5 \\
  6 & 9
  \end{bmatrix}\right)
  \begin{bmatrix}
  16 & 0
  \end{bmatrix}
  \]

  4. What matrix must you multiply by \[
  \begin{bmatrix}
  5 & 3 & 1 \\
  1 & 2 & 0
  \end{bmatrix}
  \] to get \[
  \begin{bmatrix}
  1 & .6 & .2 \\
  -5 & -1 & 0
  \end{bmatrix}
  \]?

  5. Find a single matrix for \[
  \begin{bmatrix}
  8 & 6 \\
  3 & -2 \\
  4 & -1
  \end{bmatrix}
  - \begin{bmatrix}
  -3 & 0 \\
  -1 & 6 \\
  -4 & -3
  \end{bmatrix}
  \]

  In 6 and 7, let \[
  A = \begin{bmatrix}
  2 & 3 & 4 \\
  7 & 5 & -1 \\
  1 & 2 & 0
  \end{bmatrix}
  \]
  and \[
  B = \begin{bmatrix}
  1 & -6 & 0 \\
  2 & 3 & 1 \\
  4 & 9 & 2
  \end{bmatrix}
  \]

  6. Find \(2A + B\).

  7. Find \(3A - 4B\).

  8. \[
  \begin{bmatrix}
  a & 16 \\
  10 & b
  \end{bmatrix}
  + \begin{bmatrix}
  .4 & -.1 \\
  -10 & 3.1
  \end{bmatrix}
  = \begin{bmatrix}
  2 & 15 \\
  0 & -7
  \end{bmatrix}
  \]

  9. \[
  2\begin{bmatrix}
  -1 & 9 \\
  b & -.5
  \end{bmatrix}
  - \begin{bmatrix}
  a & 7 \\
  -3 & 3
  \end{bmatrix}
  = \begin{bmatrix}
  6 & 11 \\
  13 & -4
  \end{bmatrix}
  \]

  10. \[
  \begin{bmatrix}
  a & 0 \\
  0 & b
  \end{bmatrix}
  \begin{bmatrix}
  2 \\
  -9
  \end{bmatrix}
  = \begin{bmatrix}
  10 \\
  27
  \end{bmatrix}
  \]

  11. \[
  \begin{bmatrix}
  0 & -1 \\
  1 & 0
  \end{bmatrix}
  \begin{bmatrix}
  a \\
  b
  \end{bmatrix}
  = \begin{bmatrix}
  -5 \\
  8
  \end{bmatrix}
  \]

- **Objective B:** *Determine equations of lines perpendicular to given lines.* (Lesson 4-8)

  12. Find an equation for the line through \((3, -1)\) and perpendicular to \(y = \frac{-1}{2}x + 4\).

  13. Find an equation for the line through \((7, 8)\) and perpendicular to \(x = -4\).

  14. Find an equation for the set of all points which are equidistant from \((8, 7)\) and \((-2, 9)\).

  15. Given \(A = (6, 1)\) and \(B = (-2, 3)\). Find an equation for the perpendicular bisector of \(AB\).

**PROPERTIES** deal with the principles behind the mathematics.

- **Objective C:** *Recognize properties of operations on matrices.* (Lessons 4-2, 4-6, 4-8)

  In 16 and 17, (a) is the statement true or false? (b) Give an example to back up your answer.

  16. Matrix addition is commutative.

  17. Matrix multiplication is associative.

  18. Determine whether the following products exist.

  a. \[
  \begin{bmatrix}
  1 & 6 & 4
  \end{bmatrix}
  \begin{bmatrix}
  2 \\
  8
  \end{bmatrix}
  \]

  b. \[
  \begin{bmatrix}
  3 & 1 & 6 \\
  5 & 8 & -2
  \end{bmatrix}
  \begin{bmatrix}
  1 & -1 & 0 & 7 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 5 & 2
  \end{bmatrix}
  \]
19. \( N \) and \( T \) are matrices. \( N \) has dimensions \( r \times p \) and \( T \) has dimensions \( q \times r \).
   a. Which product exists, \( NT \) or \( TN \)?
   b. What are the dimensions of your answer in part a?

20. What \( 2 \times 2 \) matrix is the identity for multiplication?

USES deal with applications of mathematics in real situations.

Objective D: Use matrices to store data. (Lesson 4-1)

21. Chuck makes handcrafted furniture. Last year he made 5 oak tables, 10 oak chairs, 3 pine tables, 12 pine chairs, 1 maple table, and 6 maple chairs. Store this data in a \( 2 \times 3 \) matrix.

22. Bogus High School has 490 freshmen boys, 487 freshmen girls, 402 sophomore boys, 416 sophomore girls, 358 junior boys, 344 junior girls, 293 senior boys, and 300 senior girls. Write a \( 4 \times 2 \) matrix to describe the school's enrollment.

23. The matrix below gives the cost of several items at three different markets. Which element gives the cost of plums in Market 1?

<table>
<thead>
<tr>
<th>Market 1</th>
<th>Market 2</th>
<th>Market 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>eggs</td>
<td>.89</td>
<td>.95</td>
</tr>
<tr>
<td>plums</td>
<td>.90</td>
<td>.79</td>
</tr>
<tr>
<td>peaches</td>
<td>1.49</td>
<td>1.50</td>
</tr>
<tr>
<td>bananas</td>
<td>.33</td>
<td>.28</td>
</tr>
</tbody>
</table>

Objective E: Use matrix addition, matrix multiplication, and scalar multiplication to solve real-world problems. (Lessons 4-2, 4-9)

24. A large pizza costs $12.50, a medium pizza costs $8.90, and a small pizza $5.20. An order for a Journalism Club party consists of 7 large pizzas, 2 medium pizzas, and 4 small pizzas. Write matrices \( C \) and \( N \) for the cost and number ordered, then calculate \( CN \) to find the total cost of the order.

25. An electronics manufacturer has two factories. Sales (in thousands) can be summarized by the following matrix.

\[
\begin{bmatrix}
\text{Factory 1} & \text{Factory 2} \\
\text{VHS} & 15 & 6 \\
\text{TV} & 10 & 8 \\
\text{CD} & 2 & 1 \\
\end{bmatrix}
\]

The selling price of a VHS recorder is $270, a TV is $320, and a compact disc player is $210. Use matrix multiplication to determine the total revenue of each factory.

26. A book company has two presses, and print runs for two years are given in the matrices below.

\[
\begin{array}{ccc}
1987 & & \\
\text{Press 1} & 250,000 & 125,000 & 312,000 \\
\text{Press 2} & 60,000 & 48,000 & 90,000 \\
\end{array}
\]

\[
\begin{array}{ccc}
1988 & & \\
\text{Press 1} & 190,000 & 100,000 & 140,000 \\
\text{Press 2} & 45,000 & 60,000 & 72,000 \\
\end{array}
\]

a. Calculate the matrix that represents the growth in production of each press from 1987 to 1988.

b. Which type of book decreased the most in production?

27. Normal fares (in $) of an airline to three cities are given in the matrix below.

\[
\begin{array}{ccc}
\text{city 1} & \text{city 2} & \text{city 3} \\
\text{first class} & 415 & 672 & 258 \\
\text{economy} & 198 & 394 & 109 \\
\end{array}
\]

To increase air travel, the airline plans to reduce fares by 40%. Find the new fares for travel to these three cities.
Objective F: Relate transformations to matrices, and vice versa. (Lessons 4-3, 4-4, 4-5, 4-6, 4-7, 4-10)

28. Translate the following matrix equation into English by filling in the blanks.

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}
\]

The reflection image of the point ? over the line ? is the point ?.

29. Multiply the matrix for \( r_y \) by itself, and explain your answer in terms of the transformation it represents.

30. The matrix \( \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \) is associated with a ? change with center ? and magnitude ?.

Objective G: Use matrices to perform transformations. (Lessons 4-3, 4-4, 4-5, 4-7, 4-8, 4-9)

In 33–35, match a matrix with each transformation.

a. \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)  
   b. \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)  
   c. \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)  
   d. \( \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \)  
   e. \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \)  
   f. \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \)  

33. \( r_y = x \)  
34. \( S_{4,6} \)  
35. \( R_{90} \)

36. Find the image of \( \begin{bmatrix} -1 & 0 & 4 & 0 \\ 3 & .5 & -1 & 5 \end{bmatrix} \) under \( r_y \).

37. \( GOLD \) has coordinates \( G = (0, 0) \), \( O = (4, 1) \), \( L = (3, 5) \), and \( D = (-1, 4) \). Find the matrix of the image \( GOLD \) under \( R_{270} \).

38. Find the matrix of the image of \( \begin{bmatrix} 6 & 8 & 2 \\ 0 & 4 & 0 \end{bmatrix} \) under \( S_y \).

39. What translation maps \( PEAR \) onto \( P'E'A'R' \) as shown at the right?

Objective H. Graph figures and their transformation images. (Lessons 4-1, 4-3, 4-4, 4-5, 4-6, 4-9)

40. Draw the polygon described by the matrix \( \begin{bmatrix} 3 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix} \).

41. Refer to the graph at the right. Write quadrilateral \( HOPE \) as a matrix.

42. Draw the polygon \( \begin{bmatrix} 1 & 5 & 3 & -2 \\ 4 & 6 & -2 & -2 \end{bmatrix} \) and its image under \( S_y \).

43. Trapezoid \( ABCD \) is represented by \( \begin{bmatrix} -1 & 6 & 5 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \). Graph the preimage and image under \( r_y \).

44. Consider the quadrilateral defined by the matrix \( \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \).

a. Graph the quadrilateral and its image under \( \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \).

b. Are the image and preimage similar?

c. Are they congruent?
CHAPTER 5

Systems

5-1: Compound Sentences
5-2: Representing Systems
5-3: The Linear-Combination Method
5-4: The Substitution Method
5-5: Inverses of Matrices
5-6: Using Matrices To Solve Systems
5-7: Systems of Linear Inequalities
5-8: Linear Programming I
5-9: Linear Programming II
In ordinary language, you can use the words *and* and *or* as conjunctions to join two or more clauses. In mathematics, these two words are used in a similar way. A sentence in which two clauses are connected by the word *and* or by the word *or* is a compound sentence.

For instance, here are postal regulations for the minimum size packages that can be sent through the mails.

All pieces must be at least 0.007 of an inch thick, *and* all pieces (except keys and identification devices) that are \( \frac{1}{4} \) inch rectangular in shape, *and* or less thick must be \[ \begin{align*}
&\text{at least } 3\frac{1}{2} \text{ inches high, } \text{and} \\
&\text{at least } 5 \text{ inches long.}
\end{align*} \]

Mathematically, we could say the conditions are: (1) thickness \( T \geq 0.007" \) and (2) if \( T \leq 0.25" \), then the parcel must be a rectangular solid with other dimensions \( h \geq 3.5" \) and \( \ell \geq 5" \). Notice that constraints may involve just one variable \( (T) \) or many variables \( (T, h, \text{ and } \ell) \).

When mathematical conditions are joined by the word *and*, the set of conditions or sentences is called a *system*. Thus, a system is a special kind of compound sentence. You have seen the most common type, systems of linear equations, in earlier mathematics courses. Here is an example:

\[
\begin{align*}
3x + 4y &= 12 \\
x - 7y &= 15
\end{align*}
\]

The history of solving systems has included some of the greatest mathematicians of all time. An efficient procedure for solving systems of linear equations with any number of variables was developed by the German mathematician Karl Friedrich Gauss in 1819. He adapted the linear combination method you will study in Lesson 5-3. In 1821, the French mathematician Jean Baptiste Joseph Fourier considered systems of linear inequalities and proved the theorem mentioned in Lesson 5-8. Solving systems by using matrices, found in Lesson 5-6, was known to Cayley in the middle of the last century.

Solving systems has always had many applications, and new ones have been developed in recent times. Gauss was trying to calculate the orbits of planets and asteroids from the sightings made by a few astronomers. In 1939, the Russian mathematician L.V. Kantorovich was the first to announce that large systems might have applications for production planning in industry. In 1945, George Stigler used systems to determine a best diet for the least cost. For these works, Kantorovich received a Nobel Prize in 1975; Stigler in 1982. (Both prizes were in economics; there is no Nobel Prize in mathematics.) You will study simplified examples of these kinds of problems in the last three lessons of this chapter.
You are already familiar with the use of compound sentences to describe intervals on the number line. For example, the sentence $4 < x < 8$ means $4 < x$ and $x < 8$.

The solution set for a compound sentence using *and* consists of the **intersection** of the solution sets to the individual sentences. The intersection of two sets is the set consisting of those values common to both sets. The graph of the intersection consists of the points common to the graphs of the individual sets. Of the three graphs above, the bottom is the intersection of the other two.

The symbol used for intersection is $\cap$. $A \cap B$ is the intersection of sets $A$ and $B$. Thus,

$$\{x: 4 < x < 8\} = \{x: x > 4\} \cap \{x: x < 8\}.$$  

This line can be read "The set of $x$ between 4 and 8 equals the intersection of the set of $x$ greater than 4 and the set of $x$ less than 8."

In contrast, the solution set for a compound sentence using *or* consists of the **union** of the solution sets to the individual sentences. The union of two sets is the set consisting of those values in either one or both sets. In other words, all points of either set are in the final graph. This meaning of the word *or* is inclusive, which is somewhat different from the ordinary, exclusive, meaning that usually implies *either, but not both*. The symbol often used for union is $\cup$. $A \cup B$ is the union of sets $A$ and $B$. For instance, the sets of ages $A$ which do not pay full fare on buses might be

$$\{A: A < 12 \text{ or } A > 65\}.$$  

The graph is the union of the graphs of the individual parts.
Compound sentences have many uses.

**Example 1**
On some interstate highways you must drive at least 45 mph but no more than 55 mph. Let \( s \) represent the speed.

a. Graph the possible legal speeds.

b. Write the possible legal speeds in set notation.

**Solution**

a. The legal speeds must satisfy both conditions at the same time, so this is an example of the intersection of the two solution sets.

"At least 45" means \( s \geq 45 \).

![Graph of \( s \geq 45 \)]

"No more than 55" means \( s \leq 55 \).

![Graph of \( s \leq 55 \)]

So, \( s \geq 45 \) and \( s \leq 55 \).

b. An appropriate expression in set notation is \( \{ s: 45 \leq s \leq 55 \} \). This can be read "The set of numbers \( s \) such that \( s \) is between 45 and 55, inclusive."

Another possible answer in set notation is \( \{ s: s \geq 45 \} \cap \{ s: s \leq 55 \} \).

Compound sentences may also describe graphs in the coordinate plane. For instance, in the coordinate plane the graph of \( x = 3 \) is a vertical line and the graph of \( y = 2 \) is a horizontal line. The ordered pair \( (3, 2) \) is the intersection of these two lines. It is the only point for which \( x = 3 \) and \( y = 2 \).
Example 2

Graph the set of all points \((x, y)\) in a plane for which

a. \(x \geq 3\) or \(y \geq 2\);

b. \(x \geq 3\) and \(y \geq 2\).

Solution

\(\{(x, y): x \geq 3\}\) has the graph:

\(\{(x, y): y \geq 2\}\) has the graph:

\[\text{a. Or tells you to find all values satisfying either one or both sentences. The result is the union of the above sets. The graph of }\]
\(\{(x, y): x \geq 3\text{ or } y \geq 2\}\) is the shaded region shown below.

\[\text{b. And tells you to find all values satisfying both sentences simultaneously. The result is all the common values, that is, the }\]
intersection of the sets \(\{(x, y): x \geq 3\}\) and \(\{(x, y): y \geq 2\}\).

The graph of \(\{(x, y): x \geq 3\text{ and } y \geq 2\}\) is the shaded region shown below.
1. A sentence consisting of clauses joined by the words and or or is called a ___.

2. The solution set to a compound sentence using or consists of the __ of the solution sets to the individual sentences.

3. Graph on a number line the solution set to the sentence $x < 6$ or $x > 11$.

4. $-2 < x < 3$ means $-2 < x$ __ $x < 3$.

5. The solution set to a compound sentence using and consists of the __ of the solution sets to the individual sentences.

6. Graph on a number line the solution set to the sentence $n \geq 2.95$ and $n \leq 3.005$.

7. Translate using set notation: the set of numbers $x$ between 0 and 10 equals the intersection of the set of numbers $x$ greater than 0 and the set of numbers $x$ less than 10.

8. Refer to Example 1. Recently some states have changed the maximum speed limit to 65 mph, while maintaining a minimum speed of 45 mph.
   a. Graph the possible legal speeds on a number line.
   b. Write the possible legal speeds in set notation.

9. In a coordinate plane, graph
   a. the intersection of the lines with equations $x = 5$ and $y = 4$.
   b. the union of the line with equation $x = 5$ and the line with equation $y = -4$.

10. a. Graph $\{(x, y): x \geq 7\} \cup \{(x, y): y \leq -9\}$.
    b. Graph $\{(x, y): x \geq 7\} \cap \{(x, y): y \leq -9\}$.

11. Why did the mathematician Gauss become interested in solving systems of equations?

12. Name two persons who received Nobel prizes for finding new applications for systems, and describe their applications.

13. Match each set at the left with its graph at the right.
   a. $\{x: x > 1$ and $x < 4\}$ (i)
   b. $\{x: x > 1$ or $x < 4\}$ (ii)
   c. $\{x: x < 1$ or $x > 4\}$ (iii)
   d. $\{x: x < 1$ and $x > 4\}$ (iv)
   (v)
14. Louise wants to buy a car. She will spend more than $8000 but less than $11,000 on a new car, or she will buy a good used car for no more than $5000. Let $c$ represent the cost of the car she will buy.
   a. Write a sentence using set notation describing the amount she may spend.
   b. Graph the possible values of $c$ on a number line.

15. Willard solved $x^2 = 4$ and wrote “$x = 2$ and $x = -2$.” What is wrong with Willard’s answer?

16. The set $\{(x, y): x > 0$ and $y > 0\}$ describes the first quadrant of the coordinate plane. What set describes the second quadrant?

17. a. Graph $\{(x, y): 2 \leq x \leq 5$ and $-2 \leq y \leq 3\}$.
   b. Describe the graph geometrically.

18. Consider the postal regulations on the first page of this chapter.
   a. Graph $\{((\ell, h): \ell \geq 5$ and $h \geq 3.5\}$.
   b. What have you graphed?

19. The words AND and OR in BASIC have the same meaning as in algebra. Consider the program below.

```
10 FOR X = 1 TO 100
20 LET Y = 3*X
30 IF Y < 250 AND Y > 200 THEN PRINT X
40 NEXT X
50 END
```
   a. What numbers will be printed when this program is run?
   b. What numbers would be printed if the word AND were changed to OR in line 30?

20. An equation relating the total surface area $T$ of a cylinder with height $h$ and radius $r$ is $T = 2\pi r^2 + 2\pi rh$. Solve this equation for $h$.
   (Lesson 1-8)

21. If $y$ varies inversely as $x^3$, how is the value of $y$ changed if $x$ is:
   a. quadrupled?  
   b. halved?  (Lesson 2-3)

22. Graph the line with equation $8x - 4y = 16$. (Lesson 3-4)

23. Triangle $ABC$ has coordinates $A = (1, -2), B = (4, 0)$, and $C = (-3, 3)$. Graph and write the matrix of the image of $\triangle ABC$ under $R_{180}$. (Lesson 4-6)

24. Normal weights are often given in a table as a range of values depending on height. Find such a table in a health book or almanac. Graph the interval of normal weights for your height.

25. In ordinary usage, replacing or by and can dramatically change a sentence. For instance, “Give me liberty and give me death.” only differs from Patrick Henry’s famous saying by that one word. Find examples of other sayings whose meaning is changed by replacing “and” with “or,” or vice-versa.
In the previous lesson you worked with compound sentences that represent situations satisfying more than one condition. A **system**, which is a set of conditions joined by the word *and*, is a special kind of compound sentence. A system is often denoted by a brace. Thus the compound sentence

\[ y = 5x + 40 \text{ and } y = 9x \]

can be written as this system:

\[
\begin{cases}
  y = 5x + 40 \\
  y = 9x
\end{cases}
\]

Systems with one or two variables can be represented graphically. The **solution set for a system** is the intersection of the solution sets for the individual sentences.

**Example 1**

The system

\[
\begin{cases}
  y = 5x + 40 \\
  y = 9x
\end{cases}
\]

is graphed below.

**a.** How many solutions are there?

**b.** Find the solution(s) from the graph.

**Solution**

**a.** The solution consists of all points of intersection. The two lines intersect in only one point, so there is only one solution.

**b.** The lines seem to intersect when \( x = 10 \) and \( y = 90 \).

**Check**

Substitute the point \((10, 90)\) into both sentences.

Is \( 90 = 5(10) + 40 \)? Yes.

Is \( 90 = 9(10) \)? Yes.

The single solution is the point \((10, 90)\).
The three ways of describing solutions to individual sentences also apply to systems. The solution to the system of Example 1 could be expressed in any of the following ways.

1. listing the solution: \((10, 90)\)
2. writing the solution set: \(\{(10, 90)\}\)
3. writing a simplified equivalent system:

\[
\begin{align*}
    x &= 10 \\
    y &= 90
\end{align*}
\]

Graphing a system can quickly indicate the number of solutions, but does not always give an exact answer. As Example 2 shows, graphing helps to approximate the solutions.

**Example 2** Bobbi owns land on a straight stretch of the Old Man River. She plans to fence in a rectangular piece of land along the river. She has 80 m of fencing material, and she wants to enclose an area of 500 square meters. The stretch along the river does not need to be fenced. What can the dimensions of this region be?

**Solution** Let \(W\) and \(S\) be the lengths of the width and sides, respectively. The perimeter of fencing is one width plus two sides. Then

\[
\begin{align*}
    W + 2S &= 80 \quad \text{(fencing)} \\
    WS &= 500 \quad \text{(area)}
\end{align*}
\]

Graph each sentence. (Key points are identified in the graph below.) Since \(W\) and \(S\) must be positive because they represent lengths, it is not necessary here to draw the third-quadrant branch of the hyperbola.

The graphs intersect at two points. So there are two solutions. One looks to be near \((65, 8)\) and the other is near \((15, 32)\). Bobbi can make the width about 65 m long and the other two sides about 8 m long, or she can make the width 15 m long and the other two sides 32 m long.
Check  Because the solutions are estimates, a check is very important. Substitute \((65, 8)\) into both sentences.

\[
\begin{align*}
65 + 2 \cdot 8 &= 81 \approx 80 \\
65 \cdot 8 &= 520 \approx 500
\end{align*}
\]

The check shows that \((65, 8)\) is an approximate solution, not an exact solution. The check of \((15, 32)\) is left to you in Question 6. The set of approximate solutions is \(\{(65, 8), (15, 32)\}\).

If you have an automatic grapher, you can estimate the solutions to a system to a high degree of accuracy by rescaling or zooming around each point of intersection. For instance, the screen below, from a function grapher, shows that a more accurate solution to the system in Example 2 is \(W = 64.5\) and \(S = 7.8\).

\[
\begin{align*}
64.5 + 2(7.8) &= 80.1 \approx 80 \\
64.5 \cdot 7.8 &= 503.1 \approx 500
\end{align*}
\]

To get a solution accurate to the nearest hundredth, you would have to zoom or rescale again.

Systems are classified into two groups depending on whether or not solutions exist. If a system has solutions, it is called **consistent**; if it has no solutions, it is **inconsistent**.

When two individual sentences are lines, there are three possibilities for the system. The lines can intersect in one point, they can be parallel and nonintersecting, or they can be identical. The possibilities are shown in the next three examples.

**Example 3**  Is the system \[
\begin{cases}
x + y = 5 \\
2x - y = 4
\end{cases}
\] inconsistent or consistent?

**Solution**  For \(x + y = 5\), the slope is \(-1\). For \(2x - y = 4\), the slope is \(2\). The two lines intersect, so the system is consistent.
Example 4  Is the system \( \begin{cases} 4x + 3y = 24 \\ 12x + 9y = 36 \end{cases} \) inconsistent or consistent?

Solution  For \( 4x + 3y = 24 \), the slope is \(-\frac{4}{3}\). For \( 12x + 9y = 36 \), the slope is also \(-\frac{4}{3}\). Thus the lines are parallel. The \( y \)-intercept of the first line is 8; for the second line, it is 4. There are no points that satisfy both equations, so the system is inconsistent.

Example 5  Is the system \( \begin{cases} 4x + 3y = 24 \\ 8x + 6y = 48 \end{cases} \) inconsistent or consistent?

Solution  Both lines have slope of \(-\frac{3}{2}\) and \( y \)-intercept of 8. So the two equations represent the same line. Every point satisfying one equation satisfies the other one. The system is consistent and there are infinitely many solutions.

Questions

1. **Multiple choice** The solution to a system consists of
   (a) the union of the solution sets of the individual sentences.
   (b) the intersection of the solution sets of the individual sentences.
   (c) all points satisfying at least one of the individual sentences.

   In 2 and 3, consider this system: \( \begin{cases} y = 3x + 2 \\ y = 20 \end{cases} \)

2. **Multiple choice** The system means
   (a) \( y = 3x + 2 \) and \( y = 20 \)
   (b) \( y = 3x + 2 \) or \( y = 20 \).

3. a. The solution to the system is ___.
   b. Verify that the ordered pair you found in part a is the solution.
In 4–6, refer to Example 2.

4. **Multiple choice** \( WS = 500 \) represents a relationship involving:
   (a) perimeter   (b) area   (c) volume

5. Why do you not need to draw the third-quadrant branch of the hyperbola?

6. Show that \((15, 32)\) is an approximate solution to the system.

7. Give an example of a system with infinitely many solutions.

8. **True or false** If two lines have the same slope, then they represent inconsistent systems.

In 9–11, the graph of a system is described. Is the system consistent or inconsistent?

9. parallel, different lines

10. lines intersecting at one point

11. identical lines

In 12–14, use the given systems and their graphs.
   (a) Tell how many solutions the system has.
   (b) Identify the system as inconsistent or consistent.
   (c) Estimate the solutions, if there are any.
   (d) Verify that your solutions satisfy all equations of the system.

12. \[
\begin{align*}
  y &= x^2 \\
  y &= x - 5
\end{align*}
\]

13. \[
\begin{align*}
  xy &= 2 \\
  2x - y &= 3
\end{align*}
\]

14. \[
\begin{align*}
  y &= x^2 \\
  xy &= 8
\end{align*}
\]
In 15 and 16, (a) graph each system; (b) tell how many solutions the system has; and (c) estimate any solutions.

15. \[
\begin{align*}
  y &= \frac{1}{2}x^2 \\
  y &= -x + 5
\end{align*}
\]

16. \[
\begin{align*}
  y &= \frac{9}{x^2} \\
  y &= x + 4
\end{align*}
\]

17. The system of Example 1 could represent the following situation: A child challenged her father to a race. The father gave her a head start of 40 m. He ran at 9 meters per second. She ran at 5 meters per second. Let \( y \) be distance and \( x \) be time in seconds.
   a. Which equation represents the father’s distance from the start after \( x \) seconds?
   b. After 1 second, how far was the father from the start? How far was the daughter?
   c. When did the father catch up to his daughter?
   d. When the father caught up to her, how far from the start were they?

18. Phillip has 150 m of fencing material and wants to surround all four sides of a rectangle with area 1300 square meters. Use graphing to estimate the dimensions of this region.

19. Use a graph to show that there do not exist two real numbers \( x \) and \( y \) whose product is 30 and whose sum is 10.
20. Graphing can be the first step in helping to solve complicated systems by search procedures. Consider the system \( \begin{cases} y = 3x^2 \\ y = 4x + 10 \end{cases} \)
A rough graph shows a solution between \( x = -2 \) and \( x = 0 \). By restricting the domain in the FOR ... NEXT loop, you can get closer and closer approximations of solutions.

```
10 PRINT "X", "Y1", "Y2"
20 FOR X = -2 TO 0 STEP .01
30 Y1 = 3 * X ^ 2
40 Y2 = 4 * X + 10
50 PRINT X, Y1, Y2
60 NEXT X
70 END
```

(Different scales on the two axes)

a. Run the program and find the solution to two decimal places.
b. Change line 20 to find the other solution, which is between \( x = 2 \) and \( x = 3 \).

21. Graph the solution set on a number line.
   \( \{x: -0.5 \leq x < 4\} \) (Lesson 5-1)

22. a. Graph \( \{x: x > 4 \text{ or } x > -2\} \).
b. Graph \( \{x: x > 4 \text{ and } x > -2\} \). (Lesson 5-1)

23. Angela has \( x \) $5.99 records, \( y \) $6.25 records, and \( z \) $7.99 records. If the total value is \( T \), write an equation relating all these variables. (Lesson 3-3)

24. Let \( A = (-2, 6) \) and \( B = (3, 7) \).
a. Find \( AB \).
b. Find the midpoint of \( \overline{AB} \). (Previous course)

25. Consider both branches of the hyperbola with equation \( y = \frac{1}{x} \). Is there any line that intersects this hyperbola in exactly one point?
In the last lesson you graphed systems to estimate or find solutions. Graphing works for estimating, but does not often give exact solutions. To find exact solutions, you usually need to use algebraic techniques. The next several lessons discuss algebraic techniques.

Systems are equivalent if and only if they have the same solutions. For example, the two systems below are equivalent because they have the same solution \((-1, 4)\).

\[
\begin{align*}
3a + 2b &= 5 \\
7a + 4b &= 9
\end{align*}
\quad \begin{align*}
\begin{array}{l}
a + b = 10 \\
-14a - 3b = 2 \\
2a - b = -6
\end{array}
\end{align*}
\]

The goal in solving is to take a more complicated system like either of those above and find the simplest equivalent system,

\[
\begin{align*}
a &= -1 \\
b &= 4.
\end{align*}
\]

One technique for solving equations uses the Addition Property of Equality.

Example 1  Solve the system \(\begin{align*}x + y &= 9 \\
2x - y &= 2\end{align*}\).

Solution  Notice that the coefficients of \(y\) are 1 and -1 which add to zero. Adding the sides of the two equations gives an equation in one variable.

\[3x = 11\]

Thus \(x = \frac{11}{3}\) or \(3\frac{2}{3}\).

Substitute the value of \(x\) into either of the two original equations to solve for \(y\). We choose the first equation.

\[\frac{11}{3} + y = 9\]

\[y = \frac{16}{3}\]

The solution is \((\frac{11}{3}, \frac{16}{3})\).

Check 1  Verify that \((\frac{11}{3}, \frac{16}{3})\) satisfies each of the given sentences.

Does \(\frac{11}{3} + \frac{16}{3} = 9\)?  Yes.

Does \(2 \cdot \frac{11}{3} - \frac{16}{3} = 2\)?  Yes, so it checks.
Check 2  Graph the lines \( x + y = 9 \) and \( 2x - y = 2 \). The coordinates \((\frac{11}{3}, \frac{16}{3})\) are reasonable for the point of intersection.

In Example 1, the coefficients of one variable were opposites. In many systems, the coefficients are not so convenient. Example 2 shows how such systems can be solved using both the Multiplication and Addition Properties of Equality.

**Example 2**  Solve the system \[ \begin{cases} 6s + 8p = 22 \\ 4s + 3p = 10 \end{cases} \]

**Solution**  To use the addition method of Example 1, the coefficients of one of the variables must be opposites. Notice that the least common multiple of 6 and 4 (the coefficients of \( s \)) is 12. Use the Multiplication Property of Equality to multiply the first equation by 2 and the second equation by -3 to get opposite coefficients for \( s \).

\[
\begin{align*}
6s + 8p &= 22 \quad \text{(multiply by 2)} & 12s + 16p &= 44 \\
4s + 3p &= 10 \quad \text{(multiply by -3)} & -12s - 9p &= -30 \\
\text{Add} & & 7p &= 14 \\
& & p &= 2.
\end{align*}
\]

Substitute \( p = 2 \) into either equation and solve for \( s \). We substitute into \( 6s + 8p = 22 \).

\[
6s + 8(2) = 22 \\
s = 1
\]

**Check**  Substitute \( p = 2 \) and \( s = 1 \) into the other sentence, \( 4s + 3p = 10 \). Does \( 4(1) + 3(2) = 10? \)  Yes.

In Example 2, we could have found the least common multiple for the coefficients of \( p \), which is 24. Then the first equation would be multiplied by 3 (because \( 8 \cdot 3 = 24 \)), and the second equation would be multiplied by -8 (because \( 3 \cdot -8 = -24 \)). Solving the system in this way would still result in the same solution.
Ration stamps were used to distribute scarce goods.

Systems arise in important practical endeavors. During World War II, the United States (and many other countries) had to ration certain foods. Mathematicians were involved in determining diets that met the minimum requirements of protein, vitamins, and minerals. Here is a simplified example.

The table shows the protein and calcium contents for a serving of spaghetti and peas. How many servings of each are needed to get 22 g of protein and 100 mg of calcium?

<table>
<thead>
<tr>
<th></th>
<th>SPAGHETTI</th>
<th>PEAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>protein (g) per serving</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>calcium (mg) per serving</td>
<td>40</td>
<td>30</td>
</tr>
</tbody>
</table>

To solve, let $s$ be the number of servings of spaghetti and let $p$ be the number of servings of peas. Then the grams of protein must satisfy

$$6s + 8p = 22,$$

and the milligrams of calcium must satisfy

$$40s + 30p = 100.$$  

If you divide the second equation by 10, the system of Example 2 results.

In Chapter 3 you learned that $Ax + By = C$ is called a linear combination in two variables $x$ and $y$. The method used to solve the problems in this lesson is often called the linear-combination method of solving systems because it involves adding multiples of the given equations.

The systems in the examples above are consistent, and each has a unique solution. The linear-combination method gives interesting results for inconsistent systems or ones with infinitely many
In Lesson 5-2, for example, graphing showed that
\[
\begin{align*}
4x + 3y &= 24 \\
12x + 9y &= 36
\end{align*}
\]
represents an inconsistent system. These results could have been found by the linear-combination method. Multiply the first equation by -3 and add:

\[
\begin{align*}
-12x - 9y &= -72 \\
12x + 9y &= 36 \\
\hline
0 &= -36
\end{align*}
\]

The result 0 = -36 is false. This indicates the original system is always false; it has no solutions. The two lines are parallel.

Graphing also showed that the system
\[
\begin{align*}
4x + 3y &= 24 \\
8x + 6y &= 48
\end{align*}
\]
has infinitely many solutions. Again this could have been found by the linear-combination method. Multiply the first equation by -2.

\[
\begin{align*}
-8x - 6y &= -48 \\
8x + 6y &= 48 \\
\hline
0 &= 0
\end{align*}
\]

The result 0 = 0 is always true. This indicates that the original system is always true; there are infinitely many solutions. Any (x, y) satisfying 4x + 3y = 24 solves the system. The two equations represent the same line.

Questions

1. When are systems of equations equivalent?

In 2 and 3, refer to Example 1.

2. The equation 3x = 11 is the result of ___ the two original equations.

3. Why is the method of solving by graphing unsatisfactory for this problem?

4. Refer to Example 2. Multiply the first equation by 3 and the second equation by -8. Solve the resulting system to show that the solution is still p = 2 and s = 1.

5. Lynne wants to get 26 grams of protein and 17.5 grams of fat from one meal of beef stew and bread. How many servings of each does she need to eat?

<table>
<thead>
<tr>
<th>Protein (g) per serving</th>
<th>Beef stew with vegetables</th>
<th>Bread</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
6. Morris was solving a system and got $0 = 0$ after adding the equations together. This result means that Morris has what kind of system?

In 7–12, use the linear-combination method to solve the system.

7. \[
\begin{align*}
5x - 6y &= 3 \\
2x + 12y &= 12
\end{align*}
\]

8. \[
\begin{align*}
2v + w &= 47 \\
8v - 4w &= 28
\end{align*}
\]

9. \[
\begin{align*}
x + 3y &= 12 \\
4x + 12y &= 48
\end{align*}
\]

10. \[
\begin{align*}
1000x + 30y &= 500 \\
x - 2y &= 11
\end{align*}
\]

11. \[
\begin{align*}
a + b &= \frac{1}{3} \\
a - b &= \frac{1}{4}
\end{align*}
\]

12. \[
\begin{align*}
5u + 4v &= -18 \\
0.04u - 0.12v &= 0.16
\end{align*}
\]

In 13–15, use the linear-combination method to determine whether the system is inconsistent or consistent.

13. \[
\begin{align*}
2x + 3y &= 4 \\
5x + 6y &= 7
\end{align*}
\]

14. \[
\begin{align*}
2x + 3y &= 4 \\
4x + 6y &= 8
\end{align*}
\]

15. \[
\begin{align*}
2x + 3y &= 4 \\
4x + 6y &= 9
\end{align*}
\]

16. At the zoo, Jay and Terri bought food for themselves and a friend. Jay bought 3 slices of pizza and 1 lemonade for $4.50. Terri paid $4.00 for 2 slices of pizza and 2 lemonades. What should their friend reimburse Terri for his lemonade?

17. \(N\) ml of a 60% salt solution are mixed with \(S\) ml of an 80% salt solution. The result is 35 ml of a 72% salt solution.
   a. Write an equation relating \(N\), \(S\), and the total number of ml.
   b. The amount of salt in the 72% solution is \(0.72(35) = 25.2\) ml. Write an equation relating the amount of salt in the 60%, 80%, and 72% solutions.
   c. Solve the system represented by your answers in parts a and b. How many ml of the 60% and the 80% solutions are needed?
18. Solve by the linear-combination method.
\[ \begin{align*}
9x^2 - 6y^2 &= 291 \\
3x^2 + 2y^2 &= 197
\end{align*} \]

19. Graph the solution set of this compound sentence in a coordinate plane. *Lesson 5-1*
\[ x \geq 9 \text{ or } y < \frac{1}{2} \]

20. Graph the solution set of each sentence in the system below and approximate the solutions. *Lesson 5-2*
\[ \begin{align*}
y &= \frac{1}{2}x^2 \\
y &= \frac{1}{3}x + 1
\end{align*} \]

21. Consider the system
\[ \begin{align*}
5x - 3y &= 6 \\
x - y &= 2
\end{align*} \]
graphed below.
\[ a. \text{ How many solutions does the system have?} \\
b. \text{ Is the system inconsistent or consistent?} \]

22. Look up the suggested daily number of grams of protein, vitamin A, and calcium recommended for your age. Find some combinations of food that will give you the recommended amounts.
The Substitution Method

If \( a = b \), then \( a \) can be substituted for \( b \) in any arithmetic or algebraic expression. This, the Substitution Property of Equality, is obvious when \( a = 10 \) and \( b = 7 + 3 \). It is not so obvious that when \( y = 6x - 17 \), for example, you can substitute \( 6x - 17 \) for \( y \). This second application of the Substitution Property is useful in solving systems.

**Example 1** Solve the system \[
\begin{align*}
  y &= 6x - 17 \\
  21x - 4y &= 10.
\end{align*}
\]

**Solution** Substitute \( 6x - 17 \) for \( y \) in the second equation.

\[
21x - 4(6x - 17) = 10
\]

The equation now has only one variable. Solve for \( x \).

\[
\begin{align*}
  21x - 24x + 68 &= 10 \\
  -3x &= -58 \\
  x &= \frac{58}{3}
\end{align*}
\]

Now substitute in either equation to find \( y \). We use the first because it is solved for \( y \). (The other equation will be the check.)

\[
\begin{align*}
  y &= 6 \cdot \frac{58}{3} - 17 \\
  &= 116 - 17 \\
  &= 99
\end{align*}
\]

**Check** Use the second equation. Does \( 21 \cdot \frac{58}{3} - 4 \cdot 99 = 10 \)?

Yes, \( 406 - 396 = 10 \).

Another situation where substitution can be used conveniently is when there are more than two variables and two equations, as Example 2 illustrates.
Example 2

An end zone has a seating capacity of 4216. There are four times as many lower-level seats as there are upper-level seats. Also, there are three times as many mezzanine seats as there are upper-level seats. How many seats of each type are there?

Solution

Let \( L \) = the number of lower-level seats,
\( M \) = the number of mezzanine seats,
\( U \) = the number of upper-level seats.

Then the system is

\[
\begin{align*}
L + M + U &= 4216 \\
L &= 4U \\
M &= 3U.
\end{align*}
\]

Substitute the expressions for \( L \) and \( M \) into the first equation.

\[
4U + 3U + U = 4216
\]

\[
8U = 4216
\]

Thus,

\[
U = 527
\]

\[
L = 4 \cdot 527 = 2108
\]

\[
M = 3 \cdot 527 = 1581.
\]

There are 527 upper-level seats, 2108 lower-level seats, and 1581 mezzanine seats.

Check

The total number of seats should add to 4216. It does:

\[
527 + 2108 + 1581 = 4216.
\]

The substitution method generally works with any system which has a linear equation and a nonlinear equation. Substitute an expression from the linear equation into the nonlinear equation. Example 3 illustrates this.

---

Example 3

Solve the system \( \begin{align*}y &= 3x \\
x \cdot y &= 48.\end{align*} \)

Solution

Substitute 3x for \( y \) in the second equation.

\[
x(3x) = 48
\]

\[
3x^2 = 48
\]

\[
x^2 = 16
\]

\[
x = 4 \text{ or } x = -4
\]

(Note the word or. The solution set is the union of all possible answers.) Each value of \( x \) yields a value of \( y \). Substitute each value of \( x \) into either of the original equations. We substitute into \( y = 3x \). If \( x = 4 \), then \( y = 3(4) = 12 \). If \( x = -4 \), then \( y = 3(-4) = -12 \). The solution set is \( \{(4, 12), (-4, -12)\} \).
Check 1  Substitute the coordinate of each point into each equation. For (4, 12): Does 12 = 3 \cdot 4? Yes. Does 4 \cdot 12 = 48? Yes. In the Questions, you are asked to check the other point.

Check 2  Graph the equations. The graph below shows that there are two solutions. One solution seems near (4,12); the other near (-4,-12).

The examples of this lesson illustrate that substitution may be an appropriate method when:

1. at least one of the equations has been or can easily be solved for one of the variables;
2. there are three or more equations and three or more variables; or
3. the system has one linear and one nonlinear equation.

Questions

1. Write the Substitution Property of Equality.

2. Solve the system \[ \begin{cases} y = 3x + 5 \\ 4x - 3y = 12 \end{cases} \]

In 3 and 4, refer to Example 2.

3. After the expressions in the second and third equations were substituted into the first equation, how many variables were in this new equation?

4. A second stadium was built to the same specifications but has a seating capacity of 6904. How many upper level seats are there?

5. Solve the system \[ \begin{cases} 3x + 2y + z = 24 \\ x = 5z - 20 \\ y = -2z \end{cases} \]
In 6 and 7, refer to Example 3.

6. Verify that (-4, -12) is a solution to the system.

7. The graph of \( y = 3x \) is a __ and the graph of \( xy = 48 \) is a __.

8. Solve the system \[
\begin{align*}
  y &= 3x \\
  xy &= 75
\end{align*}
\]

In 9 and 10, refer to the following systems.

(a) \[
\begin{align*}
  y &= 3x + 1 \\
  4x - 3y &= 12
\end{align*}
\]

(b) \[
\begin{align*}
  5x - 7y &= 12 \\
  -12x + 8y &= 19
\end{align*}
\]

(c) \[
\begin{align*}
  3x + 2y + z &= 7 \\
  x &= 5z \\
  y &= -2z
\end{align*}
\]

(d) \[
\begin{align*}
  x + 5y &= 12 \\
  4x - 7y &= 13
\end{align*}
\]

9. Which systems are written in a form that is convenient to be solved using linear combinations?

10. Which systems can be solved conveniently using substitution?

11. A sports stadium seats 60,000 people. The home team gets 4 times as many tickets as the visiting team. Let \( H \) be the number of tickets for the home team and \( V \) be the number of tickets for the visiting team.

   a. Multiple choice Which system represents the given conditions?
      (i) \[
      \begin{align*}
      4H + 4V &= 60,000 \\
      H &= 4V
      \end{align*}
      \]
      (ii) \[
      \begin{align*}
      H + V &= 60,000 \\
      V &= 4H
      \end{align*}
      \]
      (iii) \[
      \begin{align*}
      H &= 4V \\
      H + V &= 60,000
      \end{align*}
      \]

   b. Solve the correct system for \( H \) and \( V \).

12. FASTPIC offers to process a roll of film for 30¢ per print with free developing. A competitor, QUALIPRINT, will process a roll for 25¢ per print plus a $2.00 developing charge.

   a. For what number of prints will the cost be the same at FASTPIC and QUALIPRINT?

   b. What is the cost for this number of prints?

13. A recipe that makes 7 cups of French dressing uses tomato juice, vinegar, and olive oil. It calls for 3 times as much vinegar as tomato juice and 4\( \frac{1}{2} \) times as much olive oil as vinegar. How much of each ingredient should be used?

14. Six towns on a train line are in order: Achilles, Bacchus, Calypso, Daedalus, Electra, and Fates. The distance from Achilles to Bacchus is twice the distance from Bacchus to Calypso, which is three times the distance from Calypso to Daedalus. The distance from Daedalus to Electra is two miles less than the distance from Daedalus to Electra to Fates, which is twelve times longer than the distance from Daedalus to Electra. The total distance from Achilles to Fates is 112 miles. What is the distance between Achilles and Daedalus?
15. Solve the system
\[
\begin{align*}
4x - 3y &= 1 \\
5x - 6y &= 9
\end{align*}
\]
(Lesson 5-3)

16. Solve by graphing
\[
\begin{align*}
y &= 12x^2 \\
y &= 3x - 4
\end{align*}
\]
(Lesson 5-2)

In 17 and 18, multiply. (Lesson 4-2)

17. 
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
2 & \sqrt{3} & -1 \\
0 & 5.1 & 0 \\
-4 & 11 & -2
\end{bmatrix}
\]

18. 
\[
\begin{bmatrix}
2 & 4 \\
-3 & 11
\end{bmatrix}
= 
\begin{bmatrix}
7 \\
11
\end{bmatrix}
\]

19. Find the measures of all four angles of quadrilateral \textit{ROCK}
drawn below. (Previous course)

20. This problem was made up by the Indian mathematician Mahavira and
dates from about 850 A.D. "The price of nine citrons and seven fra-
grant wood apples is 107; again, the mixed price of seven citrons and
nine fragrant wood apples is 101. Oh you arithmetician, tell me
quickly the price of a citron and a wood apple here, having distinctly
separated these prices well." At this time algebra had not been de-
veloped yet. How could this question be answered by someone without
algebra?
Inverses of Matrices

In this chapter you have seen three methods of solving systems: graphing, linear combination, and substitution. A fourth method makes use of matrices.

Recall that real numbers $a$ and $b$ are multiplicative inverses if and only if $ab = ba = 1$. Recall also that the real number 0 does not have a multiplicative inverse.

Similarly, $2 \times 2$ matrices $M$ and $N$ are inverse matrices if and only if their product is the $2 \times 2$ identity matrix for multiplication,

$$MN = NM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

There are many such pairs of matrices. For instance, the inverse of

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$  

is

$$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$  

because

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$  

and

$$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

Recall that $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ is the matrix for the scale change $S_{4,3}$.

To undo the effect of $S_{4,3}$ on a figure, apply the scale change $S_{4^{-1},3}^{-1}$.

That scale change is associated with the matrix

$$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$
Other inverses may not be so obvious. The inverse of
\[
\begin{bmatrix}
2 & 5 \\
3 & 8 \\
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
8 & -5 \\
-3 & 2 \\
\end{bmatrix}.
\]
This result can be verified by matrix multiplication.
\[
\begin{bmatrix}
8 & -5 \\
-3 & 2 \\
\end{bmatrix}\begin{bmatrix}
2 & 5 \\
3 & 8 \\
\end{bmatrix} = \begin{bmatrix}
8(2) + -5(3) & 8(5) + -5(8) \\
-3(2) + 2(3) & -3(5) + 2(8) \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
Similarly, multiplication in the other order also results in the identity.
\[
\begin{bmatrix}
2 & 5 \\
3 & 8 \\
\end{bmatrix}\begin{bmatrix}
8 & -5 \\
-3 & 2 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
How can the inverse of a matrix be found? The following powerful theorem tells when an inverse matrix exists, and gives you a formula to find that inverse.

**Inverse-Matrix Theorem:**

If \(ad - bc \neq 0\), the inverse of
\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
d & -b \\
ad - bc & ad - bc \\
-c & a \\
ad - bc & ad - bc \\
\end{bmatrix}.
\]

**Proof:**

We need only to show that the product of the two matrices in either order is the identity matrix.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}\begin{bmatrix}
d & -b \\
ad - bc & ad - bc \\
-c & a \\
ad - bc & ad - bc \\
\end{bmatrix} = \begin{bmatrix}
ad & -bc \\
ad - bc & ad - bc \\
-ab & ab \\
ad - bc & ad - bc \\
\end{bmatrix}
\]

matrix multiplication

\[
= \begin{bmatrix}
ad & 0 \\
ad - bc & ad - bc \\
0 & ad - bc \\
ad - bc & ad - bc \\
\end{bmatrix}
\]

addition of fractions and
definition of subtraction

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
simplification
In Question 2, you are asked to verify the multiplication in reverse order.

**Example 1** Use the theorem to find the inverse of the matrix \[
\begin{bmatrix}
0 & -2 \\
3 & 1
\end{bmatrix}.
\]

**Solution** In \[
\begin{bmatrix}
0 & -2 \\
3 & 1
\end{bmatrix},
\]
a = 0, b = -2, c = 3, and d = 1.

So \(ad - bc = 0(1) - (-2)(3) = 6\). Substitute into the formula to get

\[
\begin{bmatrix}
\frac{1}{6} & \frac{2}{6} \\
\frac{3}{6} & \frac{0}{6}
\end{bmatrix},
\]
which can be written as \[
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} \\
\frac{1}{2} & 0
\end{bmatrix}.
\]

The multiplicative inverse of a real number \(x\) is sometimes written as \(x^{-1}\). In the same way, the multiplicative inverse of a matrix \(M\) can be written as \(M^{-1}\).

Only **square matrices**, that is, ones with the same number of rows and columns can have inverses. However, not all square matrices have inverses. If \(M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(ad - bc = 0\), then \(\frac{1}{ad - bc}\) is undefined, so \(M^{-1}\) cannot exist.

**Example 2** Verify that \[
\begin{bmatrix}
3 & 1 \\
6 & 2
\end{bmatrix}
\]
does not have an inverse.

**Solution** Suppose \[
\begin{bmatrix}
3 & 1 \\
6 & 2
\end{bmatrix}
\]
had an inverse \[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]
Then

\[
\begin{bmatrix}
3 & 1 \\
6 & 2
\end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then \(3a + c = 1\) and \(6a + 2c = 0\). This system has no solution, so the matrix can have no inverse.

**Check** In this matrix \(a = 3, b = 1, c = 6,\) and \(d = 2\). Thus \(ad - bc = 3 \cdot 2 - 1 \cdot 6 = 0\). This means there is no inverse.
The expression $ad - bc$ that is associated with the $2 \times 2$ matrix \[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \] is called the determinant of the matrix $A$, since it determines whether or not matrix $A$ has an inverse. We abbreviate the word determinant as $\text{det}$. For instance, $\text{det} \begin{bmatrix} 3 & 2 \\ -5 & 4 \end{bmatrix} = 3 \cdot 4 - 2 \cdot -5 = 22$. The idea of the determinant was first used by the German mathematician Gottfried Leibniz (1646–1716), who is also known as one of the inventors of calculus. Using this notation and scalar multiplication, the Inverse-Matrix Theorem can be written as:

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\text{det} M \neq 0$, then $M^{-1} = \frac{1}{\text{det} M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Questions

1. a. If $a$ and $b$ are real numbers that are multiplicative inverses of each other, what does $ab$ equal?
   b. If $M$ and $N$ are $2 \times 2$ matrices that are multiplicative inverses of each other, what does $MN$ equal?

2. Verify the second part of the proof of the Inverse-Matrix Theorem. That is, show that the identity matrix is the product of the two matrices in the reverse order.

In 3 and 4, is the statement true or false?
3. Only square matrices have inverses.
4. All square matrices have inverses.
5. $M$ is a matrix with nonzero determinant. What is denoted by $M^{-1}$?
6. Give an expression for $\text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
7. In Example 1, what is the determinant of the given matrix?
8. Give an example of a matrix not mentioned in this lesson that does not have an inverse.

In 9–12, a matrix is given. (a) Find its determinant. (b) Find its inverse, if it has one. (c) Check your answer to part b by multiplying.

9. $\begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}$ 10. $\begin{bmatrix} -1 & -3 \\ 4 & -8 \end{bmatrix}$ 11. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
Applying the Mathematics

13. If \( A = \begin{bmatrix} -7 & 4 \\ -9 & -4 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} \), find
   
   a. \( \det A \). 
   b. \( \det B \). 
   c. \( \det AB \).

14. a. Find the inverse of the matrix for \( R_{90} \).
   b. Explain the result to part a geometrically.

15. The inverse of a \( 2 \times 2 \) matrix can be found by solving a pair of systems. Here is how. If the inverse of \( \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix} \) is \( \begin{bmatrix} e & f \\ g & h \end{bmatrix} \),
   
   then \( \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

   This yields the systems \( \begin{align*}
   0e - 2g &= 1 \\
   3e + g &= 0 \\
   0f - 2h &= 0 \\
   3f + h &= 1
   \end{align*} \).

   a. Solve the systems above and determine the inverse matrix.
   b. Check your answer to part a by finding the inverse matrix using the Inverse-Matrix Theorem.

16. a. Solve the following system by adding: \( \begin{align*}
   5x - 3y &= 15 \\
   5x + 3y &= 15
   \end{align*} \)

   b. It can just as easily be solved by subtracting. Solve by subtracting to check your answer. (Lesson 5-3)

In 17 and 18, solve the system. (Lessons 5-3, 5-4)

17. \( \begin{align*}
   y &= 4x \\
   3x + 2y &= 22
   \end{align*} \)

18. \( \begin{align*}
   2x - 8y &= 6 \\
   -x + 4y &= 3
   \end{align*} \)

19. Name two situations in which it is convenient to solve a system by substituting. (Lesson 5-4)

20. Alan Aska wants to purchase a new air conditioner. One brand costs \$540 to purchase and \$20 a month to operate. A less efficient brand costs \$320 to purchase and \$24 a month to operate.
   
   a. Plot the costs over time of both brands on a single graph.
   b. What does the point of intersection denote? (Lessons 5-2, 5-3)

21. Graph the set of ordered pairs satisfying \( 2y < 3x - 6 \). (Lesson 3-9)

22. a. Find the area of the triangle with vertices \((0, 0), (-3, 0), \) and \((-7, 8)\).

   b. Calculate \( \frac{1}{2} \det \begin{bmatrix} -3 & -7 \\ 0 & 8 \end{bmatrix} \).

   c. Find the area of the triangle with vertices \((0, 0), (5, 2), \) and \((4, 6)\).

   d. Calculate \( \frac{1}{2} \det \begin{bmatrix} 5 & 4 \\ 2 & 6 \end{bmatrix} \).

   e. Generalize parts a-d.
   f. Test your generalization with another example.
23. In 1929–31, the mathematician Lester Hill devised a method of encoding messages using matrices. Every integer is assigned a letter according to the scheme:

\[ 1 = A, 2 = B, 3 = C, \ldots, 25 = Y, 26 = Z, 27 = A, 28 = B, \ldots, \]
and \[ 0 = Z, -1 = Y, \ldots, -24 = B, -25 = A, -26 = Z, \ldots. \]

To code or encipher the word *FOUR*, follow these steps.

**Step 1.** Put the letters into a matrix four at a time. With \[ 6 = F, 15 = O, 21 = U, 18 = R, \] use the matrix

\[
\begin{bmatrix}
6 & 15 \\
21 & 18
\end{bmatrix}
\]

**Step 2.** Multiply each \(2 \times 2\) matrix by the *coding* or *key* matrix, such as

\[
\begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
6 & 15 \\
21 & 18
\end{bmatrix}
= 
\begin{bmatrix}
21 & 18 \\
48 & 51
\end{bmatrix}
\]

**Step 3.** Change the matrix \[
\begin{bmatrix}
21 & 18 \\
48 & 51
\end{bmatrix}
\]
back to letters to write the coded message: *URVY*.

**Step 4.** Repeat this as many times as necessary to encode a longer message.

a. Code *MEET ME AT* using the key \[
\begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix}
\].

To decode or decipher a message,

**Step 1.** Break the message up into groups of four letters and write as matrices using the corresponding numbers. Each letter-group matrix should be:

\[
\begin{bmatrix}
1 \text{st letter} & 2 \text{nd letter} \\
3 \text{rd letter} & 4 \text{th letter}
\end{bmatrix}
\]

**Step 2.** Find the inverse of the key matrix and multiply each letter-group matrix by the inverse.

b. The following message was also enciphered using the key

\[
\begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix}
\]:

*YTKOFOTISBGVITWKOULO*.

What is the original message?

c. Make up a code matrix and a coded message of your own. The inverse of your coding matrix must have a determinant of 1.
Notice that \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
x + 3y \\
2x - y
\end{bmatrix}.
\]

This means that it is possible to represent the system \[
\begin{cases}
x + 3y = 22 \\
2x - y = 2
\end{cases}
\]
as a matrix equation:
\[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
22 \\
2
\end{bmatrix}
\]

This is the matrix form of the system. The matrix \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\]
represents the coefficients of the variables, so it is called the coefficient matrix. The matrix \[
\begin{bmatrix}
22 \\
2
\end{bmatrix}
\]
contains the constants on the right sides of the equations. It is called the constant matrix for this system.

A system in matrix form can be solved using matrix multiplication. Just as the Multiplication Property of Equality allows both sides of an equation to be multiplied by any number, both sides of a matrix equation can be multiplied by any matrix. To solve, we multiply by the inverse of the coefficient matrix. By the theorem in Lesson 5-5,

the inverse of \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\]
is found to be \[
\begin{bmatrix}
\frac{3}{7} & \frac{-1}{7} \\
\frac{2}{7} & \frac{1}{7}
\end{bmatrix}
\]

Multiply both sides of the matrix equation by this inverse of the coefficient matrix. Because matrix multiplication is not commutative, the inverse matrix must be at the left on each side of the equation.

\[
\begin{bmatrix}
\frac{1}{7} & \frac{3}{7} \\
\frac{2}{7} & \frac{1}{7}
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{7} & \frac{3}{7} \\
\frac{2}{7} & \frac{1}{7}
\end{bmatrix}
\begin{bmatrix}
22 \\
2
\end{bmatrix}
\]

After the matrices are multiplied, the equation becomes
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
4 \\
6
\end{bmatrix}
\]

The presence of the identity matrix verifies that the inverse matrix was calculated correctly. Thus
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
4 \\
6
\end{bmatrix},
\]
or \(x = 4\) and \(y = 6\). You are asked to check this solution in Question 3 at the end of this lesson.
In general, to solve the system \[ \begin{align*}
ax + by &= e \\
 cx + dy &= f
\end{align*} \] by using matrices, rewrite the system as a matrix equation
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
e \\
f
\end{bmatrix},
\]
which is of the form
\[
M \begin{bmatrix}
x \\
y
\end{bmatrix}
= K.
\]
Then multiply both sides of the equation by \( M^{-1} \).
\[
M^{-1}M \begin{bmatrix}
x \\
y
\end{bmatrix}
= M^{-1}K
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= M^{-1}K
\]
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= M^{-1}K
\]
The last equation shows that the solution of a system is the product of the inverse of the coefficient matrix and the constant matrix.

**Example 1** Use matrices to solve \( \begin{align*}
9x &= 3 + y \\
2x - 3y &= 5
\end{align*} \).

**Solution** Rewrite the first equation so that it can be put in matrix form.
\[
\begin{align*}
9x - y &= 3 \\
2x - 3y &= 5
\end{align*}
\]
This is equivalent to the matrix equation
\[
\begin{bmatrix}
9 & -1 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
3 \\
5
\end{bmatrix}
\]
The inverse of the coefficient matrix is
\[
\begin{bmatrix}
3 & -1 \\
25 & 25
\end{bmatrix}
\]
or \[
\begin{bmatrix}
.12 & -.04 \\
.08 & -.36
\end{bmatrix}
\]
Multiply both sides of the matrix equation by the inverse matrix; the inverse matrix is always placed on the left.
\[
\begin{bmatrix}
.12 & -.04 \\
.08 & -.36
\end{bmatrix}
\begin{bmatrix}
9 & -1 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
.12 & -.04 \\
.08 & -.36
\end{bmatrix}
\begin{bmatrix}
3 \\
5
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
.16 \\
-1.56
\end{bmatrix}
\]
So the solution is \( x = .16 \) and \( y = -1.56 \).
Check: Does $9 \cdot .16 = 3 + 1.56$? Yes, both sides equal 1.44.
Does $2 \cdot .16 - 3 \cdot 1.56 = 5$? Yes.

Matrices provide an easy way to tell when linear systems have exactly one solution. The system
\[
\begin{align*}
ax + by & = e \\
 cx + dy & = f
\end{align*}
\]

has exactly one solution only if the inverse of \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
exists. This inverse exists if and only if its determinant, \(ad - bc\), is not zero. This leads to the following theorem.

**Matrix-Solution Theorem:**

A 2 × 2 system has exactly one solution if and only if the determinant of the coefficient matrix is *not* zero.

When the determinant of the coefficient matrix is 0, there is no unique solution. To determine whether the system has infinitely many solutions or none at all, you should find a solution to one of the equations and test it in the other one. Consider the system
\[
\begin{align*}
6x - 9y & = 10 \\
62x - 93y & = 310
\end{align*}
\]

The determinant is \(ad - bc = 6 \cdot (-93) - (-9) \cdot (62) = 0\); so there is no unique solution. The point \((\frac{5}{3}, 0)\) satisfies the first equation, but not the second. Thus the system has no solution; it is inconsistent.

Computer programs can find inverses of large matrices (often with dozens or hundreds of variables) to solve linear systems. Without such programs, you will solve a system of three equations with three variables using 3 × 3 matrices. The identity matrix for 3 × 3 matrices is
\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The calculation of the inverse of a 3 × 3 matrix is complicated, so we give it.
Example 2  Solve this system

\[
\begin{align*}
2x - y + 3z &= 9 \\
x + 2z &= 3 \\
3x + 2y + z &= 10
\end{align*}
\]

by using the coefficient matrix \( M \) and its inverse \( M^{-1} \) shown below.

\[
M = \begin{bmatrix}
2 & -1 & 3 \\
1 & 0 & 2 \\
3 & 2 & 1
\end{bmatrix}
\quad M^{-1} = \begin{bmatrix}
\frac{4}{7} & -1 & 2 \\
\frac{5}{7} & 1 & 1 \\
\frac{2}{7} & 1 & -\frac{1}{7}
\end{bmatrix}
\]

Solution  Rewrite the system as a matrix equation.

\[
\begin{bmatrix}
2 & -1 & 3 \\
1 & 0 & 2 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
9 \\
3 \\
10
\end{bmatrix}
\]

Multiply both sides on the left by \( M^{-1} \) and simplify.

\[
\begin{bmatrix}
\frac{4}{7} & -1 & 2 \\
\frac{5}{7} & 1 & 1 \\
\frac{2}{7} & 1 & -\frac{1}{7}
\end{bmatrix}
\begin{bmatrix}
2 & -1 & 3 \\
1 & 0 & 2 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
\frac{4}{7} & -1 & 2 \\
\frac{5}{7} & 1 & 1 \\
\frac{2}{7} & 1 & -\frac{1}{7}
\end{bmatrix}
\begin{bmatrix}
9 \\
3 \\
10
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
5 \\
-2 \\
-1
\end{bmatrix}
\]

So the solution is \( x = 5 \), \( y = -2 \), and \( z = -1 \). This is easily checked.

You are not expected to find the inverse matrix for a \( 3 \times 3 \) system in this chapter. It will always be given.
Questions

Covering the Reading

In 1–3, refer to the system at the start of this lesson.

1. What does the matrix \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\] represent?

2. Noel Issen found the inverse \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\] and multiplied as shown here.

\[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
22 \\
2
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\]

What did Noel do wrong?

3. Check that (4, 6) is the solution of the system.

4. Solve the system of Example 1, using inverse matrices, if the order of equations is reversed.

5. How many solutions does the following system have?

\[
\begin{cases}
4x + y = 2 \\
9x - 2y = 4
\end{cases}
\]

Justify your answer.

6. Tell whether the following system has no solutions or infinitely many solutions.

\[
\begin{cases}
30x - 18y = 67 \\
35x - 21y = 76
\end{cases}
\]

In 7–9, \(M\) stands for the coefficient matrix of each system. What matrix does \(M^{-1} \cdot M\) equal for the following systems?

7. a system with two equations and two unknowns such as \[
\begin{cases}
ax + by = c \\
dx + ey = f
\end{cases}
\]

\[
\begin{cases}
4x - 2y + 3z = 1 \\
8x - 3y + 5z = 4 \\
7x - 2y + 4z = 5
\end{cases}
\]

8. the system

9. a \(3 \times 3\) system

In 10 and 11, solve each system using matrices.

10. \[
\begin{cases}
-8A - 3B = 10 \\
4A + 6B = 5
\end{cases}
\]

11. \[
\begin{cases}
10x + 15y = 30 \\
4x + 6y = 12
\end{cases}
\]
12. Refer to Example 2.
   a. Solve the system \[ \begin{align*}
   4x - 2y + 3z &= 1 \\
   8x - 3y + 5z &= 4 \\
   7x - 2y + 4z &= 5
\end{align*} \] using matrices. The coefficient matrix and its inverse are given here.
   \[
   M = \begin{bmatrix}
   4 & -2 & 3 \\
   8 & -3 & 5 \\
   7 & -2 & 4
\end{bmatrix}
   \quad
   M^{-1} = \begin{bmatrix}
   -2 & 2 & -1 \\
   3 & -5 & 4 \\
   5 & -6 & 4
\end{bmatrix}
   
   b. Check your answer.

**Applying the Mathematics**

In 13–14, for what value of \( n \) does

13. \[ \begin{align*}
   2x + 4y &= n \\
   x + 2y &= 7
\end{align*} \] have infinitely many solutions?

14. \[ \begin{align*}
   4x - 6y &= 5 \\
   2x + ny &= 2
\end{align*} \] have no solution?

15. a. Solve this matrix equation. \[ \begin{bmatrix}
   1 & 2 \\
   3 & 4
\end{bmatrix}
   \begin{bmatrix}
   w \\
   x
\end{bmatrix}
   =
   \begin{bmatrix}
   5 \\
   6
\end{bmatrix}
   
   b. What two systems does part a simultaneously solve?

**Review**

In 16 and 17, solve using any method. (Lessons 5-3, 5-4)

16. \[ \begin{align*}
   3x - 2 &= z \\
   x &= y + 9 \\
   z &= 8x
\end{align*} \]

17. \[ \begin{align*}
   a &= 20b + 11 \\
   a &= 3b - 6
\end{align*} \]

In 18 and 19, two cake mixes contain percents of the U.S. RDA (recommended daily amounts) of vitamin C as given here. (Lesson 5-3)

<table>
<thead>
<tr>
<th>Mix</th>
<th>Vitamin C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.0%</td>
</tr>
<tr>
<td>B</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

18. Duncan Crocker wants to mix some of each kind to obtain 200 oz of mixture that contains 1.5% of the U.S. RDA of vitamin C. How much of each should he use?
19. Betty Hines wants 100 oz of a mixture that contains 1.8% of the U.S. RDA of vitamin C. How should she make the mixture?

20. Write an inequality for the graph below. (Lesson 3-9)

21. Graph \( \{ x: x > 11 \} \cap \{ x: x > 12 \} \). (Lesson 5-1)

22. a. Show how the two systems
\[
\begin{align*}
3w + 4y &= 5 \\
w + 2y &= 0
\end{align*}
\quad
\begin{align*}
3x + 4z &= 1 \\
x + 2z &= 3
\end{align*}
\]

can be rewritten as a single equation using three \(2 \times 2\) matrices.

b. When can two \(2 \times 2\) systems be rewritten as in part a?

c. Can three \(2 \times 2\) systems ever be rewritten as a single matrix equation?
The graph of a linear inequality in two variables is a half-plane. So the graph of the solution to a system of linear inequalities is the intersection of coplanar half-planes. The set of solutions to a system of linear inequalities is called the **feasible set** or **feasible region** for that system. Such feasible sets are often the interiors of angles or polygons. Sometimes boundaries are included in the sets and sometimes not. The intersections of the boundaries are called **vertices** of the feasible sets.

**Example 1**

Graph the feasible set of the system \[
\begin{align*}
    y &> -2x + 6 \\
    y &\leq \frac{1}{3}x - 3.
\end{align*}
\]

**Solution**

Graph each inequality on the same set of axes. The graph of \( y > -2x + 6 \) is the set of points above and to the right of the line with equation \( y = -2x + 6 \). In the graphs below, this set is indicated by the shading **blue**. The graph of \( y \leq \frac{1}{3}x - 3 \) consists of points on or below the line with equation \( y = \frac{1}{3}x - 3 \), indicated by the shading **pink**. The part of the plane marked with both types of shading is the feasible set for this system. As shown in the graph at the bottom, in this example the feasible set is the interior and one side of an angle.
Check  Find the vertex of the feasible set by solving the system of equations

\[
\begin{align*}
y &= -2x + 6 \\
y &= \frac{1}{4}x - 3.
\end{align*}
\]

This vertex is (4, -2), which checks with the graph. Pick a point in the shaded region, such as (8, -5), and substitute it into each inequality.

- Is \(-5 > -2(8) + 6\)? Yes.
- Is \(-5 \leq \frac{1}{4} (8) - 3\)? Yes. So the solution checks.

You should also try points outside the region to show they do not work. However, this is not a fool-proof check.

The applications of systems to production planning in industry, discovered by Kantorovich in 1939, may involve systems with thousands of variables. Computers are needed to work out the problems. But simple examples can be done by hand. Here is a simplified example of a business application.

Example 2 The Biltrite Furniture company makes wooden desks and chairs. Carpenters and finishers work on each item. On the average the carpenters spend four hours working on each chair and eight hours on each desk. There are enough carpenters for up to 8000 worker-hours per week. The finishers spend about two hours on each chair and one hour on each desk. There are enough finishers for a maximum of 1300 worker-hours per week. Given the above constraints, find the feasible region for the number of chairs and desks that can be made per week.
Solution  Make a table to illustrate the information.

<table>
<thead>
<tr>
<th>CHAIRS</th>
<th>DESKS</th>
<th>TOTAL HOURS AVAILABLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours of carpentry per piece</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Hours of finishing per piece</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Then identify the variables.
Let \( x = \) number of chairs to be made per week,
\( y = \) number of desks to be made per week.

Write sentences to model each aspect of the manufacturing process. Because \( x \) and \( y \) represent pieces of furniture, they must be any non-negative integers; that is,
\[
x \geq 0 \quad \text{and} \quad y \geq 0.
\]

The carpentry hours must satisfy
\[
4x + 8y \leq 8000.
\]

The finishing hours must satisfy
\[
2x + y \leq 1300.
\]

Thus the feasible region for this situation is the solution of the system
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
4x + 8y &\leq 8000 \\
2x + y &\leq 1300.
\end{align*}
\]

Now graph the solution to the system. The first two inequalities indicate all solutions are in the first quadrant or on the positive axes. So it is sufficient to graph the last two inequalities only in the first quadrant to find the feasible region. Although only integer values are solutions, shading is used because it would be too difficult to show all the dots. In this case and many others, the context of the situation is necessary to interpret the graph accurately.
The intersection of the half-planes is the feasible region. This region is the union of the quadrilateral QRST and its interior. You can find the coordinates of each vertex of the quadrilateral by solving a system of equations. For instance, Q is the solution of the system

\[
\begin{align*}
4x + 8y &= 8000 \\
2x + y &= 1300.
\end{align*}
\]

**Check** Choose any point in the feasible region and see if it is a solution of each inequality of the system. We choose (400, 200).

Is 400 ≥ 0? Yes.
Is 200 ≥ 0? Yes.
Is 4(400) + 8(200) ≤ 8000? Yes.
Is 2(400) + 200 ≤ 1300? Yes.

So one option available to the Biltrite Furniture Company is to make 400 chairs and 200 desks per week.

As a further check, choose a point outside the shaded region, like (700, 400), and show that there is at least one inequality in which it does not work.

**Questions**

1. The solution to a system of linear inequalities can be represented by the ___ of half-planes.
2. What is the solution to a system of linear inequalities often called?

In 3–5, refer to Example 1. Verify that:

3. (10, -6) is a solution to the system.
4. (5, 0) is not a solution to the system.
5. (4, -2) is the vertex of the feasible set for the system.

In 6–10, refer to Example 2.

6. Which inequality expresses the amount of time that the company can have its finishers working?

7. Why is it sufficient to consider only the first quadrant in graphing the feasible set?

8. Find the coordinates of vertex Q.

9. What system of equations gives vertex R as its solution?

10. Could the company manufacture 600 chairs and 200 desks per week under the given operating conditions?
11. Refer to the graph below.
   a. Find the coordinates of vertex B of the feasible set.
   b. Write a system of inequalities represented by this feasible set.

   ![Graph](image)

   In 12 and 13, (a) graph the feasible region of each system of inequalities: and (b) find the coordinates of each vertex.

   12. \[
   \begin{align*}
   y & \geq 3x + 1 \\
   y & \leq -2x + 4
   \end{align*}
   \]

   13. \[
   \begin{align*}
   x + 3y & \leq 18 \\
   2x + y & \leq 16 \\
   x & \geq 0 \\
   y & \geq 2
   \end{align*}
   \]

14. A clothier makes women's suits and coats from nylon and wool. Each suit requires 2 yards of nylon lining and 3 yards of wool. Each coat requires 3 yards of nylon lining and 4 yards of wool. Only 42 yards of nylon lining and 58 yards of wool are in stock.
   a. Let \( s \) be the number of suits and \( c \) be the number of coats. Complete the translation of this situation into a system of inequalities:
      \( s \geq \_; c \_; 2s + 3c \leq \_; \_ \leq 58 \)
   b. Graph the feasible region for this system and label the vertices.
      (Let \( s \) be the independent variable.)

15. An electronics firm makes two kinds of televisions: black-and-white and color. The firm has enough equipment to make as many as 1000 black-and-white sets per month or 600 color sets per month. It takes 20 worker-hours to make a black-and-white set and 30 worker-hours to make a color set. The firm has up to 24,000 worker-hours of labor available each month. Let \( x \) be the number of black-and-white TVs and \( y \) be the number of color TV's made in a month.
   a. Translate this situation into a system of inequalities.
   b. Graph the feasible set for this system and label the vertices.

16. Name four methods of solving systems of equations. (Lessons 5-2, 5-3, 5-4, 5-6)

   In 17–19, (a) state a method which would solve each system conveniently and (b) solve each system. (Lessons 5-2, 5-3, 5-4, 5-6)

   17. \[
   \begin{align*}
   y &= 4x \\
   8x + 7y &= 18
   \end{align*}
   \]

   18. \[
   \begin{align*}
   5x + 6y &= 18 \\
   -5x + 7y &= 21
   \end{align*}
   \]

   19. \[
   \begin{align*}
   \frac{x}{y} &= 8 \\
   y &= x^2
   \end{align*}
   \]
20. Solve the following matrix equation. *(Lesson 5-6)*

\[
\begin{bmatrix}
-1 & 2 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= 
\begin{bmatrix}
-6 \\
8 \\
\end{bmatrix}
\]

21. A region of the plane is said to be **convex** if and only if any two points of the region can be connected by a line segment which is itself entirely within the region. The pentagon below is convex but the quadrilateral is not.

Which of the following shaded regions are convex? *(Previous course)*

(a) ![Shaded region A](image1.png)  
(b) ![Shaded region B](image2.png)  
(c) ![Shaded region C](image3.png)  
(d) ![Shaded region D](image4.png)

22. Solve. *(Previous course)*

a. \(x^2 + 7 = 56\)

b. \((y - 3)^2 = 81\)

23. a. At most how many pieces can you get out of a circular pie with 4 straight cuts? How many of these pieces are convex?

b. At most how many pieces can you get out of a circular pie with \(n\) straight cuts? How many of these pieces are convex?
In Example 2 of Lesson 5-7, a system of linear inequalities models some of the manufacturing operations of the Biltrite Furniture Company. The feasible set describes various linear combinations of chairs and desks that the company can make with the given constraints. Now suppose that the company also knows that it earns a profit of $15 on each chair and $20 on each desk it makes. Given the known constraints, how can the production schedule be set up to maximize the profit?

If \( x \) chairs and \( y \) desks are sold, the profit \( P \) is given by the formula:

\[
15x + 20y = P.
\]

For instance, the solutions to

\[
15x + 20y = 3000
\]

are ordered pairs that will yield a $3000 profit.

The figure below again shows the feasible region for Biltrite’s system of inequalities. The figure also shows the graphs of some lines that result from substituting different values of \( P \) into the profit formula.
All lines with equations of the form \(15x + 20y = P\) are parallel because each has slope \(-\frac{3}{2}\). Some of these lines intersect the feasible region and some do not. In the figure above, the lines above \(L_3\) do not intersect the feasible set. Lines that do intersect the feasible region represent possible profits. The greatest profit will occur when the line with equation \(15x + 20y = P\) is as high as possible, but still intersects the feasible region. This will happen when the profit line passes through vertex \(Q = (200, 900)\). This is the line \(L_3\). Thus to maximize profits, the company should manufacture 200 chairs and 900 desks per week. So the maximum profit under these conditions is $21,000.

Problems such as this one, which lead to systems of linear inequalities, are called linear-programming problems. The word “programming” does not refer to a computer; it means that the solution gives a “program,” or course of action, to follow. The most profitable “program” for Biltrite Furniture Company is to make 200 chairs and 900 desks per week.

In 1826, the French mathematician Joseph Fourier proved the following theorem:

**Linear-Programming Theorem:**

The feasible region of a linear programming problem is convex, and the maximum or minimum quantity is determined at one of the vertices of the region.

The Linear-Programming Theorem tells you where to look for the greatest or least value of a linear combination expression in a linear-programming situation, without having to draw many lines through the feasible region.

In 1945, George Stigler (then at Columbia University, now at the University of Chicago) was looking for the cheapest diet that would provide a person’s daily needs of calories, proteins, calcium, iron, vitamin A, thiamine, riboflavin, niacin, and ascorbic acid. He considered 70 possible foods and found that the lowest-cost diet was a combination of wheat flour, cabbage, and pork liver. By mixing amounts of these, a person was thought to be able to live in good health for $59.88 a year (then). Costs today are higher, so it might now cost $400 a year for that diet.

Here is a simplified diet problem, of the type first considered by Stigler.
Stuart Dent decided to investigate one of his typical meals, fried chicken and corn on the cob. He compiled the data in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Vitamin A</th>
<th>Potassium (mg)</th>
<th>Iron (mg)</th>
<th>Calories</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fried Chicken</td>
<td>100</td>
<td>0</td>
<td>1.2</td>
<td>122</td>
</tr>
<tr>
<td>Corn</td>
<td>310</td>
<td>151</td>
<td>1.0</td>
<td>70</td>
</tr>
</tbody>
</table>

Stuart let \( f \) = the number of pieces of chicken and \( e \) = the number of ears of corn. After deciding the minimum amounts of each needed from this meal he wrote the system:

\[
\begin{align*}
    f & \geq 0 \\
    e & \geq 0 \\
    100f + 310e & \geq 1000 \quad \text{(at least 1000 units vitamin A)} \\
    151e & \geq 200 \quad \text{(at least 200 mg potassium)} \\
    1.2f + e & \geq 6 \quad \text{(at least 6 mg iron)} \\
    122f + 70e & \geq 600 \quad \text{(at least 600 Calories)}
\end{align*}
\]

His graph is shown below.

a. Find the vertices of the feasible region.

b. Apply the Linear-Programming Theorem to determine the values of \( f \) and \( e \) that will meet the nutritional requirements above at the lowest cost \( C \), where \( C = .90f + .75e \). (That is, each piece of chicken costs 90¢ and an ear of corn costs 75¢.)
Solution  

a. There are three vertices: R, S, and T. Vertex R is the e-intercept of the line with equation $122f + 70e = 600$. Substituting 0 for $f$ gives

$$122(0) + 70e = 600.$$  
$$e = 8.6$$

So the coordinates of R are about (0, 8.6).

Vertex S is the intersection of the lines with equations $122f + 70e = 600$ and $100f + 310e = 1000$. First divide each equation so the coefficient of $f$ becomes 1 or -1; then solve by the linear-combination method.

$$122f + 70e = 600 \quad \text{(divide by 122)} \Rightarrow \quad f + .57e = 4.92$$
$$100f + 310e = 1000 \quad \text{(divide by -100)} \Rightarrow \quad -f - 3.10e = -10$$

(add) \Rightarrow \quad -2.53e = -5.08 \quad \Rightarrow \quad e = 2$$

Substituting 2 for e into either equation gives $f = 3.8$; so the coordinates of S are about (3.8, 2).

Finally, T is the intersection of $100f + 310e = 1000$ and $151e = 200$. By solving the second equation and substituting into the first, Stuart found that the coordinates of T are about (6, 1.3).

b. The feasible region is a convex set. The Linear-Programming Theorem says that the minimum of $C = .90f + .75e$ occurs at a vertex. Stuart evaluated C at each vertex to see which combination of $f$ and $e$ gave the minimum cost. To make sense in this context, the coordinates of each vertex must be rounded off to the next highest integer. Thus $R = (0, 9)$, $S = (4, 2)$, and $T = (6, 2)$.

At $R = (0, 9)$: $C = .90(0) + .75(9) = 6.75$
At $S = (4, 2)$: $C = .90(4) + .75(2) = 5.10$
At $T = (6, 2)$: $C = .90(6) + .75(2) = 6.90$

Thus the minimum value of $C$ is $5.10$. It occurs at vertex S, that is, when Stuart has 4 pieces of chicken and 2 ears of corn.

Of course many other vitamins, minerals, and foods are taken into account by dietitians planning well-balanced meals. Stigler could not consider a greater number of possible foods nor consider more health needs because computers were not available in 1945. Today it is possible to consider hundreds of foods and many more daily needs than he did.
Covering the Reading

In 1–5, refer to the discussion of the Biltrite Furniture Company at the beginning of this lesson.

1. What is the company trying to maximize?

2. What do 15 and 20 represent in the profit equation?

3. True or false If a line with equation $15x + 20y = P$ intersects the feasible region for the system of inequalities, it is possible for Biltrite to make a profit of $P$ dollars.

4. The maximum weekly profit Biltrite can earn is ___. This occurs when the company produces ___ chairs and ___ desks.

5. Find the profit if 199 chairs and 899 desks are made.

6. To what does the word “program” refer in a linear-programming problem?

7. In a linear-programming problem, why is it necessary to find the vertices of the feasible region?

In 8 and 9, refer to the example in this lesson.

8. a. Which linear combination must be minimized?
   b. The minimum value of $C$ satisfying the constraints of the problem is ___. It occurs at vertex ___.

9. If Stuart had needed more iron in his diet he might have written the linear combination for iron as $1.2f + e \geq 10$.
   a. Regraph the feasible region of the system with this new iron requirement.
   b. In the new feasible region, which vertex yields the minimum cost?

10. a. A diet problem like the one in the example was first modeled mathematically by whom and in what year?
    b. Why can more variables be dealt with now than could be considered when diet problems were first done?

11. Use the feasible set graphed below. Which vertex maximizes the given profit equation, $P = 3x + 4y + 250$?
In 12–14, suppose that a farmer has no more than 50 acres for planting alfalfa and soy beans and has a maximum of $1200 to spend on the planting. It costs $20 per acre to plant alfalfa and $30 per acre to plant soy beans. The profit per acre for alfalfa is $250 and for soy beans is $300. If $A$ is the number of acres of alfalfa and $S$ is the number of acres of soy beans that the farmer plants, the system for this problem is:

\[
\begin{align*}
A + S & \leq 50 \\
20A + 30S & \leq 1200 \\
A & \geq 0 \\
S & \geq 0
\end{align*}
\]

12. Match each inequality in the system with its meaning.
   a. $A + S \leq 50$  
      (i) The number of acres of soybeans is not negative.
   b. $20A + 30S \leq 1200$  
      (ii) The total number of acres is not more than 50.
   c. $A \geq 0$  
      (iii) The cost of planting must be no more than $1200.
   d. $S \geq 0$  
      (iv) The least number of acres of alfalfa is zero.

13. Graph the feasible region. Let $A$ be the independent variable.

14. a. Find the vertices of the feasible region.
   b. The profit formula is $P = 250A + 300S$. At which vertex is $P$ maximized?

15. A landscaping contractor uses a combination of two brands of fertilizers, each containing different amounts of phosphates and nitrates, as shown in the table below. A certain lawn requires a mixture of at least 24 lb of phosphates and at least 16 lb of nitrates.

<table>
<thead>
<tr>
<th></th>
<th>Phosphate content per package</th>
<th>Nitrate content per package</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brand A</td>
<td>4 lb</td>
<td>2 lb</td>
</tr>
<tr>
<td>Brand B</td>
<td>6 lb</td>
<td>5 lb</td>
</tr>
</tbody>
</table>

If $x$ is the number of packages of Brand $A$ and $y$ is the number of packages of Brand $B$, then the conditions of the problem can be modeled by the following system of inequalities:

\[
\begin{align*}
x & \geq 0 \\
y & \geq 0 \\
4x + 6y & \geq 24 \\
2x + 5y & \geq 16
\end{align*}
\]

a. Graph the feasible region.
   b. If a package of Brand $A$ costs $6.99 and a package of Brand $B$ costs $17.99, then the cost $C$ is found by the equation $C = 6.99x + 17.99y$. Which pair $(x, y)$ in the feasible region gives the lowest cost?
16. Refer to Question 15 in Lesson 5–7. Suppose the electronics firm earns a profit of $25 on each black-and-white TV and $40 on each color TV.
   a. Write a formula for the monthly profit earned.
   b. What combination of black-and-white and color TV’s will maximize profit?

17. Find an equation of the line through the points (-2, -4) and (5, 7).  
   (Lesson 3-5)

18. Multiple choice Which of the following systems describes the graph below?  
   (Lesson 5-7)

   (a) $2y + x \geq 6$  
   (b) $2y + x \leq 6$  
   (c) $2y + x < 6$  
   $y - 3x \leq 3$  
   $y - 3x \geq 3$  
   $y - 3x > 3$

19. The strength $S$ of a rectangular beam varies directly as its width $w$ and the square of its depth $d$, and varies inversely as its length $L$. Suppose a beam can support 1750 pounds, and its dimensions are $w = 4''$, $d = 8''$, and $L = 20$ feet. What is the strength of a beam of the same material where $w = 4''$, $d = 8''$, and $L = 25$ feet?  
   (Lesson 2-9)

20. Suppose $a + b + c + d = 100$, and $a$, $b$, $c$, and $d$ are all nonnegative. What are the largest and smallest possible values of $abcd$?
In Lesson 5–8, you practiced using the Linear-Programming Theorem for a given feasible region and linear combination expression to be maximized or minimized. In this lesson you will learn to solve linear-programming problems from scratch. Because linear-programming problems are long and involved, you must be very organized and neat. To solve a linear programming problem:

1. Identify the variables.
2. Translate the constraints of the problem into a system of inequalities relating the variables. If necessary, make a table.
3. Graph the system of inequalities; find the vertices of the feasible set.
4. Write a formula or an expression to be maximized or minimized.
5. Apply the Linear-Programming Theorem.
6. Interpret the results.

The bolder type in the solution to the example is what you should write. The rest is explanation, or what you might think as you solve the problem.

**Example** Some students make necklaces and bracelets in their spare time and sell all that they make. Every week they have available 10,000 g of metal and 20 hours to work. It takes 50 g of metal to make a necklace and 200 g to make a bracelet. Each necklace takes 30 minutes to make and each bracelet takes 20 minutes. The profit on each necklace is $3.50, and the profit on each bracelet is $2.50. The students want to earn as much money as possible. Because you are taking this algebra course, they ask you to give them advice. What numbers of necklaces and bracelets should they make each week?

**Solution** 1. Identify the variables.

   Let \( x \) = the number of necklaces to be made per week
   and \( y \) = the number of bracelets to be made per week.
2. The following table summarizes the information about production.

<table>
<thead>
<tr>
<th></th>
<th>Metal Used</th>
<th>Time to Make</th>
</tr>
</thead>
<tbody>
<tr>
<td>for each necklace</td>
<td>50 g</td>
<td>30 min</td>
</tr>
<tr>
<td>for each bracelet</td>
<td>200 g</td>
<td>20 min</td>
</tr>
<tr>
<td>Total available</td>
<td>10,000 g</td>
<td>20 hours</td>
</tr>
</tbody>
</table>

Translate the constraints into a system. Negative numbers cannot be used; thus:

\[ x \geq 0 \]
\[ y \geq 0 \]

The amount of metal (in grams) used satisfies

\[ 50x + 200y \leq 10,000. \]

The amount of time (in minutes) needed satisfies

\[ 30x + 20y \leq 1200. \]

3. Graph the system and find the vertices. Only the feasible set is shown below.

The vertices are found by solving systems of equations. For instance, \( Q \) is found by solving \( 50x + 200y = 10,000 \) and \( 30x + 20y = 1200 \).

4. Write a formula to be maximized. The profit formula is

\[ P = 3.50x + 2.50y. \]

\( P \) is to be maximized.
5. Apply the Linear-Programming Theorem. Substitute the coordinates of each vertex into the profit formula.

\[
\begin{align*}
\text{For } (0,0) & \quad P = 3.5(0) + 2.5(0) = 0 \\
\text{For } (40, 0) & \quad P = 3.5(40) + 2.5(0) = 140 \\
\text{For } (8, 48) & \quad P = 3.5(8) + 2.5(48) = 148 \\
\text{For } (0, 50) & \quad P = 3.5(0) + 2.5(50) = 125
\end{align*}
\]

6. Interpret the results. The maximum profit of $148 occurs at vertex \( Q = (8, 48) \). If they want to satisfy all the conditions, the students should make 8 necklaces and 48 bracelets each week.

Linear programming is often used in industries in which all the competitors make the same product (such as oil, gasoline, paper, milk, and so on). Efficiency in the use of labor and materials determines the amount of profit. These situations can involve as many as 5000 variables and 10,000 inequalities. Although we use graphing to solve linear programming problems in this book, more efficient methods of solution are used by computers. The most important procedure is the simplex algorithm invented in 1947 by the econometrician Leonid Hurwicz and the mathematicians George Dantzig and T.C. Koopmans, all from the United States. For this work, Koopmans shared the Nobel Prize with Kantorovich in 1982.

Questions

In 1–4, refer to the example in this lesson.

1. What are the students trying to find?

2. What is the \( x \)-intercept of the equation that limits the amount of metal to be used? Is it in the feasible region?

3. How much more profit do the students make with the linear combination at \( (8, 48) \) than at the next best vertex?

4. a. What would be the profit if the students made 9 necklaces and 47 bracelets?
   b. Why can they not do this?
5. Suppose the students in the example decide to put semi-precious gems in their jewelry: six in each necklace and one in each bracelet. They can use 150 gems each week.
   a. Translate this constraint into an inequality.
   b. The entire system, including this new constraint, is graphed below. Find the new vertices, T and U.
   c. Do the students need to change their program to keep profits at a maximum? Justify your answer.

   ![Graph of the system of inequalities]

6. a. Name a method for solving linear programming problems without graphing.
   b. Who developed this method, and when?

7. Some parents shopping for their family want to know how much hamburger and how many potatoes to buy. From a food-value table they find that one ounce of hamburger has .8 mg of iron, 10 units of vitamin A, and 6.5 grams of protein. One medium potato has 1.1 mg of iron, 0 units of vitamin A, and 4 grams of protein. For this meal the parents want to serve at least 5 mg of iron, 30 units of vitamin A, and 35 grams of protein. One potato costs $0.05 and 1 ounce of hamburger costs $0.11. The parents want to be economical (minimize their costs), yet meet daily requirements. They need a program for the quantity of hamburger and potatoes to buy for the family.
   a. Identify the variables for this problem.
   b. Translate the constraints of the problem into a system of inequalities. (You should have five inequalities; a table may help.)
   c. Graph the system of inequalities in part b, and find the vertices of the feasible set.
   d. Write an expression for the cost (to be minimized).
   e. Apply the Linear-Programming Theorem to determine which vertex minimizes the cost expression of part d.
   f. Interpret your answer to part e. What is the best program for this family?
8. A company makes two kinds of tires: model \( R \) (regular) and model \( S \) (snow). Each tire is processed on three machines, \( A \), \( B \), and \( C \). To make one model \( R \) requires \( \frac{1}{2} \) hour on machine \( A \), 2 hours on \( B \), and 1 hour on \( C \). To make one model \( S \) requires 1 hour on \( A \), 1 hour on \( B \), and 4 hours on \( C \). During the next week machine \( A \) will be available for at most 20 hours, machine \( B \) for at most 60 hours, and machine \( C \) for at most 60 hours. If the company makes a $10 profit on each model \( R \) tire and a $15 profit on each model \( S \) tire, about how many of each tire should be made to maximize the company’s profit?

9. An alloy containing 65% aluminum is made by melting together two alloys that are 25% aluminum and 75% aluminum. How many kilograms of each alloy must be used to produce 160 kilograms of the 65% alloy? (Lesson 5-3)

10. A formula for the \( n \)th term of a sequence is

\[
a_n = 10 - .5(n - 1).
\]

a. Write the first three terms.
b. Write a recursive formula for this sequence.
c. Solve the explicit formula for \( n \).
d. One term of the sequence is -39. Which term is it? (Lessons 1-3, 1-4, 1-8)

11. In the NFL passer rating system, a quarterback who completes \( x\% \) of his passes is awarded \( p \) points as follows:

- if \( x \leq 30 \), \( p = 0 \)
- if \( 30 < x < 77.5 \), \( p = \frac{1}{20} (x - 30) \)
- if \( x \geq 77.5 \), \( p = 2.375 \)

Graph the ordered pairs \((x, p)\). (Note: Quarterbacks also get points for touchdowns and yards gained and lose points for interceptions.) (Lesson 3-8)

12. **Skill sequence Solve.** (Previous course)

   a. \( x^2 = 49 \)
   b. \( x^2 + 2 = 51 \)
   c. \( x^2 + 2 = 49 \)

13. In this and the last lesson, several situations are given that lead to linear-programming problems. For instance, one situation is found in the example of this lesson, and one is in Question 7 above. Make up another situation that would lead to this kind of problem.
Summary

When two or more sentences are joined by the words and or or, a compound sentence results. The solution set to \( A \) or \( B \) is the union of the solution sets of \( A \) and \( B \). If the word joining them is and, the compound sentence is called a system. The solution set to \( A \) and \( B \) is the intersection of the solution sets of \( A \) and \( B \).

Systems have many applications and may contain any number of variables. If the system contains one variable, then its solutions can be graphed on a number line. If the system contains two variables, then its solutions can be graphed in the plane. Graphing in the plane often tells you the number of solutions but may not yield the exact solutions.

This chapter deals primarily with ways of solving systems of linear equations and inequalities in two variables. Three methods for solving linear equations use linear combinations, substitution, and matrices. The matrix method converts a system of two equations in two unknowns to a single matrix equation. To get the solution, both sides of the equation are then multiplied by the inverse of the coefficient matrix.

The graph of a single linear inequality is a half-plane. For a system of two linear inequalities, if the boundary lines intersect, then the graph is the interior of an angle and perhaps one or both of its sides.

Systems with two variables but more than two inequalities arise in linear-programming problems. In a linear-programming problem, you look for a solution to the system that maximizes or minimizes the value of a particular expression. You first find the set of solutions to the system. This feasible set is always a convex region. The Linear-Programming Theorem states that the desired point must be a vertex of the feasible set, so all vertices are tried. Applications of linear programming are a recent development in mathematics and are quite important in industry.

Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description and a specific example.

**Lesson 5-1**
compound sentence, * union of sets
* intersection of sets

**Lesson 5-2**
system, * solution set for a system
consistent, inconsistent system

**Lesson 5-3**
* equivalent systems
linear-combination method

**Lesson 5-4**
substitution method

**Lesson 5-5**
* inverse \( M^{-1} \) of a matrix \( M \)
Inverse-Matrix Theorem, square matrix
* determinant of a \( 2 \times 2 \) matrix \( M \), det \( M \)

**Lesson 5-6**
matrix form of a system
coefficient matrix, constant matrix
Matrix-Solution Theorem

**Lesson 5-7**
* feasible set, feasible region
* vertices of feasible region
convex region

**Lesson 5-8**
linear-programming problem
Linear-Programming Theorem

300
Take this test as you would take a test in class. Use graph paper and a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. On a number line, graph \( \{ x : x \geq -5 \} \cap \{ x : x \geq 7 \} \).

2. A graph of the system \[
\begin{align*}
y &= \frac{1}{2}x - 2 \\
y &= -x^2
\end{align*}
\]
is shown below. Approximate the solutions to the system.

3. Consider the system \[
\begin{align*}
2x - 9y &= 8 \\
2x - 9y &= -7
\end{align*}
\]
a. Is this system inconsistent?
b. Why or why not?

4. \[
\begin{align*}
s &= 4t \\
r &= t + 11
\end{align*}
\]

5. \[
\begin{align*}
-3x + 3y &= 2 \\
-4x - 2y &= 3
\end{align*}
\]

6. At Eggs-N-Links Restaurant you can get a Double Duo Breakfast of 2 eggs with 2 sausage links for $2.78 and a Triple Quad Breakfast of 3 eggs with 4 sausage links for $4.99. From this information, what might Eggs-N-Links charge for 1 egg?

In 7 and 8, consider the system \[
\begin{align*}
8x + 3y &= 41 \\
6x + 5y &= 39
\end{align*}
\]

7. Find the inverse of the coefficient matrix.

8. Use a matrix equation to solve the system.

9. **Multiple choice** The graph below shows the feasible set for which system?

   (a) \[
   \begin{align*}
y &\leq x \\
x &\geq 2
   \end{align*}
   \]

   (b) \[
   \begin{align*}
y &> x \\
x &\leq 2
   \end{align*}
   \]

   (c) \[
   \begin{align*}
y &> x \\
x &< 2
   \end{align*}
   \]

   (d) \[
   \begin{align*}
y &< x \\
x &< 2
   \end{align*}
   \]

In 10–12, a furniture manufacturer makes upholstered chairs and sofas. On the average it takes carpenters 7 hours to build a chair and 4 hours to build a sofa. There are enough carpenters for no more than 133 worker-hours per day. Upholsterers average 2 hours per chair and 6 hours per sofa. There are enough upholsterers for no more than 72 worker-hours per day. The profit per chair is $80 and the profit per sofa is $70. How many sofas and chairs should be made per day to maximize the profit?

10. Translate the constraints into a system of linear inequalities.

11. Graph the system of inequalities and find the vertices of the feasible set.

12. Apply the Linear-Programming Theorem and interpret the results.

13. Which boundary is included in the solution set of \[
\begin{align*}
y &\leq 7 \\
y &> x + 2
\end{align*}
\]


Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

- **Objective A:** Solve systems using the linear-combination or substitution method. (Lessons 5-3, 5-4)

1. **Multiple choice** The system \( \begin{cases} 2x + 3y = 19 \\ 4x - y = 17 \end{cases} \) becomes \(-7y = -21\) if you:
   
   (a) multiply the first equation by -2 and add.
   
   (b) multiply the second equation by 3 and add.
   
   (c) multiply the first equation by 2, the second equation by -1 and add.
   
   (d) multiply the second equation by 3 and subtract.

2. Which choice in Question 1 does not help to solve the system?

In 3–8, solve and check.

3. \( \begin{cases} 2a - 4b = 18 \\ 3a - b = 22 \end{cases} \)

4. \( \begin{cases} 3m + 10n = 16 \\ m = -6n \end{cases} \)

5. \( \begin{cases} y = x - 4 \\ 2x - y = -2.5 \end{cases} \)

6. \( \begin{cases} 3x + 6y = -3 \\ -5x - 8y + 22 = 0 \end{cases} \)

7. \( \begin{cases} 2r + 15t = 6 \\ r = 3s \\ t = \frac{2}{3} s \end{cases} \)

8. \( \begin{cases} a = 3b - 2 \\ b = 4c + 5 \\ c = 5a + 1 \end{cases} \)

9. Consider the system \( \begin{cases} y = 5x \\ -3x + 2y = -28 \end{cases} \)

   a. Name three methods you can use to solve this system.

   b. Solve and check the system.

- **Objective B:** Find the inverse and determinant of a 2 \( \times \) 2 matrix. (Lesson 5-5)

In 10–15, give: (a) the determinant; (b) the inverse, if it exists.

10. \( \begin{bmatrix} 1 & 9 \\ -7 & 6 \end{bmatrix} \)  
11. \( \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \)

12. \( \begin{bmatrix} 6 & 4 \\ -3 & 2 \end{bmatrix} \)

13. \( \begin{bmatrix} 1 & 4 \\ -3 & 6 \end{bmatrix} \)

14. \( \begin{bmatrix} 2 & -4 \\ 5 & -10 \end{bmatrix} \)

15. \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

- **Objective C:** Use matrices to solve systems of equations. (Lesson 5-6)

In 16–18, solve each system using matrices.

16. \( \begin{cases} 2x - 9y = 14 \\ 6x - y = 42 \end{cases} \)

17. \( \begin{cases} 4a - 5b = -19 \\ 3a + 7b = 18 \end{cases} \)

18. \( \begin{cases} 3m = 4n + 5 \\ 2m = 3n - 6 \end{cases} \)

19. \( \begin{cases} \frac{1}{2} = 3x - 4y \\ 3 = x + 8y \end{cases} \)
Objective D: Recognize properties of systems of equations. (Lessons 5-2, 5-3, 5-4, 5-6)

20. Are the systems \[ \begin{cases} 3x - y = 19 \\ 5x + 2y = 39 \end{cases} \] and \[ \begin{cases} x = 7 \\ x + y = 9 \end{cases} \] equivalent?

21. Give the simplest system equivalent to \( 3x = 6 \) and \( x + y = 10 \).

22. A system with no solutions is called _inconsistent_.

In 23–25, (a) identify each system as inconsistent or consistent; (b) give the number of solutions.

23. \[ \begin{cases} 3x + 5y = 15 \\ 3x + 5y = 45 \end{cases} \]

24. \[ \begin{cases} 6m - 4n = 9 \\ -3m = -2n - \frac{9}{2} \end{cases} \]

25. \[ \begin{cases} 8a - 5b = 40 \\ 2a + b = -6 \end{cases} \]

26. For what value of \( k \) does \( \begin{cases} 2x + ky = 6 \\ 14x + 7y = 42 \end{cases} \) have infinitely many solutions?

27. For what value of \( t \) does \( \begin{cases} 3x + 9y = t \\ 4x + 12y = 7 \end{cases} \) have infinitely many solutions?

Objective E: Recognize properties of systems of inequalities. (Lessons 5-7, 5-8)

28. True or false The boundaries are included in the solution set of \( \begin{cases} y > 2 \\ y < 4 - x \end{cases} \).

29. Which two of the following could be feasible regions in a linear-programming situation?

(a) \[ (c) \]

(b) \[ (d) \]

30. A system of inequalities was graphed as shown below. Are the coordinates listed below possible solutions to the system? Justify your answers.

a. (3, 2)

b. (6, 2)

31. Where in a feasible set are the possible solutions to a linear-programming problem?

32. Does the point \( M \) in the region below represent a possible solution to a linear-programming problem?
Objective F: Use linear systems to solve real-world problems. (Lessons 5-3, 5-4)

33. At Kit’s Kitchen, the Big Deal costs $3.50 for two hamburgers and one order of fries. The family pack costs $12.00 for six burgers and six orders of fries. If the prices are constant, how much does one hamburger cost?

34. One night at the circus the big-top attraction sold out, selling all 3,050 seats. There are four times as many lower-level seats as upper-level seats. How many upper-level seats are there?

35. Billy likes to mix two cereals for breakfast. He wants to reduce his sugar intake without giving up Sugar-O’s, his favorite cereal. Sugar-O’s contains 20% sugar while Health-Nut contains 5% sugar. How much of each cereal should he eat to fill a bowl with a 25 g mixture which is 10% sugar?

Objective G: Use linear programming to solve problems in the real world. (Lessons 5-8, 5-9)

36. Jocelyn’s Jewelry Store makes rings and pendants. Every week the staff uses at most 500 g of metal and spends at most 80 hours making jewelry. It takes 5 g of metal to make a ring and 20 g to make a pendant. Each ring takes 1.5 hours to make and each pendant takes 1 hour. The profit on each ring is $90 and the profit on each pendant $40. The store wants to earn as much profit as possible.
   a. Identify the variables and translate the constraints into a system of inequalities.
   b. Graph the system and find the vertices of the feasible set.
   c. Write an expression to be maximized.
   d. Apply the Linear-Programming Theorem and interpret the results.

37. Brenda is studying hard for two final exams, one in her Spanish class, the other in algebra. She figures that each Spanish vocabulary word she learns will mean an increased score of about 0.04 point. (That is, she expects that about 1 in 25 words will be on the test.) Each algebra question she reviews will mean an increased score of about 0.10 point. (About 1 in 10 questions will be on the test.) She has two difficulties: there are only 4 hours to study between now and the test and she has only 12 sheets of notebook paper, each with 35 lines. Each new vocabulary word takes about 2 minutes to learn. An algebra question averages about 4 minutes. A vocabulary word takes 3 lines of notebook paper (she writes them down again and again to work on them). A typical algebra question takes 10 lines.
   a. How much time should Brenda spend on Spanish and how much on algebra to maximize the increase in her total score?
   b. If she spends the time, what increase in score can she expect in Spanish and what increase in algebra?
Objective H: Graph compound sentences. (Lesson 5-1)
In 38–41, graph on a number line.
38. \(x: x > 9 \text{ and } x < 14\)
39. \(t: -2 \leq t < 7\) \(\cap \) \(t: t \geq 0\)
40. \(n: n > 5\) \(\cup\) \(n: n > 3\)
41. \(y: y \leq 4\) \(\cup\) \(y: 5 \leq y \leq 6\)
42. Write the compound sentence in the variable \(x\) that is graphed below.

In 43 and 44, graph on a coordinate plane.
43. \(x < -2\) \(\text{ or } y \geq 0\)
44. \(x \geq 5\) \(\text{ and } y \geq 12\)
45. Use a compound sentence to describe the region graphed below.

Objective J: Estimate solutions to systems by graphing. (Lesson 5-2)
In 49 and 50, estimate all solutions by graphing.
49. \(\begin{cases} x - 2y = -4 \\ y = x^2 \end{cases}\)
50. \(\begin{cases} 3x + 5y = -20 \\ xy = 6 \end{cases}\)
51. Multiple choice Which of the following systems describes the region below?
(a) \(\begin{cases} y < \frac{1}{2}x + 2 \\ y < \frac{1}{2}x - 3 \end{cases}\)
(b) \(\begin{cases} y > 2x + 2 \\ y < 2x - 3 \end{cases}\)
(c) \(\begin{cases} y \leq 2x + 2 \\ y \geq 2x - 3 \end{cases}\)
(d) \(\begin{cases} y \leq 2x - 1 \\ y \geq 2x + 1.5 \end{cases}\)
**Parabolas and Quadratic Equations**

**Quadratic expressions** are expressions that contain one or more terms in $x^2$, $y^2$, or $xy$, but no higher powers of $x$ or $y$. An expression that can be simplified into the form $ax^2 + bx + c$ is called a quadratic expression in the variable $x$. Equations that involve quadratic expressions are called quadratic equations. You have probably solved quadratic equations in previous courses.

Many diverse situations lead to quadratic expressions. The area formulas $A = s^2$ (for a square) and $A = \pi r^2$ (for a circle) obviously involve quadratic expressions. All other area formulas also involve the product of two lengths. Even the area of a rectangle with length $x$ and width $y$ is a quadratic expression in the two variables $x$ and $y$. You will study quadratics and area in Lesson 6-1.

Quadratic expressions also arise from studying the paths of objects. The distance an object such as a basketball travels depends on its velocity and acceleration. Lesson 6-2 discusses equations which arise from velocity and acceleration.

---

6-1: Squares and Square Roots  
6-2: Graphing $y = ax^2 + bx + c$  
6-3: The Parabola  
6-4: The Graph-Translation Theorem  
6-5: Completing the Square  
6-6: The Quadratic Formula  
6-7: Analyzing Solutions to a Quadratic  
6-8: The Imaginary Number $i$  
6-9: Complex Numbers  
6-10: Solving All Quadratics
The formula for the distance \( d \) between two points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

Underneath the radical sign are two quadratic expressions. The curve known as the parabola can be defined in terms of distance and, as a result, all parabolas can be described by quadratic equations. You will see how this is done in Lesson 6-3.

The remainder of the chapter is devoted to quadratic equations and the parabolas which are their graphs. The study of quadratic equations began in ancient times and is related to some of the most important developments in all of mathematics. As you learn about quadratic equations, you will also learn some of the history of mathematics.
The simplest quadratic equations are of the form $x^2 = k$. As you know, if $k \geq 0$ the solutions to $x^2 = k$ are called the square roots of $k$, namely $\sqrt{k}$ and $-\sqrt{k}$.

Caution: The square root or radical sign $\sqrt{}$ stands only for the non-negative square root of a number; $\sqrt{16} = 4$. To write the negative square root, write $-\sqrt{16} = -4$.

A particularly tricky expression is $\sqrt{x^2}$. It can be simplified, but cases must be considered separately.

- If $x$ is positive, then $\sqrt{x^2} = x$.
- If $x$ is negative, then $\sqrt{x^2} = -x$.
- If $x = 0$, then $\sqrt{x^2} = 0$.

This proves a surprising relationship between square roots and absolute value.

**Theorem:**

For all real numbers $x$, $\sqrt{x^2} = |x|$.

For example, $\sqrt{(-4)^2} = \sqrt{16} = 4$ and $|-4| = 4$.

$\sqrt{(8.18)^2} = \sqrt{66.9124} = 8.18$ and $|8.18| = 8.18$

Questions about squares intrigued the ancient Greeks. They wondered: What should be the radius of a circle if it is to have the same area as a given square? With algebra, it is easy to answer this question.

**Example 1** A square and circle have the same area. The square has side of length 10. What is the radius of the circle?
Solution  The area of the square is 100. Thus, if \( r \) is the radius of the circle with area 100, we find \( r \) as follows:

\[
\pi r^2 = 100 \\
\frac{100}{\pi} = r^2 \\
\frac{100}{\pi} = r \\
\]

We can ignore the negative solution here. A calculator gives \( r \approx 5.64 \).

Check  The diameter of the circle should be greater than the side of the square. Since \( d = 2r \), \( d = 11.26 \), which is greater than 10.

Expressions you might write for finding areas of rectangles may be quadratic expressions.

Example 2  Suppose a swimming pool 50 m by 20 m is to be built with a walkway around it. If the walkway is \( w \) meters wide, write an expression for the total area of the pool and walkway.

Solution  Draw a picture. The pool with walkway occupies a rectangle with length 50 + 2\( w \) meters and width 20 + 2\( w \) meters. The area of this rectangle is \((50 + 2w)(20 + 2w)\) square meters.

By using the Distributive Property, the expression \((50 + 2w)(20 + 2w)\) can be expanded. You have done this kind of multiplication in your earlier study of algebra. Remember that the idea is to think of \((50 + 2w)\) as a single quantity and to apply the Distributive Property.

\[
(50 + 2w)(20 + 2w) = (50 + 2w) \cdot 20 + (50 + 2w) \cdot 2w \\
\]

Again use the Distributive Property.

\[
= 1000 + 40w + 100w + 4w^2 \\
= 1000 + 140w + 4w^2 \\
\]

The expanded expression is in the familiar form \( ax^2 + bx + c \).
The expressions $50 + 2w$ and $20 + 2w$ are binomials. If the two binomials to be multiplied are equal, then the result is the square of the binomial.

**Example 3** Expand $(x + y)^2$.

**Solution**

\[
(x + y)^2 = (x + y)(x + y) = (x + y)x + (x + y)y = x^2 + yx + xy + y^2 = x^2 + 2xy + y^2.
\]

This expansion occurs so often it is identified as a theorem.

**Binomial-Square Theorem:**

For all real numbers $x$ and $y$:

\[
(x + y)^2 = x^2 + 2xy + y^2 \quad \text{and} \quad (x - y)^2 = x^2 - 2xy + y^2.
\]

You are asked to prove the second part of the theorem in Question 15. Of course, the theorem holds for any numbers or expressions. It is important that you be able to apply it rather automatically.

**Example 4** Expand $\left(3x - \frac{k}{4}\right)^2$.

**Solution** The idea is to use the second part of the Binomial-Square Theorem, with $3x$ in place of $x$ and $\frac{k}{4}$ in place of $y$.

\[
\left(3x - \frac{k}{4}\right)^2 = (3x)^2 - 2(3x)\left(\frac{k}{4}\right) + \left(\frac{k}{4}\right)^2 = 9x^2 - \frac{3k}{2}x + \frac{k^2}{16}
\]

**Check** Let $x = 3$ and $k = 8$. The left side is then $7^2$ or 49. The right side is

\[
9 \cdot 3^2 - \frac{3 \cdot 8}{2} \cdot 3 + \frac{8^2}{16}.
\]

You should do the calculation to verify that its value is 49.
Squares of binomials occur in the formula for the distance \( d \) between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the plane.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

If the points are on the same horizontal line, then \( y_2 = y_1 \) and so \( y_2 - y_1 = 0 \). Then \( (y_2 - y_1)^2 = 0 \) and the formula can be simplified to

\[
d = \sqrt{(x_2 - x_1)^2}.
\]

By the theorem on page 308, this is equivalent to the formula

\[
d = |x_2 - x_1|,
\]

which you learned in geometry as the formula for the distance between two points on a number line. For instance, the distance between \((17, 5)\) and \((3, 5)\) is:

\[
\sqrt{(3 - 17)^2 + (5 - 5)^2} = \sqrt{(3 - 17)^2} = \sqrt{(-14)^2} = 14
\]

The result, 14, is the same as \(|3 - 17|\). In this way absolute value, distance, squares, and square roots are closely related.

Questions

1. **Multiple choice** Which is not a quadratic equation?
   - a. \( y = \frac{1}{2}x^2 \)
   - b. \( xy = 4 \)
   - c. \( x^2 + y^2 = 10 \)
   - d. \( y = 2x \)

2. The square roots of 5 are \( \) and \( \).

3. Solve for \( t \): \( t^2 = 400 \).

4. A circle has the same area as a square of side 6. What is the radius of the circle?

5. A swimming pool 50 m by 25 m is to be built with a walkway \( w \) meters wide around it. Write the total area of the pool and walkway in expanded form.
In 6–14, expand and simplify.

6. \((4x + y)(2x + 3y)\)  
7. \((x + 1)(x - 2)\)  
8. \((2 - y)(3 - y)\)

9. \((x + y)^2\)  
10. \((x - y)^2\)

11. \((5n + 8p)^2\)

12. \((a - 8)^2\)  
13. \((2w - \frac{1}{2})^2\)

14. \((x - 1)^2 + 1^2\)

15. Prove the second part of the Binomial-Square Theorem.

16. Calculate:
   a. \(|3|\)  
   b. \(|-3|\)  
   c. \(-|-3|\)  
   d. \(-(-3)\)

17. When \(x < 0\), \(\sqrt{x^2} = \) ___.

18. When \(x < 0\), \(|x| = \) ___.

In 19–22, give the distance between the two points.

19. \((a, b)\) and \((c, d)\)  
20. \((8, 2)\) and \((-3, 2)\)

21. \((1, 5)\) and \((-1, -5)\)  
22. \((0, 0)\) and \((5, 12)\)

23. If \(\sqrt{(x - 3)^2} = |k|\), then \(k = \) ___.

24. Refer to the walkway around the swimming pool mentioned in this lesson. What is the area of the walkway?

In 25–27, expand.

25. \(\frac{1}{2}n(n + 1)\), the sum of integers from 1 to \(n\)

26. \(\frac{1}{6}x(x + 1)(2x + 1)\), the sum of the squares of the integers from 1 to \(x\)

27. \((x + y)^2 - (x - y)^2\)

28. Find the perimeter of the triangle with vertices at \((0, 0)\), \((0, 1)\), and \((2, 3)\).

29. On a brand-name pizza box, the directions read: “Spread dough to edges of pizza pan or onto a 10” by 14” rectangle on cookie sheet.” How big a circular pizza could you make with this dough, assuming it is spread the same thickness as for the rectangular pizza?

30. Solve: \(x^2 + 36 = 49\).
31. Graph \( y = \frac{1}{2}x^2 \) and \( y = -2x^2 \) on the same set of axes. (Lesson 2-5)

32. Find an equation for line \( l \) which goes through (5, 2) perpendicular to the line with equation \( 3x - 4y = 12 \). (Lesson 4-6)

33. Use the drawing below. Draw the segment whose length is the distance from point \( P \) to line \( l \). (Previous course)

34. Copy the figure below.
   a. Draw or construct a line \( m \) through \( Q \) perpendicular to \( l \).
   b. Find a point \( P \) on \( m \) so that \( FQ = QP \). (Previous course)

35. Let \( A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 0.2 & -0.4 \\ 0.1 & 0.3 \end{bmatrix} \).
   a. Find \( AB \).
   b. How are \( A \) and \( B \) related? (Lesson 5-5)

36. In classical times the problem of finding a square with the same area as a circle was called "squaring the circle" by the Greeks. Another problem they were interested in was called "duplicating a cube." Determine what this problem was.
LESSON

Graphing $y = ax^2 + bx + c$

Should a thunderstorm travel 30 mph in an easterly direction for $2\frac{1}{2}$ hours, it will wind up 75 miles east of where it started. In general, if it travels at a velocity $v$ for $t$ hours, its distance $d$ in miles from the start equals $vt$.

In the 17th century, Isaac Newton discovered that the same idea holds for objects thrown into the air or moving in space—but with an additional term needed. He found that if a ball is thrown straight up at a velocity of 44 feet per second (which is 30 mph), after $t$ seconds it will go up

$$44t$$ feet,

except that a force he called gravity would reduce its height by $16t^2$ feet. Thus after $t$ seconds, its height $h$ in feet would be

$$44t - 16t^2$$ feet.

If the thrower’s hand was 5 feet above the ground when the ball was released, the height in feet above ground would be

$$44t - 16t^2 + 5.$$

We usually write this as $h = -16t^2 + 44t + 5$, with the powers of $t$ in decreasing order. By substituting values for $t$, the height $h$ can be found after any number of seconds. The pairs $(t, h)$ can be graphed.
Example 1  If $h = -16t^2 + 44t + 5$,

a. find $h$ when $t = 0, 1, 2,$ and $3$,

b. graph the pairs $(t, h)$, and

c. interpret the graph in the situation of throwing a ball.

Solution

a. When $t = 0, \quad h = 5.$

When $t = 1, \quad h = -16 \cdot 1^2 + 44 \cdot 1 + 5$

$= 33.$

When $t = 2, \quad h = -16 \cdot 2^2 + 44 \cdot 2 + 5$

$= 29.$

When $t = 3, \quad h = -7.$

b. The points are plotted below.

c. $(0, 5)$ means that at 0 seconds, the time of release, the ball is 5 feet above the ground. The solution $(1, 33)$ means the ball is 33 feet high after 1 second, and $(2, 29)$ means the ball is 29 feet high after 2 seconds (it is already on its way down). After 3 seconds, the value of $h$ is 7 feet below ground level. Unless the ground is not level, it has already hit the ground.

The points in Example 1 do not lie on a line and, in fact, do not tell much about the shape of the graph. More points are needed. By calculating $h$ for other values of $t$, or letting a function grapher do the work, you can obtain a graph similar to the one below.

Note that the domain of the graph is the set of nonnegative real numbers.

LESSON 6-2  Graphing $y = ax^2 + bx + c$  315
The graph is part of a parabola. In fact, if \( h = -16t^2 + 44t + 5 \) is graphed for all real numbers \( t \), it is a translation image of the graph of \( y = -16x^2 \).

Two natural questions about the thrown ball are related to questions about this parabola: 1. How high does the ball get? That asks for the largest possible value of \( h \). From the graph, it seems to be about 35 feet. 2. When does the ball hit the ground? That asks for the larger \( t \)-intercept of the graph. It is between 2 and 3, nearer 3. In later lessons, you will learn how to determine more precise answers to these questions.

Newton found a general formula for the height \( h \) of an object at time \( t \) with an initial upward velocity \( v_0 \) and initial height \( h_0 \). That formula is

\[
h = -\frac{1}{2}gt^2 + v_0t + h_0.
\]

where \( g \) is the gravitational constant, also called the acceleration due to gravity. Recall that velocity is based on speed. It is the rate of change of distance with respect to time. Velocity involves units like miles per hour, feet per second, or meters per second. Acceleration measures how fast the velocity changes. This "rate of a rate" involves units like feet per second per second or meters per second\(^2\). The acceleration due to gravity varies depending on how close the object is to the center of a massive object. Near the surface of the earth, the earth's gravitational constant is about 32 ft/sec\(^2\) or 9.8 m/sec\(^2\).

Caution! The equation

\[
h = -\frac{1}{2}gt^2 + v_0t + h_0
\]

represents the height \( h \) of the ball at time \( t \). It does not describe the path of the ball. However, the actual path of a ball thrown up into the air at any angle except straight up or straight down is almost parabolic, and an equation for the path is similar to the ones studied in this lesson.

Some parabolas have simpler equations. For instance, when a ball is dropped (not thrown downward), its initial velocity is 0. Thus \( v_0 = 0 \), and the formula \( h = -\frac{1}{2}gt^2 + v_0t + h_0 \) becomes \( h = -\frac{1}{2}gt^2 + h_0 \).

---

**Example 2**  A ball is dropped from the top of a building 20 meters tall.  
 a. Find an equation describing the relation between \( h \), the ball's height above the ground, and time \( t \).  
 b. Graph its height \( h \) after \( t \) seconds.  
 c. Estimate how much time it takes the ball to fall to the ground.
Solution

a. Because the unit of height is meters, use \( g = 9.8 \text{ m/sec}^2 \). The ball is dropped, so \( v_0 = 0 \). Because the ball started 20 meters up, \( h_0 = 20 \). So the height in this situation is determined by the equation

\[
h = -\frac{1}{2}(9.8)t^2 + (0)t + 20
\]

or

\[
h = -4.9t^2 + 20.
\]

b. Negative values of \( t \) are not in the domain of \( t \), so the graph is to the right of the vertical axis.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( h = -4.9t^2 + 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>15.1</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>-24.1</td>
</tr>
<tr>
<td>0.5</td>
<td>18.775</td>
</tr>
<tr>
<td>1.5</td>
<td>8.975</td>
</tr>
</tbody>
</table>

Notice that the rate of change of the curve becomes more and more negative as \( t \) increases from 0 to 2. This reflects the increasing speed of the ball as it falls.
The equations \( h = -16t^2 + 44t + 5 \) and \( h = -4.9t^2 + 20 \) are of the form \( y = ax^2 + bx + c \), a **quadratic equation** when \( a \neq 0 \). Later in this chapter, you will learn that the graph of \( y = ax^2 + bx + c \) is a parabola congruent to the graph of \( y = ax^2 \). Recall that when \( a < 0 \) the graph of \( y = ax^2 \) opens down; when the coefficient \( a > 0 \), the parabola opens up. The graph of \( h = -16t^2 + 28t + 3 \) is congruent to the graph of \( h = -16t^2 \), so it opens down.

### Example 3

Graph \( y = x^2 - 3x + 2 \) for values of \( x \) between -4 and 4.

**Solution**  Make a table of values by substituting for \( x \) and finding \( y \). Adjust the scale on the \( y \)-axis so that the entire curve is included. Plot points by hand or use a function grapher.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>30</td>
</tr>
<tr>
<td>-3</td>
<td>20</td>
</tr>
<tr>
<td>-2</td>
<td>12</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

**Check**  The graph looks like the parabolic graph of \( y = x^2 \). The parabola will continue as indicated by the dashed extension. The entire parabola, which extends forever, is reflection-symmetric. The line of symmetry in Example 3 is the vertical line with equation \( x = 1\frac{1}{2} \), midway between the two points of intersection with the \( x \)-axis.

### Questions

1. If an object travels at a velocity \( v \) for \( t \) seconds, how far will it go?

   In 2–4, use the equation \( h = -\frac{1}{2}gt^2 + v_0t + h_0 \).

2. Give the meaning of each of the following variables.
   
   a. \( h \)  
   b. \( h_0 \)  
   c. \( v_0 \)  
   d. \( t \)  
   e. \( g \)

3. If \( v_0 \) is measured in meters per second, what value of \( g \) should be used?
4. What is the value of $v_0$ if a ball is dropped?

In 5–7, refer to the graph of Example 1.

5. About how high is the ball after 2.5 seconds?

6. When the ball hits the ground, the value of $h$ is ?.

7. About when will the ball be 15 feet from the ground? (There are two answers.)

In 8 and 9, refer to Example 2.

8. What point corresponds to the time the ball is dropped?

9. Tell whether the ball is above or below ground at $t = 2.1$. Justify your answer.

10. In Example 3, (a) give an equation for the line of symmetry of the graph and (b) estimate the coordinates of the lowest point on the graph.

In 11–14, graph the given equation for $-4 \leq x \leq 4$. If you use a function grapher, copy its graph onto your own paper.

11. $y = x^2 - 4x + 3$

12. $y = -2x^2 + 10x$

13. $y = \frac{1}{2}x^2 + \frac{x}{2}$

14. $y = -\frac{x^2}{4} + 2$

15. A certain circus juggler throws a ball from her hand at a height of 1 m with an initial upward velocity of 10 meters per second.
   a. Write an equation to describe the height of the ball after $t$ seconds.
   b. How high will the ball be after 1 second?
   c. Graph the equation from part a.
   d. Estimate the maximum height the ball reaches.
16. I. M. Chisov of the USSR set a record in January 1942 for the highest altitude from which someone survived after bailing out of an airplane without a parachute. He bailed out at 21,980 feet.
   a. Write an equation describing his height at \( t \) seconds.
   b. Graph the equation in part a.
   c. About how long did his fall take?

17. A picture frame \( w \) inches wide is to surround a picture that is 8" by 12".
   a. What is the area of the picture with its frame? (Lesson 6-1)
   b. What is the perimeter of the picture frame? (Previous course)

18. Evaluate for \( x = -0.5 \). (Lesson 6-1)
   a. \( |x| \)  
   b. \( |-x| \)  
   c. \( -(x) \)  
   d. \( -|x| \)  
   e. \( x^2 \)  
   f. \( (-x)^2 \)  
   g. \( \sqrt{x^2} \)  
   h. \( -\sqrt{x^2} \)  

19. Expand and simplify. (Lesson 6-1)
   a. \( (x - 2)^2 \)  
   b. \( 3(x - 2)^2 \)  
   c. \( 3(x - 2)^2 - 12 \)  

20. In football, a touchdown is worth 6 points, a field goal is worth 3 points, a safety is worth 2 points, and a point-after-touchdown (PAT) is worth 1 point. (Lessons 3-3, 3-9, 2-5)
   a. If a team gets \( T \) touchdowns, \( F \) field goals, \( S \) safeties, and \( P \) PATs, how many total points does it have?
   b. A team has no safeties and no PATs, and a total of at most 27 points. Graph the set of possible ways this could happen.
   c. Is the graph in part b discrete or continuous?
21. Given the feasibility region below, find the vertex at which
$80x + 120y = P$ is maximized. (Lesson 5-8)

![Graph showing the feasibility region with vertices at (15, 30) and (40, 10).]

22. Use this program, which generates values of quadratic expressions.

```
10 PRINT "VALUES OF Y = A * X ^ 2 + B * X + C"
20 INPUT "COEFFICIENTS A,B,C:"; A, B, C
30 PRINT "Y = "; A; "X ^ 2 + "; B; "X + "; C
40 PRINT "X", "Y"
50 FOR X = 0 TO 3 STEP 0.1
60 LET Y = A * X ^ 2 + B * X + C
70 PRINT X, Y
80 NEXT X
90 END
```

a. Use the program to generate the values of $y = -16x^2 + 44x + 5$
for the values of $x$ from 0 to 3, increasing by 0.1. (Hint: For this
equation, $A = -16$, $B = 44$, and $C = 5$.)

b. In part a, change the value of $C$ from 5 to some other value. Run
the program again and compare your results with those from part
a. Relate the results you get to the situation of Example 1.
We have asserted that the path of a tossed object is a parabola. For instance, the path of a basketball shot is part of a parabola from the time it leaves the shooter's hands until it hits some other object.

In order to determine whether a curve is or is not a parabola, a definition of **parabola** is necessary. Parabolas can be defined geometrically.

**Definition:**

Let \( \ell \) be a line and \( F \) be a point not on \( \ell \). A parabola is the set consisting of every point in the plane of \( F \) and \( \ell \) whose distance from \( F \) equals its distance from \( \ell \).

\( F \) is called the **focus** and \( \ell \) the **directrix** of the parabola. Thus a parabola is the set of points in a plane equidistant from its focus and its directrix. Neither the focus nor directrix is on the parabola. Below is a sketch of a parabola. Four points \( V, P_1, P_2, \) and \( P_3 \) are identified on the parabola. Note that each is equidistant from the focus \( F \) and the directrix \( \ell \).
To understand the definition, you must remember that the distance from a point \( P \) to a line \( \ell \) is the length of the perpendicular from \( P \) to \( \ell \). In the sketch, \( P_1Q_1 \perp \ell \) and \( P_1Q_1 = P_1F \). Also, \( P_2Q_2 \perp \ell \) and \( P_2Q_2 = P_2F \), and so on. The point \( V \) is special. It lies on the line from \( F \) perpendicular to line \( \ell \) and is called the vertex of the parabola.

**Example 1** Trace the figure. Find five points on the parabola with focus \( G \) and directrix \( m \).

![Diagram of a parabola with focus \( G \) and directrix \( m \).]

**Solution** Five points \( V, P_1, P_2, P_3, \) and \( P_4 \) are shown.

1. The vertex \( V \) is the midpoint of the perpendicular segment \( GH \) from \( G \) to \( m \).
2. Points \( P_1 \) and \( P_2 \) are third vertices of squares with \( GH \) as one side. Each is the same distance \( d \) from \( G \) as from \( m \).
3. To find \( P_3 \) and \( P_4 \), first draw a circle with center \( G \) and with any radius \( r > \frac{d}{2} \). Then draw a line that is parallel to \( m \) and a distance \( r \) from it. The intersection of that line with the circle gives two points, \( P_3 \) and \( P_4 \), on the parabola.

To find other points, use circles with different radii (always \( > \frac{d}{2} \)).
To find an equation for a parabola, the distance from a point to a line in the coordinate plane must be calculated. Notice that the distance \(d\) from \((x, y)\) to the line \(y = k\) is the distance from \((x, y)\) to \((x, k)\). By the distance formula,
\[
  d = \sqrt{(x - x)^2 + (y - k)^2} = \sqrt{(y - k)^2}.
\]

This idea is used in the next example, with \(k = -5\).

**Example 2** Find an equation for the parabola with focus \((0, 5)\) and directrix \(y = -5\).

**Solution** Draw a picture. Let \(P = (x, y)\) be any point on the parabola. If \(Q = (x, -5)\), then \(PF = PQ\).

\[
\begin{align*}
  PF &= PQ \\
  \sqrt{(x - 0)^2 + (y - 5)^2} &= \sqrt{(x - x)^2 + (y - (-5))^2} \\
  x^2 + (y - 5)^2 &= (y + 5)^2 \\
  x^2 + y^2 - 10y + 25 &= y^2 + 10y + 25 \\
  x^2 - 10y &= 10y \\
  x^2 &= 20y \\
  y &= \frac{1}{20}x^2
\end{align*}
\]
Check We need to verify that a point on \( y = \frac{1}{20}x^2 \) is equidistant from (0, 5) and \( y = -5 \). We use \( A = (30, 45) \).

\[
AF = \sqrt{(30 - 0)^2 + (45 - 5)^2} = \sqrt{30^2 + 40^2} = \sqrt{2500} = 50
\]

The distance from \( A \) to \( y = -5 \) is the distance from (30, 45) to (30, -5), which is 50 also.

In Example 2, if you were to replace (0, 5) by (0, \( \frac{1}{4} \)) and \( y = -5 \) by \( y = -\frac{1}{4} \), the equation for the parabola would be \( y = x^2 \). If (0, 5) is replaced by \( \left(0, \frac{1}{4a}\right) \) and \( y = -5 \) is replaced by \( y = -\frac{1}{4a} \), then the parabola has equation \( y = ax^2 \). The derivation for both of these follows the idea in Example 2 and demonstrates the following theorem.

**Theorem:**

The graph of \( y = ax^2 \) is a parabola.

When \( a < 0 \), you have learned that the parabola opens down. In that case the directrix is above the \( x \)-axis, the focus below.

If a parabola is rotated in space around its line of symmetry, the three-dimensional figure it creates is called a **paraboloid**. The focus of a paraboloid is the focus of the rotated parabola. Two examples are:

- the union of the paths of the jets of a whirling sprinkler
- a satellite receiving dish
Questions

1. Define parabola.

In 2 and 3, trace the figure and draw five points on the parabola with focus $F$ and directrix $\ell$.

2. 

3. 

In 4–6, refer to the parabola below. $F$ is its focus, $\ell$ its directrix.

4. True or false
   a. $P_1F = P_3G_3$ 
   b. $FP_4 = G_4P_4$

5. Name the vertex.

6. Are the focus and directrix on the parabola?

7. a. Graph the parabola with equation $y = \frac{1}{20}x^2$.
   b. Give its focus, vertex, and directrix.
   c. Verify that the point $(2, 0.2)$ is equidistant from the focus and directrix.

8. Verify that the graph of $y = x^2$ is a parabola with focus $(0, \frac{1}{4})$ and directrix $y = -\frac{1}{4}$ by choosing a point on the graph and showing that two appropriate distances are equal.

In 9–11, tell whether the parabola opens up or down.

9. $y = 4x^2$ 
10. $y = -4x^2$ 
11. $y = \frac{1}{4}x^2$

12. a. What is a paraboloid?
    b. Give an example from the real world.
In 13 and 14, use the information given in Question 8.

13. The graph of \( y = x^2 + 3 \) is a parabola congruent to and three units above the graph of \( y = x^2 \). What are the focus and directrix of \( y = x^2 + 3 \)?

![Graph of \( y = x^2 + 3 \)]

14. Give the focus, vertex, and directrix of the parabola with equation \( y = -x^2 \).

15. Given \( F = (0, 2) \) and line \( \ell \) with equation \( y = -2 \).
   a. What is an equation for the set of points equidistant from \( F \) and \( \ell \)?
   b. Check your answer.

16. Consider the parabola with focus \( F = (0, 3) \) and directrix \( y = -3 \).

![Graph of \( x = y^2 + 3 \)]

a. What are the coordinates of its vertex? The drawing can help you.
   b. A square has been drawn. The focus and a point \( P \) on the parabola are two vertices of the square. Find the coordinates of \( P \).
   c. Use symmetry to find the coordinates of another point on the parabola.
   d. Trace the figure and sketch the parabola.
   e. Find an equation for this parabola.
17. Two concentric circles are shown below. The smaller has radius \( r \), and the larger has radius \( r + h \).
   a. Find the area of the shaded region.
   b. Prove that the circumference of the larger is \( 2\pi h \) more than the circumference of the smaller. (Previous course, Lesson 6-1)

\[ r \quad h \]

18. a. Graph \( y = 3x - x^2 \) for values of \( x \) between -5 and 5.
   b. Estimate the coordinates of the vertex of the graph. (Lesson 6-2)

19. The KTHI-TV transmitting tower between Fargo and Blanchard, North Dakota, is about 629 meters tall.
   a. If a hammer were dropped from the top, what will its height be after \( t \) seconds?
   b. In about how many seconds would it hit the ground? (Lesson 6-2)

In 20 and 21, graph both sentences by hand, by computer, or by graphing calculator. Compare the graphs. (Lesson 6-2)

20. \( y = x^2 \) and \( y = (x + 2)^2 \)

21. \( y = x^2 - 6 \) and \( y = (x - 5)^2 - 6 \)

In 22–24, recall that for all nonnegative real numbers \( a \) and \( b \), \( \sqrt{ab} = \sqrt{a} \sqrt{b} \). Use this property to simplify each expression. (Previous course)

22. \( \sqrt{3} \sqrt{12} \)  
23. \( \sqrt{20} \sqrt{50} \)  
24. \( (\sqrt{7})^2 \)

25. Parabolas can be formed without equations or graphs. Follow these steps to see how to make a parabola by folding paper.
   a. Take a sheet of unlined paper. Fold it in half as shown at the right. Cut or tear along the fold to make two congruent pieces. On one piece mark a point \( P \) about one inch above the center of the lower edge. Fold the paper so that the lower edge touches \( P \), and crease well as shown at the right. Repeat 10 to 15 times, each time folding in a different direction. The creases represent the tangents to a parabola. What are its focus and directrix?
   b. On the other piece of paper mark a point \( Q \) approximately in the center. Repeat the procedure used in part a. Where are the focus and directrix for this parabola?
   c. The two parabolas formed in parts a and b illustrate the property that as the distance between the focus and the directrix increases, the parabola opens more slowly.
The equation \( x^2 = 9 \) has two solutions, 3 and -3. Now consider the related equation

\[ (x - 8)^2 = 9. \]

To solve this equation, a first idea might be to expand \((x - 8)^2\), but that is the hard way. The easier way is to take the square roots of each side. If \((x - 8)^2\), then there are four possibilities: \(x - 8 = 3\), \(x - 8 = -3\), \(-(x - 8) = 3\), or \(-(x - 8) = -3\). Two of the pairs are equivalent; so

\[ x - 8 = 3 \text{ or } x - 8 = -3. \]

To solve these equations, add 8 to each side.

\[ x = 11 \text{ or } x = 5 \]

The solutions to \((x - 8)^2 = 9\) are larger by 8 than the solutions to \(x^2 = 9\). This is to compensate for subtracting 8 in that equation. When the solutions are graphed on the number line, they are 8 units to the right. In other words, the solutions to \((x - 8)^2 = 9\) are the images of the solutions to \(x^2 = 9\) under a translation 8 to the right.

In general, the following theorem holds.

**Theorem:**

In a sentence to be solved for \(x\), replacing \(x\) by \(x - h\) increases the solutions by \(h\).
Remember that if \( h \) is negative, the translation can be viewed as being “negative right,” which is left.

The direction of the translation may seem the reverse of what you expect. But consider the two parabolas \( y = x^2 \) and \( y = (x - 8)^2 \). The first has vertex \((0, 0)\). The second has vertex \((8, 0)\). The graph of \( y = (x - 8)^2 \) is a translation image of \( y = x^2 \), 8 units to the right. The graphs are congruent parabolas.

The graphs are congruent parabolas.

Translations of graphs up and down result from replacing \( y \) by \( y - k \). For example, the equation
\[ y - 3 = x^2 \]
is equivalent to \( y = x^2 + 3 \). Its graph is 3 units above \( y = x^2 \). Again, if \( k \) is negative, then the translation is “negative up,” which is down.

Translations can be done horizontally and vertically at the same time. The general idea is simple yet very powerful. Recall that the translation \( T_{h,k} \) slides a figure \( h \) units to the right and \( k \) units up.

**Graph-Translation Theorem:**

In a sentence for a graph, replacing \( x \) by \( x - h \) and \( y \) by \( y - k \) causes the graph to undergo the translation \( T_{h,k} \).
A **corollary** is a theorem that follows immediately from another theorem. The Graph-Translation Theorem has many corollaries.

**Corollary:**

The image of the parabola \( y = ax^2 \) under the translation \( T_{h,k} \) is \( y - k = a(x - h)^2 \).

**Example 1** Sketch the graph of \( y - 7 = 3(x - 6)^2 \).

**Solution** The graph is 6 units to the right and 7 units above \( y = 3x^2 \). So the graph is a parabola with vertex \((6, 7)\). Because \( y = 3x^2 \) opens up, the parabola opens up. To find some other points start with the vertex and use symmetry. The \( x \)-value 1 unit to the left of the axis of symmetry is \( x = 5 \). Substitute \( x = 5 \) into the equation of the parabola to get \( y = 10 \). Then by symmetry a point 1 unit to the right of the axis of symmetry is \((7, 10)\). In a similar manner you can verify that \((4, 19)\) and \((8, 19)\) are on the parabola. The graph is below.

The equation \( y - k = a(x - h)^2 \) is the **vertex form of an equation for a parabola**. The parabola has vertex \((h, k)\). If \( a > 0 \), then the parabola opens up and the graph has a minimum. If \( a < 0 \), then the parabola opens down and the graph has a maximum. When the equation for a parabola is in vertex form, the parabola can be graphed quickly.
Example 2 Sketch the graph of \( y = \frac{1}{2}(x + 2)^2 \).

Solution Put the equation in vertex form: \( y - 0 = \frac{1}{2}(x - 2)^2 \). The graph is the parabola two units to the left of \( y = \frac{1}{2}x^2 \). Thus it opens upward, and its vertex is \((-2, 0)\). The minimum value is \( y = 0 \). Find other points by substituting values near \(-2\) for \( x \).

\[
\begin{array}{c|c}
 x & y \\
-2 & 0 \\
-1.5 & 1.25 \\
-3 & 2 \\
0 & 2 \\
-4 & 2 \\
\end{array}
\]

Check Use an automatic grapher to plot solutions to \( y = \frac{1}{2}x^2 \) and \( y = \frac{1}{2}(x + 2)^2 \) on the same set of axes.

Questions

1. Solve: a. \( x^2 = 100 \); b. \((x - 3)^2 = 100 \).
2. Solve: a. \( y^2 = 6 \); b. \((y - 5)^2 = 6 \).
3. How are the graphs of \( y = x^2 \) and \( y = (x - 8)^2 \) related?
4. The graph of \( y - k = a(x - h)^2 \) is units above and units to the right of the graph of \( y = ax^2 \).
5. Refer to Example 1.
   a. The focus of \( y = 3x^2 \) is \((0, \frac{1}{12})\). What is the focus of \( y - 7 = 3(x - 6)^2 \)?
   b. The directrix of \( y = 3x^2 \) is \( y = -\frac{1}{12} \). What is the directrix of \( y - 7 = 3(x - 6)^2 \)?
6. a. The equation \( y - k = a(x - h)^2 \) is in the form of an equation for a parabola.
   b. The vertex is .
7. Use the equation \( y + 2 = -3(x + 7)^2 \).
   a. Give the coordinates of the vertex.
   b. Give an equation for the axis of symmetry.
   c. Tell whether the parabola opens up or down.
   d. Graph the solution set to the equation.
8. Suppose the parabola with equation \( y = 2x^2 \) undergoes the translation \( T_{-2, -3} \). Find an equation for its image.
9. One solution to \( x^2 + 5x + 3 = 87 \) is 7. Use that information to get a solution to \((x - 4)^2 + 5(x - 4) + 3 = 87\).

10. One solution to \( 39x - 21y = 1200 \) is \((41, 19)\). Use this information to get a solution to \( 39(x - 5) - 21(y - 3) = 1200 \).

11. Solve: 
   a. \( |z| = 34 \) 
   b. \( |z + 1| = 34 \)

12. The parabola graphed at the right is congruent to \( y = 3x^2 \) and has vertex \((-2, 2)\). What is an equation for it?

13. A parabola has vertex \((2, -5)\) and opens down. If the parabola is congruent to \( y = 7x^2 \), what is an equation for the parabola?

14. The point-slope form of a line, \( y - y_1 = m(x - x_1) \), can be thought of as the image of the line with equation \( \) under the translation \( T_{a,b} \), where \( a = \) \( b = \) \( \).

15. Define: parabola. \((Lesson 6-3)\)

16. A parabola has focus \((0, 1)\) and directrix \(y = -1\).
   a. What is its vertex?
   b. State whether the vertex is a maximum or a minimum.
   c. Give the coordinates of two other points on the parabola. \((Lesson 6-3)\)

17. Expand.
   a. \( (x + 4)^2 \) 
   b. \( 2(x + 4)^2 \) 
   c. \( 2(x + 4)^2 + 3 \) \((Lesson 6-1)\)

18. A rectangular lot 100' by 60' is in a town that allows no building closer than 2 feet to the edge of a lot.
   a. How much room is there to build?
   b. If no building closer than \( x \) feet were allowed, how much room would be left? \((Lesson 6-1)\)

19. It takes \( \frac{1}{2}n(n - 1) \) handshakes for each of \( n \) people at a party to shake hands with everyone else. Expand this expression. \((Lesson 1-5)\)

20. The parabola with equation \( x = y^2 \) is graphed at the right.
   a. Give the coordinates of five points on this parabola.
   b. Graph its image under the translation \( T_{3,-1} \).
   c. Write an equation for the image.
   d. Does the Graph-Translation Theorem hold for parabolas that open to the side?
You have now seen two forms for an equation of a parabola.

\[ y = ax^2 + bx + c \]  expanded form
\[ y - k = a(x - h)^2 \]  vertex form

The vertex form, as its name suggests, shows the vertex and so gives the maximum or minimum point on the parabola. In the next lesson, you will learn that the expanded form is more convenient for finding the intercepts of the graph. Because each form is useful, converting from one form to the other is helpful.

**Example 1**  Convert \( y - 3 = 4(x + 6)^2 \) to expanded form.

**Solution**  Expand the binomial.

\[
y - 3 = 4(x^2 + 12x + 36) = 4x^2 + 48x + 144
\]

So

\[ y = 4x^2 + 48x + 147. \]

To convert from expanded form to vertex form, a process known as **completing the square** is used. Remember that

\[(x + h)^2 = x^2 + 2hx + h^2.\]

The trinomial \( x^2 + 2hx + h^2 \) is called a **perfect-square trinomial**.

**Example 2**  What number should be added to \( x^2 + 10x \) to make a perfect-square trinomial?

**Solution**  Compare \( x^2 + 10x + \_ \) with the perfect-square trinomial \( x^2 + 2hx + h^2 \). The first terms, \( x^2 \), are identical. To make the second terms equal, set

\[ 10x = 2hx. \]

So

\[ h = 5. \]

The term added should be \( h^2 \) or 25.

**Check**  \( x^2 + 10x + 25 = (x + 5)^2. \)
To generalize Example 2, consider the expression 
\[ x^2 + bx + \square. \]

What must be put in the blank so the result is a perfect-square trinomial?
\[ x^2 + bx + \square = x^2 + 2hx + h^2 \]

Since \( b = 2h \), \( h = \frac{1}{2}b \). Then \( h^2 = (\frac{1}{2}b)^2 \). This proves:

**Theorem:**

To complete the square on \( x^2 + bx \), add \((\frac{1}{2}b)^2\).

Now you are ready to find the vertex of a parabola from its expanded-form equation.

**Example 3** Find the vertex of the parabola with equation \( y = x^2 + 10x + 8 \).

**Solution**

Step 1) Rewrite the equation so that only terms with \( x \) are on one side.
\[ y - 8 = x^2 + 10x + \square \]

Step 2) Complete the square on \( x \). Here \( b = 10 \), so \((\frac{1}{2}b)^2 = 25\).

Step 3) Add 25 to both sides.
\[ y - 8 + 25 = x^2 + 10x + 25 \]

Step 4) Put in vertex form.
\[ y + 17 = (x + 5)^2 \]

The vertex of the parabola is \((-5, -17)\).

**Check** If the graph has vertex \((-5, -17)\), the parabola can be sketched. Now let \( x = -4 \). Then
\[ y = (-4)^2 + 10 \cdot -4 + 8 = -16. \]

If \( x = -6 \), \( y = -16 \) also. This indicates that \((-5, -17)\) is the lowest point on the parabola.
Example 2 and Example 3 indicate how to complete the square on an expression of the form $x^2 + bx$, where the coefficient of $x^2$ is 1. If the coefficient of $x^2$ is not 1, two extra steps are needed.

Example 4  Find the vertex of the parabola with equation $y = 2x^2 - 5x - 9$.

Solution  Again complete the square.
1. Add 9 to both sides, as before, to remove a constant term from the right side.

   $y + 9 = 2x^2 - 5x$

2. Divide both sides of the equation by 2, the coefficient of $x^2$.

   $\frac{y + 9}{2} = x^2 - \frac{5}{2}x$

3. Complete the square on the right side. Here $b = -\frac{5}{2}$, so

   \[ \left(\frac{b}{2}\right)^2 = \left(-\frac{5}{4}\right)^2 = \frac{25}{16}. \]

   This number must be added to both sides.

   $\frac{y + 9}{2} + \frac{25}{16} = x^2 - \frac{5}{2}x + \frac{25}{16}$

   $\frac{y + 9}{2} + \frac{25}{16} = \left(x - \frac{5}{4}\right)^2$

4. It looks complicated, but do not fear! Multiply both sides by the same number 2 you divided by in step 2. This makes the coefficient of $y$ again equal to 1.

   $y + 9 + \frac{25}{8} = 2\left(x - \frac{5}{4}\right)^2$

5. Put the equation into vertex form.

   $y + \frac{97}{8} = 2\left(x - \frac{5}{4}\right)^2$

6. Read the vertex from the vertex form. The vertex is $\left(\frac{5}{4}, -\frac{97}{8}\right)$.

Check 1  Substitute $\frac{5}{4}$ for $x$ in the original equation. Does $y = -\frac{97}{8}$?

   $2\left(\frac{5}{4}\right)^2 - 5 \cdot \frac{5}{4} - 9 = 2 \cdot \frac{25}{16} - \frac{25}{4} - 9$

   $= \frac{25}{8} - \frac{25}{4} - 9$

   $= \frac{25}{8} - \frac{50}{8} - \frac{72}{8}$

   $= -\frac{97}{8}$

   If the vertex is at $x = \frac{5}{4}$, then $x = 1$ and $x = 1.5$ should give the same values of $y$. Try them.

Check 2  Graph the parabola. This is left for you to do also.
Completing the square helps to find key points on graphs involving quadratic expressions. Its most important application is in the proof of the Quadratic Formula, which you shall see in the next lesson.

Questions

In 1–4, convert the equation to expanded form.

1. \( y = (x + 3)^2 + 2 \) \hspace{1cm} 2. \( y = \left(x - \frac{b}{2}\right)^2 \)

3. \( y = 2(x - 4)^2 - 1 \) \hspace{1cm} 4. \( \frac{y}{6} = (x + \frac{1}{2})^2 \)

In 5–7, find a number to put in each blank to make each expression a perfect-square trinomial.

5. \( x^2 + 18x + \underline{\hspace{2cm}} \) \hspace{1cm} 6. \( z^2 - 3z + \underline{\hspace{2cm}} \) \hspace{1cm} 7. \( x^2 + bx + \underline{\hspace{2cm}} \)

In 8–11, find the vertex of the parabola represented by each equation.

8. \( y = x^2 + 18x + 6 \) \hspace{1cm} 9. \( y = x^2 - 3x + 1 \)

10. \( y = 2x^2 - 6x + 4 \) \hspace{1cm} 11. \( y = 3x^2 + 4x + 5 \)


13. a. Find the vertex of the parabola \( h = -16t^2 + 44t + 5 \) graphed in Lesson 6-2.
   b. What is the maximum height of the ball?

14. What term must be added to the expression \( a^2 + 7ay + \underline{\hspace{2cm}} \) to make a perfect square?

15. A student stated that \( y^2 - 6y + 9 = (y - 3)^2 \). Another student stated that \( y^2 - 6y + 9 = (3 - y)^2 \). Their teacher stated that both students were correct. Explain this mystery.

16. a. Give the sum of the areas of the three rectangles.
   b. What number must be added to this sum to complete the square?
   c. Interpret your answer to part b geometrically.
In 17–19, solve. *(Previous course, Lesson 6-4)*

17. $(x + 16)^2 = 36$  
18. $y^2 + 16 = 36$  
19. $(z - 5)^2 = 2$

In 20 and 21, refer to the graph of a bird’s flight shown below. *(Lesson 3-8)*

20. When was the bird on the ground?

21. What was the bird’s average speed during its first descent?

---

22. Graphed below is $y = x^2$ and its directrix $y = -\frac{1}{4}$.
   a. What are the coordinates of the focus $F$?
   b. Calculate $d_1$, $d_2$, $d_3$, and $d_4$. *(Lesson 6-3)*
In 23–25, is the statement true or false? (Previous course)

23. $\sqrt{40,000} = 200$  
24. $\sqrt{50} = 5\sqrt{10}$  
25. $\sqrt{48} = 4\sqrt{3}$

26. $ABCD$ is a square of sides $a + b + c$. The areas of three regions inside have been given.

a. Find the areas of the other six rectangles.
b. Use the drawing to expand $(a + b + c)^2$.
c. Show a drawing to expand $(a + b + c + d)^2$.
d. Generalize parts b and c.

![Diagram of a square with areas of its parts labeled a², ab, ac, etc.]
Pop Fligh, the famous baseball player, hits a pitch that is 3.5 ft high. The ball travels towards the outfield along a nearly parabolic path, described by the equation

\[ h = -.005x^2 + 2x + 3.5, \]

where \( x \) is the distance (in feet) of the ball from home plate, and \( h \) is the height (in feet) of the ball at that instant. When is the ball 8 feet high?

Because we wish to know the horizontal distance \( x \) when the height is 8, substitute 8 for \( h \) in the above equation.

\[ 8 = -.005x^2 + 2x + 3.5 \]

By adding -8 to each side, the equation is put in standard form \( ax^2 + bx + c = 0 \).

\[ 0 = -.005x^2 + 2x - 4.5 \]

This equation can be solved by rewriting it in vertex form, but the arithmetic is messy. It is much easier to solve the general equation \( ax^2 + bx + c = 0 \). The result is called the Quadratic Formula. This formula is very important—\textit{you must memorize it.} The Quadratic Formula is a theorem; that is, it can be proved from the basic properties of algebra.

**Quadratic-Formula Theorem:**

If \( ax^2 + bx + c = 0 \) and \( a \neq 0 \), then

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

The proof of the Quadratic Formula requires completing the square.
Proof

Given is the equation $ax^2 + bx + c = 0$, where $a \neq 0$.

1. Divide both sides by $a$ so the coefficient of $x^2$ is 1.

\[ x^2 + \frac{b}{a} x + \frac{c}{a} = 0 \]

On the right side, $\frac{0}{a} = 0$.

2. Add $-\frac{c}{a}$ to each side.

\[ x^2 + \frac{b}{a} x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \]

3. To complete the square, add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ to both sides.

\[ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \]

4. Write the left side as a binomial squared.

\[ x + \frac{b}{2a} = \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

5. Add the fractions on the right side.

6. Take the square root of each side.

7. Add $\frac{b}{2a}$ to both sides.

---

Example 1

Solve $3x^2 + 11x - 4 = 0$.

Solution

The Quadratic Formula lets you solve any quadratic equation. Here $a = 3$, $b = 11$, and $c = -4$.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ = \frac{-11 \pm \sqrt{121} - 4 \cdot 3 \cdot -4}{2 \cdot 3} \]

\[ = \frac{-11 \pm \sqrt{121} - 48}{6} \]

\[ = \frac{-11 \pm 13}{6} \]

The $\pm$ sign here means there are two solutions, one with the $+$ sign, one with the $-$ sign.

\[ x = \frac{-11 + 13}{6} \quad \text{or} \quad x = \frac{-11 - 13}{6} \]

So

\[ x = \frac{-11 + 13}{6} \quad \text{or} \quad x = \frac{-11 - 13}{6} \]

Check

Each solution should be checked.

Does $3 \cdot \left(\frac{1}{3}\right)^2 + 11 \cdot \frac{1}{3} - 4 = 0$? Yes, $\frac{1}{3} + \frac{11}{3} - 4 = 0$.

Does $3 \cdot (-4)^2 + 11 \cdot -4 - 4 = 0$? Yes, $48 - 44 - 4 = 0$. 

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Example 2  The 3-4-5 right triangle has sides whose lengths are consecutive integers. Are there any other right triangles with this property?

![Diagram of a right triangle with sides labeled n, n+1, and n+2.]

**Solution**  If $n$ is an integer, then $n$, $n + 1$, and $n + 2$ are consecutive integers. By the Pythagorean Theorem, we get

$$n^2 + (n + 1)^2 = (n + 2)^2.$$ 

Therefore

$$n^2 + n^2 + 2n + 1 = n^2 + 4n + 4.$$ 

or

$$2n^2 + 2n + 1 = n^2 + 4n + 4.$$ 

To rewrite the equation in standard form, add $-n^2 - 4n - 4$ to each side.

$$n^2 - 2n - 3 = 0$$

Now the quadratic formula can be used. Here $a = 1$, $b = -2$, and $c = -3$. Therefore

$$n = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{16}}{2}$$

$$= \frac{2 \pm 4}{2}.$$ 

So $n = \frac{2 + 4}{2} = 3$ or $n = \frac{2 - 4}{2} = -1$.

If $n = 3$, $n + 1 = 4$, and $n + 2 = 5$. The second solution must be rejected because a side of a triangle must have positive length. Thus the only right triangle with consecutive-integer dimensions is the 3-4-5 right triangle.

In Examples 1 and 2, the number $b^2 - 4ac$ under the radical sign is a perfect square; so the solutions are integers or simple fractions. In applications, however, the numbers are not always so nice. Still the Quadratic Formula works, but a calculator is needed. Here is the problem posed at the beginning of this lesson.
Example 3  Find out when Pop Fligh's baseball is 8 feet high.

Solution  We need to solve 

\[-.005x^2 + 2x - 4.5 = 0,\]

Thus, in this situation \(a = -.005\), \(b = 2\), and \(c = -4.5\). Substitute into the formula:

\[
x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-.005) \cdot (-4.5)}}{2 \cdot (-.005)}
\]

\[
= \frac{-2 \pm \sqrt{4 - .09}}{-.01}
\]

\[
= \frac{-2 \pm \sqrt{3.91}}{-.01}
\]

Use a calculator to estimate the square root and separate the two solutions.

\[
x = \frac{-2 + 1.977}{-.01} \quad \text{or} \quad x = \frac{-2 - 1.977}{-.01}
\]

So \(x \approx 2.3\) \quad \text{or} \quad x = 397.7.

As you might expect, there are two places where the ball is 8 ft high. The first is when the ball is about 2.3 ft away from home plate and on the way up. The second is when the ball is about 398 ft away from home plate and on the way down.

Questions

1. If \(ax^2 + bx + c = 0\), give the two values of \(x\) in terms of \(a\), \(b\), and \(c\).

2. The proof of the Quadratic Formula is based on what idea?

3. Multiple choice  The Quadratic Formula is a
   (a) theorem.  (b) postulate.  (c) definition.

   In 4–6, refer to the proof of the quadratic formula.

4. Expand \((x + \frac{b}{2a})^2\).

5. Write a fraction equal to \(\frac{c}{a}\) but with denominator \(4a^2\).

6. One square root of \((x + \frac{b}{2a})^2\) is \(x + \frac{b}{2a}\). What are the square roots of \(\frac{b^2 - 4ac}{4a^2}\)?
In 7 and 8, find all solutions using the Quadratic Formula.
7. \(10x^2 + 13x + 3 = 0\)  
8. \(6v^2 - 5v - 3 = 0\)

In 9–11, consider the equation \(h = \cdot005x^2 + 2x + 3.5\) and the situation at the start of this lesson.
9. What do \(h\) and \(x\) represent?

10. Manny Walker is the pitcher on the team playing against Pop. Manny is standing on the pitcher’s mound, about 60 ft from home plate. How high is the ball when it is over Manny’s head?

11. When will Pop’s hit be 100 ft high?

12. In Example 3, we could have multiplied both sides of the equation by -1 and solved \(.005x^2 - 2x + 4.5 = 0\). Find the solutions to this equation to the nearest tenth.

13. Refer to Example 2. If \(n\) is the first of three consecutive integers, what are the other two?

14. Find all right triangles whose sides are consecutive even integers \(n\), \(n + 2\), and \(n + 4\).

In 15–17, (a) put the equation in standard form and identify \(a\), \(b\), \(c\); (b) solve the equation using the quadratic formula.
15. \(0 = -11x^2 + 20x + 4\)
16. \(n^2 + 9 = 6n\)
17. \(4(m^2 - 3m) = -9\).

18. a. Why cannot \(a\) equal 0 in the quadratic formula?  
b. Solve \(0x^2 + bx + c = 0\).

19. Consider the equation \(ax^2 + bx + c = 0\), where \(c = 0\).  
a. Solve for \(x\) in this special case.  
b. Solve \(5y^2 + 8y = 0\).

20. Consider the parabola with equation \(y = x^2 + 6x - 1\).  
a. Find the values of \(x\) for which \(y = 0\). On a graph these points are called the _?_.  
b. Find the vertex of this parabola.  
c. Graph the parabola.  
d. Give an equation for the axis of symmetry.

21. Alice tried to solve the equation 
\[3x^2 - 8x + 5 = 0\]
using the quadratic formula and the calculator key sequence:
\[8 \pm \sqrt{8} \div 4 \times 3 \times 5 \div \sqrt{3} \div 2 \times 3 \div \ast \ast\].

Her friends, Lois and Carol, starred the two places where she made mistakes. Correct Alice’s mistakes.
22. Madilyn Hadder said that she solved the equation

\[ 3x^2 - 8x + 5 = 0 \]

in three steps.

Step 1) Calculate the square root first and store it in the memory:

\[ 8 \text{ } \left[ \begin{array}{c} x^2 \cr - \end{array} \right] - 4 \times 3 \times 5 = \sqrt{2} \text{ STO} \]

Step 2) Find \( \frac{-b + \sqrt{c}}{2a} \):

\[ 8 \text{ } \left[ \begin{array}{c} \text{RCL} \cr + \end{array} \right] = \left[ \begin{array}{c} \text{RCL} \cr + \end{array} \right] \left[ \begin{array}{c} 2 \times 3 \end{array} \right] = \]

Step 3) Find \( \frac{-b - \sqrt{c}}{2a} \):

\[ 8 \text{ } \left[ \begin{array}{c} \text{RCL} \cr - \end{array} \right] = \left[ \begin{array}{c} \text{RCL} \cr + \end{array} \right] \left[ \begin{array}{c} 2 \times 3 \end{array} \right] = \]

a. What are the two answers M. Hadder’s key sequences give? Are they the correct solutions?

b. Solve \( 5x^2 - 7x - 12 = 0 \) using the key sequences above. You may need to use \( \text{M+} \) for \( \text{STO} \) and \( \text{MR} \) for \( \text{RCL} \).

In 23–25, find the slope of the line:

23. containing the points (8, 9) and (-5, 2);

24. with equation \( 2y = 1.2x - 3 \);

25. through (1, -4) and parallel to the \( x \)-axis. (Lessons 3-1, 3-2)

26. a. Graph the solution sets to \( y = \frac{36}{x} \) and \( y + 2 = \frac{36}{x} \) on the same axes.

b. Find equations for the lines of symmetry for the image and pre-image. (Lessons 2-5, 6-5)

27. Without graphing, explain why the equations \( y = 3x^2 + 24x + 50 \) and \( y - 2 = 3(x + 4)^2 \) have the same graph. (Lessons 6-4, 6-5)

28. Make up some questions you could ask about the path of the ball hit by Pop Fligh in this lesson. If you can, answer those questions.
Even as early as 1700 B.C., ancient mathematicians considered geometry problems that today we would solve using quadratic equations. However, the ancients did not have our modern notation. Euclid, who lived around 300 B.C., would have phrased the problem $x^2 - 5x = 20$ geometrically:

"If a certain straight line be diminished by five, the rectangle of the whole and the diminished segment equals twenty."

Five hundred years after Euclid, in about 250 A.D., the Greek mathematician Diophantus was the first to use symbols. Diophantus had no single symbol for an unknown: $x$ is $\xi$ and $x^2$ is $\Lambda \gamma$. He would have written the equation $x^2 - 5x = 20$ as follows:

\[
\begin{align*}
\text{square of } x \text{ less } 5x & \text{ equals } 20 \\
\Lambda \gamma \land \xi \eta & = 20
\end{align*}
\]

One of the first general descriptions of a method to solve quadratic equations was given by the Arab mathematician Al-Khowarizmi in 825 A.D., in a book entitled *Hisab al-jabr w'al muqabalah*. Our modern word "algebra" is derived from the second word of the title. Khowarizmi solved quadratics by completing the square, but his solutions were entirely in words. Around 1200 this book was translated into Latin by Fibonacci and European mathematicians had a method for solving quadratics.
The first to use letters and coefficients the way we do was François Viète, a French mathematician, in the late 1500s. For \( x^2 - 5x = 20 \), Viète would write “\( IAQ - 5A \) aequatur 20”. Our modern notation, with exponents, is first found in a book by René Descartes published in 1637. Today’s notation makes it relatively easy to solve any quadratic equations.

The history of quadratic equations is connected to the history of number ideas. Recall that a **real number** is a number that can be represented as a decimal and so graphed on a number line. Real numbers are either positive, negative, or zero. The Greeks thought there was only one solution to \( x^2 = 9 \), the “square root of 9.” They did not consider negative numbers because to them the unknown could stand only for a length.

The Greeks at first thought that all numbers could be written as **simple fractions** in the form \( \frac{p}{q} \) where \( p \) and \( q \) are integers, \( q \neq 0 \).

However, the Pythagoreans discovered that their solution to an equation like \( x^2 = 8 \) could not be represented as a simple fraction. They called these numbers **irrational**. Today we know that any square root of a whole number that is not a perfect square is irrational. Irrational numbers are exactly those numbers that have infinite nonrepeating decimals. Every real number is either rational or irrational.

<table>
<thead>
<tr>
<th>real number</th>
<th>as a fraction</th>
<th>as a decimal</th>
<th>examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>rational</td>
<td>can be written as a simple fraction</td>
<td>is either finite or infinitely repeating</td>
<td>977.54, ( \frac{2}{3} ), 8.12, 17( \frac{3}{32} ), ( \sqrt{9} )</td>
</tr>
<tr>
<td>irrational</td>
<td>cannot be written as a simple fraction</td>
<td>is infinite and non-repeating</td>
<td>( \sqrt{2} ), ( \pi ), -3 + ( \sqrt{45} )</td>
</tr>
</tbody>
</table>

Due to their geometric origins, solutions to quadratic (and some other) equations are sometimes called **roots**. The roots of even the simplest quadratic equations include all types of numbers.

<table>
<thead>
<tr>
<th>equation</th>
<th>number of real roots</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 = 9 )</td>
<td>2</td>
<td>rational</td>
</tr>
<tr>
<td>( x^2 = 8 )</td>
<td>2</td>
<td>irrational</td>
</tr>
<tr>
<td>( x^2 = 0 )</td>
<td>1</td>
<td>rational</td>
</tr>
<tr>
<td>( x^2 = -4 )</td>
<td>0</td>
<td>rational</td>
</tr>
</tbody>
</table>
When the quadratic equation is more complicated, the roots can still be counted and classified without completely solving the equation. Suppose \(a, b,\) and \(c\) are real numbers with \(a \neq 0,\) and \(ax^2 + bx + c = 0.\) Then the Quadratic Formula gives

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

If \(b^2 - 4ac\) is positive, there are two real solutions, as you found in the last lesson. If \(b^2 - 4ac\) is negative, then the Quadratic Formula results in the square root of a negative number. So there are no real solutions. If \(b^2 - 4ac = 0,\) then

\[
x = \frac{-b}{2a},
\]

and there is only one solution, \(x = \frac{-b}{2a}.\)

Thus the expression \(b^2 - 4ac\) determines the number of real roots. Accordingly, \(b^2 - 4ac\) is called the discriminant of the quadratic equation \(ax^2 + bx + c = 0.\)

**Discriminant Theorem:**

If \(a, b,\) and \(c\) are real and \(a \neq 0,\) then the equation \(ax^2 + bx + c = 0\) has:

- a. two real roots, if \(b^2 - 4ac > 0\)
- b. one real root, if \(b^2 - 4ac = 0\)
- c. zero real roots, if \(b^2 - 4ac < 0.\)

**Example 1** Determine the number of real roots of the equation \(4x^2 - 12x + 9 = 0.\)

**Solution** Use the Discriminant Theorem. Here \(a = 4, b = -12,\) and \(c = 9.\) The discriminant \(b^2 - 4ac\) is

\[
(-12)^2 - 4(4)(9) = 0.
\]

Thus, there is one real solution or root.

**Example 2** How many real roots does \(2x^2 + 3x + 4 = 0\) have?

**Solution** Here \(a = 2, b = 3,\) and \(c = 4,\) so \(b^2 - 4ac = 9 - 4 \cdot 2 \cdot 4 < 0.\) There are no real roots.
The $x$-intercepts of the graph of $y = ax^2 + bx + c$ are the roots of $ax^2 + bx + c = 0$. It should therefore not be surprising that the discriminant has information about the graph of $y = ax^2 + bx + c$. The number of real solutions to $ax^2 + bx + c = 0$ is equal to the number of $x$-intercepts of the graph of $y = ax^2 + bx + c$.

The following are graphs of parabolas whose equations are related to Examples 1 and 2 on page 348.

If $a$, $b$, and $c$ are rational and $b^2 - 4ac$ is a perfect square, then $\sqrt{b^2 - 4ac}$ is rational and both roots are rational. Thus the discriminant tells you quickly much information about the nature of the solutions to a quadratic equation.

**Example 3** Does $10t^2 - 3t = 4$ have any rational solutions?

**Solution** First, rewrite the equation in standard form.

$10t^2 - 3t - 4 = 0$

Evaluate the discriminant. Here $a = 10$, $b = -3$, and $c = -4$. So $b^2 - 4ac = 9 - 4 \cdot 10 \cdot -4 = 169$. Since 169 is positive and is a perfect square, there are two real solutions, both rational.

**Check** Finish solving the equation. $t = \frac{3 \pm \sqrt{169}}{20} = \frac{3 \pm 13}{20}$; so $t = \frac{4}{5}$ or $t = -\frac{1}{2}$. Do these numbers work in the original equation?

That is left for you to check.
Questions

In 1–4, match the idea at the left with the estimated length of time it has been known.

1. completing the square  a. about 3700 years
2. today’s notation for quadratics  b. about 2300 years
3. problems leading to quadratics  c. about 1750 years
4. first use of symbols for unknowns  d. about 1150 years
5. The word “algebra” is descended from the Arabic word _?_.

In 6–11, tell whether the number is (a) real and rational, (b) real and irrational, or (c) not real.

6. \(\sqrt{10}\)  7. \(\sqrt{100}\)
8. -5  9. \(\frac{203}{317}\)
10. \(\sqrt{-4}\)  11. 0

12. Give (a) the discriminant and (b) the roots of the quadratic equation \(ax^2 + bx + c = 0\).

13. Why are there no real roots when the discriminant to a quadratic equation is negative?

In 14 and 15, what does the discriminant tell you about the roots to the equation?

14. \(3x^2 - 4x + 5 = 0\)
15. \(5y^2 - 10y + 5 = 0\)

16. What information does the discriminant give you about the graph of \(y = ax^2 + bx + c\)?

17. The discriminant of the equation of a parabola is -1000. What do you know about the graph?

In 18 and 19, determine the number of x-intercepts the graph has, and whether they are rational or irrational.

18. \(y = -3x^2 + 2x + 1\)
19. \(y = x^2 + 12x + 36\)
In 20 and 21, true or false. If true, prove the statement; if false, give a counterexample.

20. Every parabola that has an equation of the form \( y = ax^2 + bx + c \) has a \( y \)-intercept.

21. By symmetry, whenever a parabola has one \( x \)-intercept, it also has a second \( x \)-intercept.

22. Pop Fligh wanted to know if the ball he hit (in Lesson 6-6) went higher than the top of the stadium, 200 feet high. He knew that the ball was \( h \) feet high \( x \) feet from home plate, where \( h = -.005x^2 + 2x + 3.5 \). Was the ball ever 200 feet high? (Hint: You can do this without finding the vertex.)

23. Find the value(s) of \( k \) for which the graph of the quadratic equation \( y = x^2 + kx + 9 \) will have exactly one \( x \)-intercept. (Hint: When is the discriminant zero?)

In 24 and 25, the following program in BASIC uses the discriminant to solve quadratic equations of the form \( ax^2 + bx + c = 0 \), where \( a \neq 0 \).

```
90 PRINT "SOLVE A*X^2 + B*X + C = 0."
100 INPUT "COEFFICIENTS"; A, B, C
200 DISC = B^2 - 4*A*C
250 IF DISC < 0 THEN 700
300 X1 = (-B + SQR(DISC))/(2*A)
400 X2 = (-B - SQR(DISC))/(2*A)
500 PRINT "THE ROOTS ARE ", X1, " OR ", X2, "."
600 GO TO 999
700 PRINT "THERE ARE NO REAL ROOTS."
999 END
```

24. Check to see that this BASIC program gives the correct solutions to the equations.
   a. \(-2x^2 + 40x = 0\)
   b. \(5x^2 - 150x + 1185 = 0\)

   (Note: If your computer does not recognize "-B" in lines 300 and 400 and gives you an error message, try "-1*B" instead.)

25. a. Describe what happens when you input \( a = 0 \).

   b. Modify the program so it tests whether \( a \) is 0, and prints "NOT A QUADRATIC EQUATION" when \( a = 0 \).
26. Solve the quadratic equation in Question 15. (Lesson 6-6)

27. Find the lengths of the sides of the rectangle shown at the beginning of this lesson. (Lesson 6-6)

28. a. Solve the quadratic equation \( x^2 + 12x + 36 = 100 \).
   b. What do the solutions have to do with the parabola of Question 19? (Lessons 6-2, 6-6)

29. Janice did not want to deal with a fraction when solving \( n^2 - 5n + \frac{25}{4} = 0 \), so she multiplied both sides by 4 and solved \( 4n^2 - 20n + 25 = 0 \) instead. Do both equations have the same roots? Why or why not? (Previous course, Lesson 1-6)

In 30–32, simplify. (Previous course)

30. \( \sqrt{20} \sqrt{5} \)  
31. \( \frac{\sqrt{63}}{\sqrt{9}} \)  
32. \( \frac{\sqrt{84}}{2} \)

33. Thousands of years ago the Pythagoreans investigated \textit{figurate numbers}. You are familiar with the sequence of squares,

\[
\begin{align*}
1 & \quad 4 & \quad 9 & \quad 16 & \quad 25 \\
\end{align*}
\]

The \( n \)th square number is \( n^2 \). The \textit{triangular numbers} are

\[
\begin{align*}
1 & \quad 3 & \quad 6 & \quad 10 & \quad 15 \ldots \\
\end{align*}
\]

\( T_n \), the \( n \)th triangular number, is \( \frac{n(n + 1)}{2} \).

a. Find the 15th triangular number.
b. Which triangular number is 8128? (Lessons 1-3, 6-6)

34. Each of these words or phrases is named after a mathematician who is mentioned in this lesson. Tell its meaning and the name of the mathematician. (Look in a reference book, if necessary.)

a. \textit{algorithm}  
b. \textit{Diophantine equation}  
c. \textit{Cartesian coordinate}
Imagine your surprise if you picked a number from a machine and got $\sqrt{-1}$. Until the 1500s, mathematicians were also puzzled by square roots of negative numbers. They knew that if they solved certain quadratics, they would get negative numbers under the radical sign. They did not know what to do with them.

One of the first to work with these numbers was Girolamo Cardano in a book called *Ars Magna* (“Great Art”) published in 1545. He and other mathematicians of his time reasoned as follows:

When $k$ is positive, the equation $x^2 = k$ has two solutions, $\sqrt{k}$ and $-\sqrt{k}$. If we solve the equation $x^2 = -k$ in the same way, then the two solutions are $\sqrt{-k}$ and $-\sqrt{-k}$. In this way, they defined symbols for the square roots of negative numbers.

**Definition:**

For $k > 0$, the two solutions to $x^2 = -k$ are denoted $\sqrt{-k}$ and $-\sqrt{-k}$.

By the definition, $(\sqrt{-k})^2 = -k$. This means that we can say, for all real numbers $r$,

$$\sqrt{r} \cdot \sqrt{r} = r.$$ 

Cardano and others then assumed that these numbers had the same properties as other numbers (but with certain exceptions, as you shall soon see). Descartes called these square roots of negatives *imaginary numbers*, in contrast to the numbers everyone understood which he called “‘real numbers.’” In his famous *Algebra* of 1777, Euler used the symbol $i$ to denote $\sqrt{-1}$.

**Definition:**

$$i = \sqrt{-1}.$$
That is, \( i^2 = -1 \). Now consider multiples of \( i \), such as \( 5i \). By the definition of \( i \), \( 5i = 5 \sqrt{-1} \). If we assume that multiplication is commutative and associative, then

\[
(5i)^2 = 5i \cdot 5i = 5^2 \cdot i^2 = 25 \cdot -1 = -25.
\]

Thus \( 5i \) is a square root of \(-25\). We say that \( 5i = \sqrt{-25} \) and then \(-5i = -\sqrt{-25}\). In general, if \( k > 0 \), \( \sqrt{-k} = i\sqrt{k} \) and so the square roots of any negative number are multiples of \( i \).

---

**Example 1**  Show that \( i\sqrt{3} \) is a square root of \(-3\).

**Solution**  Multiply \( i\sqrt{3} \) by itself.

\[
\begin{align*}
(i\sqrt{3})^2 & = i \cdot i \cdot \sqrt{3} \cdot \sqrt{3} = i^2 \cdot 3 = -1 \cdot 3 = -3
\end{align*}
\]

The other square root of \(-3\) is \(-i\sqrt{3}\).

Due to the long history of quadratics, solutions to them are described in different ways. The following all refer to the same numbers.

- the solutions to \( x^2 = -41 \)
- the square roots of \(-41\)
  - \( \sqrt{-41} \) and \(-\sqrt{-41}\)
  - \( i\sqrt{41} \) and \(-i\sqrt{41}\)

The next example shows how to multiply square roots of negative numbers expressed in radical form. Notice that square roots are taken before multiplying.

---

**Example 2**  Simplify \( \sqrt{-16} \cdot \sqrt{-25} \).

**Solution**  Convert to multiples of \( i \).

\[
\begin{align*}
\sqrt{-16} \cdot \sqrt{-25} & = i\sqrt{16} \cdot i\sqrt{25} = 4i \cdot 5i = 20i^2 = -20
\end{align*}
\]
You are familiar with the property $\sqrt{ab} = \sqrt{a} \sqrt{b}$ for nonnegative real numbers $a$ and $b$. Does this property hold when $a$ and $b$ are both negative? Consider Example 2. If we assume $\sqrt{a} \sqrt{b} = \sqrt{ab}$, then
\[
\sqrt{-16} \sqrt{-25} = \sqrt{(-16)(-25)} \\
= \sqrt{400} \\
= 20.
\]

Clearly, this is different from the answer in Example 2. This counterexample shows that 
\[\sqrt{ab} \neq \sqrt{a} \sqrt{b}.\]

when $a$ and $b$ are negative numbers.

If you try to evaluate a number like $\sqrt{-16}$ on most calculators, an error message will be displayed because most scientific calculators are programmed to operate only with real numbers. So you must understand the principles of computation with imaginary numbers. The commutative, associative, and distributive postulates of addition and multiplication hold for imaginary numbers, as do all theorems based on these postulates. Consequently, working with multiples of $i$ is much like working with any other numbers. For instance,
\[
\sqrt{-9} - \sqrt{-25} = 3i - 5i = -2i \\
\frac{\sqrt{-9}}{\sqrt{-25}} = \frac{3i}{5i} = \frac{3}{5}.
\]

Questions

1. Multiple choice About when did mathematicians begin to use roots of negative numbers as solutions to equations?
   (a) sixth century (b) twelfth century (c) sixteenth century twentieth century

2. Name the solutions of $x^2 + 1 = 0$.

3. True or false Not all negative numbers have square roots.

4. $\sqrt{-b} = i\sqrt{b}$ when $b \geq 0$.

5. Who first used the term “imaginary number”?

6. Who was the first person to suggest using $i$ for $\sqrt{-1}$?

7. Show that $i\sqrt{5}$ is a square root of $-5$.

8. Show that $-3i$ is a square root of $-9$.

In 9 and 10, solve for $x$. Write the solutions both with and without radical signs.

9. $x^2 + 16 = 0$  

10. $x^2 - 16 = 0$
In 11–16, write as real numbers or as multiples of \(i\).
11. \(\sqrt{-7}\)  
12. \(\sqrt{-144}\)  
13. \(\sqrt{-2} \cdot \sqrt{2}\)  
14. \(\sqrt{-3} \cdot \sqrt{-3}\)  
15. \(\sqrt{-6} \cdot \sqrt{-3}\)  
16. \(\sqrt{-96}\)

17. When does \(\sqrt{xy} \neq \sqrt{x} \cdot \sqrt{y}\)?

In 18–23, perform the indicated operations.
18. \(3i + 4i\)  
19. \(8i - i\)  
20. \(2\sqrt{9} + \sqrt{49}\)
19. \(\sqrt{9} + \sqrt{49}\)
22. \(\frac{\sqrt{-16}}{-4}\)
23. \(\frac{2i + 3i}{i}\)

In 24 and 25, simplify.
24. \(\sqrt{-434281}\)  
25. \(\sqrt{-8} + \sqrt{-2}\)

26. Solve.
   a. \(x^2 + 15 = 6\)  
   b. \((x - 3)^2 + 15 = 6\)

27. True or false. If false, give a counterexample.
   a. The sum of two imaginary numbers is imaginary.
   b. The product of two imaginary numbers is imaginary.

In 28 and 29, (a) give the number of real solutions to the quadratic, (b) give the number of rational solutions, and (c) find all solutions. (Lessons 6-6, 6-7)
28. \(4m^2 - 4m + 1 = 0\)  
29. \(10x^2 + 11x + 2 = 0\)

30. A ball is thrown upwards from a height of 3 feet with an initial velocity of 28 feet per second.
   a. What is the height of the ball after \(t\) seconds? (Lesson 6-2)
   b. What is the maximum height of the ball? (Lesson 6-5)
   c. When does the ball hit the ground? (Lesson 6-6)

In 31 and 32, let \(y = \frac{x - 1}{\frac{2}{x - 2}}\).

31. When \(x = \frac{1}{3}\), is \(y\) positive or negative? (Lesson 1-5)
32. Give the value of \(y\) when \(x = 6\). (Lesson 1-5)

33. By definition you know that \(i^2 = -1\). So \(i^3 = i^2 \cdot i = -1 \cdot i = -i\) and \(i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1\).
   a. Continue this pattern to evaluate and simplify each of \(i^5\), \(i^6\), \(i^7\), and \(i^8\).
   b. Generalize your result to predict the value of \(i^{1988}\), \(i^{1989}\), and \(i^{2001}\).
The oasis in this picture is a mirage. A mirage is an optical illusion—it does not exist. Many years ago mathematicians thought imaginary numbers didn’t exist.

In the last lesson you studied a set of numbers of the form $bi$, called imaginary numbers. When a real number and an imaginary number are added, the sum is called a **complex number**.

**Definition:**

A complex number is a number of the form $a + bi$ where $a$ and $b$ are real numbers and $i = \sqrt{-1}$; $a$ is called the real part and $b$ is called the imaginary part.

For example, the **real part** of $-3 + 4i$ is $-3$; the **imaginary part** is $4$ (not $4i$).

Two complex numbers $a + bi$ and $c + di$ are **equal** if and only if their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if and only if $a = c$ and $b = d$. For example, if $x + yi = 2i - 3$, then $x = -3$ and $y = 2$.

All postulates for real numbers except those for inequality (see the appendices) also hold for the set of complex numbers. Thus, we can use these properties to operate with complex numbers in a manner consistent with real-number operations.

**Example 1** Simplify: $(3 + 4i) + (7 + 8i)$.

**Solution** Addition of complex numbers is performed by adding real parts together and adding imaginary parts together.

\[
(3 + 4i) + (7 + 8i) = (3 + 7) + (4i + 8i) = 10 + (4 + 8)i = 10 + 12i
\]

The Distributive Property can be used to multiply a complex number by a real number or by an imaginary number.
Perform the operations: \(2i(8 + 5i)\).

**Solution**

\[
\begin{align*}
2i(8 + 5i) &= 2i(8) + 2i(5i) \\
&= i(2 \cdot 8) + (2 \cdot 5)(i \cdot i) \\
&= i(16) + 10(i^2) \\
&= 16i + 10(-1) \\
&= -10 + 16i
\end{align*}
\]

In Example 2 notice that \(i^2\) was simplified using the fact that \(i^2 = -1\). In an answer, \(i\) should never be left to any power other than the first. Generally, all answers to complex number operations should be put in the form \(a + bi\). This makes it easy to identify the real and imaginary parts.

Example 3 illustrates how the FOIL Theorem is used to multiply two complex numbers.

Perform the operations: \((10 - 7i)(4 + 9i)\).

**Solution**

Use the FOIL Theorem.

\[
\begin{align*}
(10 - 7i)(4 + 9i) &= 10 \cdot 4 + 10 \cdot 9i - 7i \cdot 4 - 7i \cdot 9i \\
&= 40 + 90i - 28i - 63i^2 \\
&= 40 + 62i - 63(-1) \\
&= 103 + 62i
\end{align*}
\]

The previous examples illustrate the following theorem:

**Theorem:**

Given two complex numbers \(a + bi\) and \(c + di\).

- **Complex Addition:**
  \[
  (a + bi) + (c + di) = (a + c) + (b + d)i
  \]

- **Complex Multiplication:**
  \[
  (a + bi)(c + di) = (ac - bd) + (ad + bc)i
  \]

It is not necessary to memorize this theorem. Rather, remember to combine like terms when adding and to use the FOIL Theorem when multiplying.
Example 4 Multiply and simplify: \((8 - 5i)(8 + 5i)\).

Solution \((8 - 5i)(8 + 5i) = 64 + 40i - 40i - 25i^2\)

\[= 64 - 25i^2\]

\[= 64 - 25(-1)\]

\[= 89\]

The complex numbers in Example 4 are called complex conjugates of each other. In general, the complex conjugate of \(a + bi\) is \(a - bi\). The product of two complex conjugates is a real number.

Every real number \(a\) is a complex number because \(a = a + 0i\). Thus, the real numbers are a subset of the complex numbers. Likewise, every imaginary number \(bi\) equals \(0 + bi\), so the imaginary numbers are also a subset of the complex numbers.

This diagram shows how various kinds of complex numbers are related.

The first use of the term "complex number" is generally credited to Carl Friedrich Gauss (1777–1855). Gauss applied complex numbers to the study of electricity. Later in the 19th century, applications were found in geometry and acoustics. New applications continue to be discovered; since 1975 a new field called dynamical systems has arisen, in which complex numbers play a pivotal role. Among the offshoots of this field are gorgeous computer-generated drawings which have won awards in art competitions.
Questions

1. A complex number is a number of the form \( ? \) where \( a, b \) are \( ? \) numbers.

2. The term "complex number" was coined by \( ? \).

In 3–5, give the real and imaginary parts of the complex number.

3. \( 14 + 5i \)
4. \( 3 - i\sqrt{2} \)
5. \( i \)

6. The complex numbers \( 3 + 7i \) and \( 3 - 7i \) are called \( ? \).

In 7–10, perform the operations and write your answer in \( a + bi \) form.

7. \( (11 - 4i) + (18 - 3i) \)
8. \( 3i(2 + 9i) \)
9. \( (2 - i)(2 + i) \)
10. \( (-8 + i)(5 - 3i) \)

11. Provide reasons for each step.

\[
(12 + 7i)(8 + 4i) = 96 + 48i + 56i + 28i^2 \\
= 96 + 104i + 28i^2 \\
= 96 + 104i + 28(-1) \\
= 96 - 28 + 104i \\
= 68 + 104i
\]

a. \( ? \)
b. \( ? \)
c. \( ? \)
d. \( ? \)

arithmetic

12. If \( x + 12i = 7 - yi \), find \( x \) and \( y \).

13. True or false Every real number is also a complex number.

14. Name two fields of technology in which complex numbers are applied.

In 15–20, let \( z = 9 - 4i \), \( w = 16 + 2i \), and \( v = 5 + 13i \). Write the following in \( a + bi \) form.

15. \( z + w \)
16. \( z - v \)
17. \( v - z \)
18. \( 2v + 3w \)
19. \( w^2 \)
20. \( vz + wz \)

21. a. The additive identity for the complex numbers is \( ? \).
b. The multiplicative identity for the complex numbers is \( ? \).

22. Write \( \sqrt{-9} \) in \( a + bi \) form.

23. Consider the complex conjugates \( a - bi \) and \( a + bi \).

a. Calculate their sum.
b. Prove that their product is \( a^2 + b^2 \).

Division by complex numbers is performed by multiplying both numerator and denominator by the conjugate of the denominator. For example,

\[
\frac{3 - 4i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{6 - 15i - 8i + 20i^2}{4 - 10i + 10i - 25i^2} = \frac{-14 - 23i}{-29} = \frac{-14}{-29} - \frac{23}{-29} i
\]
In 24 and 25, divide using that method.

24. \( \frac{5}{6 - 2i} \)  
25. \( \frac{3 + i}{4 - 7i} \)

26. Let \( z = 1 + i \).
   a. Calculate \( z^2 \).
   b. Multiply \( z^2 \) by itself to calculate \( z^4 \).
   c. From this, complete the sentence: \( 1 + i \) is a 4th root of \( \) ___.

27. a. Find the vertex of the parabola with equation \( y + 4 = 2(x + 8)^2 \).
    b. Find the x-intercepts of the parabola with equation \( y + 4 = 2(x + 8)^2 \).
    c. Graph the parabola. (Lessons 6-1, 6-2, 6-4)

28. A ball is thrown upwards at a velocity of 10 meters per second from an initial height of 1.5 meters.
   a. Write an equation for its height \( h \) after \( t \) seconds.
   b. When does the ball reach its maximum height?
   c. How high does it reach? (Lessons 6-1, 6-5)

29. Who was the first to work with square roots of negative numbers? (Lesson 6-8)

30. Multiple choice Which is not a square root of \( -9 \)? (Lesson 6-8)
    (a) \(-3i\)  (b) \(3i\)  (c) \(-3\)  (d) \(\sqrt{-9}\)

31. \( x^2 + 3x = 2 \)
32. \( 4a^2 = 2 + 4a \)
   \( x^2 + 3x - 1 = 0 \) \( y \) \( \frac{2\pm \sqrt{9}}{2} \)
   \( 4a^2 - 4a - 2 = 0 \)
   \( -3 \pm \sqrt{9} \) \( -1 \)

33. A complex number \( a + bi \) is graphed as the point \( (a, b) \) with the x-axis as the real axis and the y-axis as the imaginary axis.
   a. Graph \( z = 1 + i \) as the point \( (1, 1) \).
   b. Compute and graph \( z^2, z^3, \) and \( z^4 \).
   c. What pattern emerges? Can you predict where \( z^5 \) will be?
In the past few lessons, you have studied the Quadratic Formula, which gives solutions to all quadratic equations (Lesson 6-6); the number $i$, which allows square roots of negatives to be considered (Lesson 6-8); and the complex numbers, which allow real numbers and square roots of negatives to be combined (Lesson 6-9). You now have the machinery which enables you to put any solution to a quadratic in $a + bi$ form.

#### Example 1

Solve $5x^2 + 8x + 10 = 0$.

**Solution**

Use the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \cdot a}$$

Now use the definition of $i$,

$$= \frac{-8 \pm i\sqrt{136}}{10}$$

So, in $a + bi$ form,

$$x = -\frac{4}{5} + \frac{i\sqrt{136}}{10} \text{ or } x = -\frac{4}{5} - \frac{i\sqrt{136}}{10}.$$  

**Check**

Since $\sqrt{136} \approx 11.7$, $x \approx -0.8 + 1.2i$ or $x = -0.8 - 1.2i$.

These can be substituted into the original equation.

Is $5(-0.8 + 1.2i)^2 + 8(-0.8 + 1.2i) + 10 = 0$?

The operations you learned in Lesson 6-9 now help to check.

Is $5(0.64 - 1.92i - 1.44) - 6.4 + 9.6i + 10 = 0$?

Is $-4 - 9.6i - 6.4 + 9.6i + 10 = 0$?

The left side simplifies to $-0.4$, quite close.

Similarly, the other root can be checked.

One way of simplifying this arithmetic is to rewrite $\sqrt{136}$ as a multiple of a smaller integer-square root.

$$\sqrt{136} = \sqrt{2 \cdot \sqrt{68}}$$

$$\sqrt{136} = \sqrt{4 \cdot \sqrt{34}} = 2\sqrt{34}$$

$$\sqrt{136} = \sqrt{8 \cdot \sqrt{17}}$$

The solutions in Example 1 can be written as follows.

$$x = -\frac{4}{5} + \frac{i\sqrt{136}}{10} \text{ or } x = -\frac{4}{5} - \frac{i\sqrt{136}}{10}$$

or

$$x = -\frac{4}{5} + \frac{2i\sqrt{34}}{10} \text{ or } x = -\frac{4}{5} - \frac{2i\sqrt{34}}{10}$$

or

$$x = -\frac{4}{5} + \frac{i\sqrt{34}}{5} \text{ or } x = -\frac{4}{5} - \frac{i\sqrt{34}}{5}$$

The last of these is not much simpler, but a little bit simpler.
We say "simplify $\sqrt{136}$" and then get $2\sqrt{34}$. Is $2\sqrt{34}$ really simpler than $\sqrt{136}$? It depends on the use you have. For evaluating the square root with a calculator, $\sqrt{136}$ is simpler. But for work with the quadratic formula as in Example 1, $2\sqrt{34}$ may be simpler.

Checking solutions to quadratic equations can be tedious, but there is an easier way to check than by substitution. The actual problem that led Cardano to deal with complex numbers provides the path to this new way of checking.

**Example 2** Find two numbers whose sum is 10 and whose product is 40.

**Solution** Let one number be $x$. Then the other is $10 - x$. Because their product is 40,

$$x(10 - x) = 40.$$  

This is a quadratic equation. Rewrite it in standard form.

$$10x - x^2 = 40$$

$$-x^2 + 10x - 40 = 0$$

Multiply both sides by -1 to make the arithmetic easier.

$$x^2 - 10x + 40 = 0$$

Use the quadratic formula.

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(40)}}{2(1)} = \frac{10 \pm \sqrt{-60}}{2}$$

Now "simplify" $\sqrt{-60}$.

$$x = \frac{10 \pm 2\sqrt{15}}{2} = \frac{10}{2} \pm \frac{2\sqrt{15}}{2}$$

So

$$x = 5 + i\sqrt{15} \text{ or } x = 5 - i\sqrt{15}.$$  

When $x = 5 + i\sqrt{15}$, $10 - x = 5 - i\sqrt{15}$, so these two roots are the solutions to the problem.

**Check** Is the sum of the numbers 10?

$$(5 - i\sqrt{15}) + (5 + i\sqrt{15}) = 10.$$  

Is the product of the numbers 40?

$$(5 - i\sqrt{15})(5 + i\sqrt{15}) = (5)^2 - (i\sqrt{15})^2 = 25 - 15^2 = 25 + 15 = 40.$$  

Compare the equation in Example 2 to its roots. The sum and the product of the roots appear in the equation. These are instances of the following theorem first recorded in its general form by Francois Viete in 1591.
Sum and Product of Roots Theorem:

The numbers $r_1$ and $r_2$ are the roots of the equation $ax^2 + bx + c = 0$, with $a \neq 0$, if and only if

$$r_1 + r_2 = \frac{-b}{a} \text{ and } r_1r_2 = \frac{c}{a}.$$  

Proof

To prove that if $r_1$ and $r_2$ are the roots, then $r_1 + r_2 = \frac{-b}{a}$ and $r_1r_2 = \frac{c}{a}$, start with the roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$  

(1) $r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$

(2) $r_1r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$

To prove that if $r_1 + r_2 = \frac{-b}{a}$ and $r_1r_2 = \frac{c}{a}$, then $r_1$ and $r_2$ are the roots of $ax^2 + bx + c = 0$, begin with

$$r_1 + r_2 = \frac{-b}{a} \text{ and } r_1r_2 = \frac{c}{a}.$$  

Multiplying both sides of both equations by $a$ gives

$$ar_1 + ar_2 = -b \text{ and } ar_1r_2 = c.$$  

So

$$ar_1 = -b - ar_2 \text{ and } ar_1r_2 = c.$$  

Substitute $-b - ar_2$ for $ar_1$: $( -b - ar_2)r_2 = c.$

Distribute and add $-c$ to both sides: $-br_2 - ar_2^2 - c = 0.$

Multiply both sides by $-1$ and put in standard form:

$$ar_2^2 + br_2 + c = 0$$

Thus $r_2$ is a solution to $ax^2 + bx + c = 0$.

Similarly $r_1$ is a solution, which completes the proof.
Example 3  Alonzo believes $\frac{2}{3}$ and $\frac{4}{3}$ are the solutions to $9x^2 - 18x + 8 = 0$. Is he correct?

Solution  Here the sum of the roots is $\frac{6}{3}$ or 2 and their product is $\frac{8}{9}$. In the equation, $a = 9$, $b = -18$, and $c = 8$. So $\frac{b}{a} = \frac{-18}{9} = (-2) = 2$ and $\frac{c}{a} = \frac{8}{9}$. The solutions check.

Questions

1. Solve for $x$ and write the solutions in $a + bi$ form: $x^2 - 4x + 5 = 0$.
2. State the problem that caused Cardano to deal with complex numbers.
3. Find two numbers whose sum is 4 and whose product is 29.
4. Find two numbers whose sum is 10 and whose product is 24.

In 5–7, simplify.

5. $\frac{20 + \sqrt{600}}{2}$
6. $-6 \pm \frac{\sqrt{-288}}{3}$
7. $\frac{-40 + \sqrt{3200}}{-60}$

8. Name a situation in which $3\sqrt{2}$ is not simpler than $\sqrt{18}$.
9. The sum of the roots of $ax^2 + bx + c = 0$ is $\underline{?}$, and the product of the roots is $\underline{?}$.
10. What are the sum and product of the roots of $7x^2 - 3x - 2 = 0$?

11. In a quadratic equation with real coefficients, the roots will be complex conjugates when the discriminant is $\underline{?}$.

In 12–14, solve.

12. $x^2 + 100 = 0$
13. $-7 - 3y^2 = 5y$
14. $9t^2 - 12t + 229 = 0$

15. a. Graph $y = x^2 - 10x + 30$. Observe that there are no $x$-intercepts.
   b. Write an equation for the axis of symmetry of this graph.
   c. Solve $x^2 - 10x + 30 = 0$.
   d. How is the average of the roots from part c related to the equation from part b?
   e. Generalize part d.
16. Some Greek temples were constructed in the shape of a special rectangle. In such a rectangle, the ratio \( \frac{w}{h} \) of width to height equals the ratio \( \frac{h}{w-h} \) of height to width after a square is removed. Let \( h = 1 \) unit and find \( w \), and thus find this ratio, known as the golden ratio. (Lesson 6-6)

17. Find the vertex, x-intercepts, and axis of symmetry of the parabola \( y = -x^2 + 2x + 5 \). Graph this parabola. (Lessons 6-2, 6-6)

18. A coin is dropped into an empty well. If it takes 2 seconds for the coin to hit the bottom, how deep is the well? (Lesson 6-2)

19. The Pauli spin matrices \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \), and \( C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), where \( i = \sqrt{-1} \), are used in studying electron spin. Show that

a. \( AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot BA. \)

b. \( CB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot BC. \) (Lessons 6-9, 4-2)

20. What is the 20th term of the arithmetic sequence \( 2 + i, 5 + 3i, 8 + 5i, \ldots \)? (Lessons 3-6, 6-9)

21. Multiple choice Which of these is a solution to the equation \( x^2 = i \)?

(a) \( \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \)  
(b) \( \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2} \)

(c) \( \frac{1 + i}{2} \)  
(d) \( \frac{1 - i}{2} \)
Quadratic expressions involve one or more terms in $x^2$, $y^2$, or $xy$, but no higher powers of $x$ or $y$. They arise from a number of situations; acceleration, paths of objects, distance, and area are the subject of problems in this chapter.

When $a$, $b$, and $c$ are real numbers, $a \neq 0$, the graph of the equation $y = ax^2 + bx + c$ is a parabola. Using the process known as completing the square, this equation can be rewritten in vertex form $y - k = a(x - h)^2$. This parabola is a translation image of the parabola $y = ax^2$ you have seen in earlier chapters. Its vertex is $(h, k)$, its line of symmetry is $x = h$, and it opens up if $a > 0$ and opens down if $a < 0$.

The parabola crosses the $x$-axis at the values of $x$ for which $ax^2 + bx + c = 0$, the $x$-intercepts of the parabola. These values are found by the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with an *. For all other terms you should be able to give a general description or a specific example.

Lesson 6-1
quadratic expression
quadratic equation
square root, radical sign
expanded form
Binomial-Square Theorem

Lesson 6-2
*quadratic equation
gravitational constant
\( h = -\frac{1}{2}gt^2 + v_0t + h_0 \)
velocity
acceleration

Lesson 6-3
*parabola
directrix, focus
vertex
paraboloid

Lesson 6-4
Graph-Translation Theorem
vertex form of an equation of a parabola

Lesson 6-5
expanded form of an equation of a parabola
completing the square
perfect-square trinomial

Lesson 6-6
Quadratic Formula
standard form of a quadratic equation

Lesson 6-7
*root of an equation
real number
*rational number, *irrational number
*discriminant of a quadratic equation
Discriminant Theorem

Lesson 6-8
*\( \sqrt{-x} \)
*\( \sqrt{-1}, i \)
imaginary number

Lesson 6-9
*complex number
*real part, imaginary part
*equal complex numbers
*complex conjugate

Lesson 6-10
Sum and Product of Roots Theorem
Take this test as you would take a test in class. Use graph paper and a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

In 1–3, consider the parabola with equation \( y = x^2 - 8x + 12 \).

1. Put the equation in vertex form.
2. What is the vertex of this parabola?
3. What are the \( x \)-intercepts of this parabola?

In 4–7, perform the operations and simplify.

4. \( 2i \cdot i \)  
5. \( \sqrt{-8} \cdot \sqrt{-2} \)
6. \( \frac{4 + \sqrt{-8}}{2} \)  
7. \( (3i + 2)(6i - 4) \)

8. If \( z = 2 - 4i \) and \( w = 1 + 5i \), what is \( z - w \)?

9. **Multiple choice** How does the graph of \( y = -2 = -(x + 1)^2 \) compare to the graph of \( y = -x^2 \)?
   (a) It is 1 unit to the right and 2 units below.
   (b) It is 1 unit to the right and 2 units above.
   (c) It is 1 unit to the left and 2 units above.
   (d) It is 1 unit to the left and 2 units below.

10. Graph the solution set to \( y = -2 = -(x + 1)^2 \).

11. The vertex of the parabola below is \( (3, 5) \). The directrix is the line \( y = 6 \). What are the coordinates of the focus?

In 12 and 13, find all solutions.

12. \( 3x^2 + 14x - 5 = 0 \)  
13. \( (m + 40)^2 = 2 \)

In 14 and 15, use the equation \( y = ax^2 + bx + c \) for \( a \neq 0 \) and \( a, b, \) and \( c \) integers.

14. How many \( x \)-intercepts does the graph have?
   a. if its discriminant is 0?
   b. if its discriminant is 1?

15. If \( y = 0 \), describe the nature of the roots
   a. if its discriminant is 0.
   b. if its discriminant is -5.

In 16 and 17, expand.

16. \( (a - 3)^2 \)
17. \( (8v + 1)^2 \)

18. A ball is thrown upwards from an initial height of 20 meters at an initial velocity of 10 meters per second. Write an equation for the height \( h \) of the ball at time \( t \).

In 19 and 20, the height in feet \( h \) of a ball at time \( t \) is given by \( h = -16t^2 + 12t + 4 \).

19. How high is the ball .5 seconds after it is thrown?
20. When does the ball hit the ground?

21. A rectangular piece of metal is 40 cm by 30 cm. Squares \( s \) cm on a side are cut from the corners so the metal can be bent and folded into an open box.
   a. What is the volume of that box in terms of \( s \)?
   b. What is the volume of the box if it is 2 cm deep?

22. a. Draw the figure below. Locate five points on the parabola with focus \( F \) and directrix \( \ell \).

   ![Diagram of a parabola with focus F and directrix \ell]

   \( F \) 1.5 in.

b. Sketch the parabola.

23. **Multiple choice** Which parabola is not congruent to the others?
   a. \( y = 2x^2 \)
   b. \( y = x^2 + 2 \)
   c. \( y = (x + 2)^2 \)
   d. \( y + 2 = x^2 \)
SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers.

- **Objective A:** Expand squares of binomials. *(Lesson 6-1)*
  
  In 1–6, expand.
  
  1. \((a + x)^2\)
  2. \((y - 11)^2\)
  3. \((3x + 4)^2\)
  4. \(2(x - 2)^2\)
  5. \(9(t - 5)^2\)
  6. \(3(a + b)^2 - 4(a - b)^2\)

- **Objective B:** Transform quadratic equations from vertex form to standard form, and vice-versa. *(Lesson 6-5)*
  
  In 7 and 8, transform into standard form.
  
  7. \(y = 3(x + 2)^2 - 10\)
  8. \(y + 8 = \frac{1}{2}(x - 4)^2\)
  
  In 9 and 10, transform each equation into vertex form.
  
  9. \(y = x^2 + 10x - 6\)
  10. \(4y = 2x^2 - 6x - 1\)
  
  11. **Multiple choice** An equivalent form of the equation \(y = 2x^2 - 4x + 3\) is
  
  a. \(y - 1 = 2(x + 1)^2\)
  b. \(y - 1 = 2(x - 1)^2\)
  c. \(y - 3 = 2(x + 1)^2\)
  d. \(y - 2 = 2(x - 1)^2\)

- **Objective C:** Simplify expressions involving imaginary numbers. *(Lessons 6-8, 6-10)*
  
  In 12–19, simplify.
  
  12. \(-i^2\)
  13. \(\sqrt{-36}\)
  14. \(\sqrt{-16} \cdot \sqrt{-49}\)
  15. \(\sqrt{2} \cdot \sqrt{-2}\)
  16. \(10\sqrt{-50}\)
  17. \(3i \cdot i\)
  18. \(\frac{4 \pm \sqrt{-80}}{2}\)
  19. \(-5 \pm \sqrt{-25}\)

- **Objective D:** Solve quadratic equations. *(Lessons 6-6, 6-8, 6-10)*
  
  In 20–33, solve.
  
  20. \((x - 3)^2 = 0\)
  21. \(d^2 - 48 = 0\)
  22. \(z^2 = -8\)
  23. \(w^2 = -9\)
  24. \(x^2 + x - 1 = 0\)
  25. \(10y^2 - 7y = 6\)
  26. \(z^2 - 8z + 11 = -5\)
  27. \(0 = 4a^2 + 3a + 2\)
  28. \(k^2 = 4k + 2\)
  29. \(3x^2 + 2x + 6 = 2x^2 + 4x - 3\)
  30. \(x^2 + 25 = 0\)
  31. \(x(x + 1) = 1\)
  32. \(3 = 5p + 2p^2\)
  33. \(2(3n^2 + 2) = 4(n - 9)\)
**Objective E:** Perform operations with complex numbers. (Lesson 6-9)

In 34–37, perform the operations and write the answer in $a + bi$ form.

34. $(3 + 7i) + (-2 + 5i)$
35. $(8 + i) - (8 - i)$
36. $i(10 + 6i)$
37. $(4 + i)(9 - i)$

**Objective F:** Use the Graph-Translation Theorem to interpret equations and graphs. (Lesson 6-4)

42. The graph of $y = x^2$ is translated 7 units to the left and 5 units down. What is an equation for its image?

43. Multiple choice Which of the following is not true for the graph of the parabola with equation $y - 5 = -2(x + 1)^2$?
(a) The vertex is $(-1, 5)$.
(b) The maximum point is $(-1, 5)$.
(c) The equation of the axis of symmetry is $x = -1$.
(d) The graph opens up.

In 44 and 45, assume that parabola $A$ is congruent to parabola $B$ in the graph below.

44. What translation maps parabola $A$ onto $B$?

45. What is the equation of parabola $B$ if parabola $A$ has equation $y = -10x^2$?

**Objective G:** Use the discriminant of a quadratic equation to determine the nature of the solutions to the equation. (Lesson 6-7)

In 49–52, a. evaluate the discriminant, b. give the number of real solutions, and c. tell whether the solutions are rational, irrational, or complex.

49. $8x^2 + 9x + 6 = 0$
50. $9 + 4y^2 - 12y = 0$
51. $z^2 = 100z + 100$
52. $6 + t = t^2 - 5$
53. How many real solutions does $2y^2 = 3y$ have?

**Objective H:** Classify complex numbers. (Lessons 6-8, 6-9)

In 54–61, tell whether the number is real, nonreal, rational, or irrational.

54. $i$
55. 17
56. $-\frac{2}{3}$
57. $\sqrt{45.3}$
58. $3 + i$
59. $\sqrt{-4}$
60. $\pi$
61. 6.831
Objective I: Use quadratic equations to solve problems dealing with velocity and acceleration. (Lesson 6-2)

62. Suppose a ball is thrown upward from a height of 4.5 feet with an initial velocity of 21 feet per second.
   a. Write an equation relating the time $t$ and height $h$ of the ball.
   b. When will the ball hit the ground? $t = \ldots$

63. A package of supplies is dropped from a helicopter hovering 100 m above the ground. Its parachute does not open. After how many seconds will the package reach the ground? (Neglect air resistance.)

64. A ball is dropped from the roof of a house 25 feet high. To the nearest hundredth of a second, how long will it take the ball to hit the ground?

65. A ball is hit by a bat when 3 feet off the ground. It is caught at the same height 300 feet away from the batter. How far from the batter did it reach its maximum height? 150 feet

Objective J: Solve area problems which can be modeled by quadratic equations. (Lesson 6-1)

66. A 20" by 36" picture is to be surrounded by a frame $w$ inches wide.
   a. What is the total area of the picture and frame?
   b. If the total area is to be $\frac{3}{2}$ the area of the picture, how wide should the frame be?

67. Miriam wants to construct a rectangular pen alongside her house for her puppy Webster. Her father said she could use the 22 meters of chicken wire he had in the garage. What should be the dimensions of the pen if Miriam wants Webster to have as much running area as possible?

Representations deal with pictures, graphs or objects that illustrate concepts.

Objective K: Graph parabolas and interpret them. (Lessons 6-2, 6-3, 6-4, 6-6)

In 68–71, graph the parabola, identifying its vertex and $x$-intercepts.

68. $y = 5x^2 - 20x$
69. $y - 4 = \frac{1}{3} (x + 2)^2$
70. $y + 4 = 3(x - 1)^2$
71. $y = -2x^2 + 4x - 1$
In 72 and 73, refer to the parabolas shown below.

72. Which are not graphs of solutions to an equation of the form \( y - k = a(x - h)^2 \)?

- (a) \( y \)
- (b) \( y \)
- (c) \( y \)
- (d) \( y \)

73. Given \( y - k = a(x - h)^2 \), for which of the above graphs is \( a \) negative?

In 74 and 75, the height of a baseball thrown upwards at time \( t \) is shown on the graph below.

74. When did the ball reach its maximum height? About how high did it get?
75. When was the ball 10 feet high?

■ Objective L: Use the discriminant of a quadratic equation to determine the number of \( x \)-intercepts of the graph (Lesson 6-7)

In 76 and 77, give the number of \( x \)-intercepts of the graph of the parabola.

76. \( y = 3x^2 + 2x - 2 \)
77. \( y = \frac{1}{2}(x + 5)^2 - 3 \)

78. Does the parabola \( y = 6x^2 - 12x \) ever intersect the line \( y = -5 \)?

79. If the graph of \( y = -\frac{1}{4}x^2 \) has one \( x \)-intercept, how many \( x \)-intercepts does the graph of \( y = -\frac{1}{4}(x - a)^2 \) have?

■ Objective M: Find points on a parabola given its focus and directrix. (Lesson 6-3)

80. Graph the set of points equidistant from the point \( (3, 2) \) and line \( y = -2 \) below.

81. Find five points on the parabola with focus \( F \) and directrix \( d \), including the vertex of the parabola.
CHAPTER 7

Functions

7-1: Function Notation
7-2: Graphs of Functions
7-3: Composition of Functions
7-4: Step Functions
7-5: Other Special Functions
7-6: Reflections and Inverses
7-7: Inverse Functions
A function is a correspondence between two variables such that each value of the first variable corresponds to exactly one value of the second variable.

You already know a great deal about many functions, though we have not used the name before. In Chapter 2 you studied functions of variation. In Chapter 3 you studied linear functions, and in Chapter 6 you studied quadratic functions.

**function of variation:** \( y = \frac{4}{x} \)

The variable \( x \) corresponds to a second variable \( y \) whose value is four divided by \( x \).

**linear function:** \( d = -3c + 1 \)

In this function, each value of \( c \) corresponds to only one value of \( d \) which is found by multiplying \( c \) by \(-3\) and then adding \( 1 \).

**quadratic function:** \( y = 2x^2 - 8x + 7 \)

Since the value of \( y \) depends on the value of \( x \), the second or dependent variable is \( y \) and the first or independent variable is \( x \).

In this chapter you will learn about some other special functions and about general properties of functions.
The table of values below was distributed several years ago by the Highway Code of Great Britain. It gives information about three distances (thinking, braking, and stopping distances), each of which is a function of a car’s speed.

<table>
<thead>
<tr>
<th>Speed of car (mph)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thinking distance (ft)</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>Braking distance (ft)</td>
<td>5</td>
<td>20</td>
<td>45</td>
<td>80</td>
<td>125</td>
<td>180</td>
</tr>
<tr>
<td>Stopping distance (ft)</td>
<td>15</td>
<td>40</td>
<td>75</td>
<td>120</td>
<td>175</td>
<td>240</td>
</tr>
</tbody>
</table>

The thinking distance is the distance a car travels after a driver is told to stop but before he or she applies the brakes. The braking distance is the distance the car needs in order to come to a complete stop after the driver applies the brakes. The stopping distance is the sum of the thinking and braking distances.

The data in the table above can be described with formulas. Suppose the car’s speed (in mph) = \(x\).

Then the thinking distance (in feet) = \(x\)

and the braking distance (in feet) = \(\frac{x^2}{20}\)

so the stopping distance (in feet) = \(x + \frac{x^2}{20}\).

If we let \(y = x\), \(y = \frac{x^2}{20}\), and \(y = x + \frac{x^2}{20}\), all the \(x\)'s and \(y\)'s would be too confusing, so formulas often are written using Euler’s \(f(x)\) notation. The notation \(f(x)\) is read “\(f\) of \(x\).” This notation is attributed to Leonard Euler (1707–1780, pronounced “oiler”), a Swiss mathematician who wrote one of the most influential algebra books of all time.

The thinking distance at speed \(x\) is

\[ T(x) = x \] which is read “\(T\) of \(x\) equals \(x\).”

The braking distance at speed \(x\) is

\[ B(x) = \frac{x^2}{20} \] which is read “\(B\) of \(x\) equals \(\frac{x^2}{20}\).”

The stopping distance at speed \(x\) is

\[ S(x) = x + \frac{x^2}{20} \] which is read “\(S\) of \(x\) equals \(x + \frac{x^2}{20}\).”
The parentheses in Euler’s notation do not stand for multiplication. Instead, they enclose the independent variable. We say that T, B, and S are functions. The numbers T(x), B(x), and S(x) stand for the values of these functions, that is, the values of the dependent variable.

Example 1 Evaluate B(45).

Solution Substitute $x = 45$ into the equation $B(x) = \frac{x^2}{20}$.

$$B(45) = \frac{45^2}{20} = 101.25$$

Check Refer to the table on page 376. Braking distance for a car going 45 mph should be between that for cars going 40 mph and cars going 50 mph; that is, between 80 ft and 125 ft. The answer 101.25 feet is a reasonable value of the function at $x = 45$.

In addition to standard formulas and Euler’s notation, functions may be described by the arrow or mapping notation used with transformations. Like Euler’s notation, mapping notation states both the name of the function and the independent variable. For instance, if T, B, and S are names of the thinking, braking, and stopping distance functions, mapping notation for these functions is:

- $T: x \rightarrow x$ which is read “T maps x onto x.”
- $B: x \rightarrow \frac{x^2}{20}$ which is read “B maps x onto $\frac{x^2}{20}$,” and
- $S: x \rightarrow x + \frac{x^2}{20}$ which is read “S maps x onto $x + \frac{x^2}{20}$.”

Example 2 Use the function S defined above to complete the statement:

$$S: 57 \rightarrow ?.$$  

Solution Substitute $x = 57$ into the mapping notation for stopping distance.

$$S: x \rightarrow x + \frac{x^2}{20}$$

$$S: 57 \rightarrow 57 + \frac{57^2}{20}$$

$$S: 57 \rightarrow 219.45$$

This is read “S maps 57 onto 219.45.”

Check The stopping distance should be between 175 ft and 240 ft, so 219.45 is a reasonable value of the function.

You should be able to express formulas for functions with either Euler’s notation or mapping notation.
Example 3 The number of diagonals $d$ of a polygon with $n$ sides is a function of $n$. Rewrite the formula

$$ d = \frac{n(n - 3)}{2} $$

a. using Euler’s notation;

b. using mapping notation.

Solution The independent variable is $n$. Let $f$ be the function’s name.

a. $f(n) = \frac{n(n - 3)}{2}$

b. $f: n \rightarrow \frac{n(n - 3)}{2}$

Notice that $d = f(n)$.

Every function is a relation, that is, a correspondence between a dependent and an independent variable. However, not all relations are functions. We can restate the definition of function given at the beginning of this chapter as follows.

Definition:

A function is a relation in which, for each ordered pair, the first coordinate has exactly one second coordinate.

Example 4 Does the table below represent a function? Why or why not?

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

Solution This relation is not a function, because $x = 4$ is paired with two different $y$-values, $y = 3$ and $y = -3$.

Questions

1. A function is a relation in which for each value of the _?_ variable there is only one value of the _?_ variable.

In 2–4, how is each read?

2. $f(x)$

3. $B: x \rightarrow \frac{x^2}{20}$

4. $S(x) = x + \frac{x^2}{20}$

In 5–7, refer to the functions in this lesson.

5. $B: 50 \rightarrow _$

6. $S(55) =$_

7. $T(10) =$_
8. The value of a function is a value of which variable, dependent or independent?

9. The usual "rule of thumb" is to maintain one car length (about 16 feet) between your car and the car in front of you for every 10 mph. How realistic is this estimate at a speed of 60 mph?

In 10 and 11, refer to Example 3.

10. Evaluate $f(12)$.

11. State whether $n$, the number of sides, is the dependent or independent variable.

In 12 and 13, suppose that $g: x \rightarrow 5x - 2$.

12. Rewrite this formula using Euler's notation.

13. $g: 3 \rightarrow _?$

In 14 and 15, is the relation a function? Why or why not?

14. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

15. \{(16, 2), (8, 1), (0, 0), (8, -1), (16, -2)\}

In 16–18, refer to the graphs at the right. The two graphs give information about the number of farms and the average size (in acres) of these farms across the United States from 1982 to 1987.

16. If the average size function is called $A$, then $A: 1985 \rightarrow _?$.

17. If the number of farms function is called $N$, then $N(1987) = _?$.


In 19 and 20, Margo painted her bedroom. It took her 10 hours. Each hour she wrote down the percent of the job she thought she had finished.

<table>
<thead>
<tr>
<th>Hours</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent finished</td>
<td>5</td>
<td>20</td>
<td>35</td>
<td>50</td>
<td>50</td>
<td>65</td>
<td>70</td>
<td>80</td>
<td>95</td>
<td>100</td>
</tr>
</tbody>
</table>

19. If $T(h)$ is the percent finished after $h$ hours, what is $T(5)$?

20. If Margo began at 8 AM, in which hour did she not work at all?
21. If \( f(x) = \frac{9}{x^2} \), find:
   
   a. \( f(4) \)
   
   b. \( f(x + 4) \)
   
   c. \( f(4x) \)
   
   d. \( f\left(\frac{4}{x}\right) \)

22. Find a formula \( f(x) = \_\) for the direct-variation function described below.

   | \( x \) | 1 | 4 | -1 | 6 | 10 | ...  
   | \( f(x) \) | 3 | 48 | 3 | 108 | 300 | ...  

23. Use the equation \( t = \frac{k}{w} \), where \( t \) = the time in hours an appliance can be run on 1 kilowatt-hour (kwh) of electricity. (1 kwh of electricity runs a 1000-watt appliance for one hour.) \( w \) = the wattage rating of the appliance.

   a. \( t \) varies \( \_ \) as \( \_ \).

   b. Rewrite the equation using Euler's notation. Let \( g \) be the name of the function.

24. Let \( f(x) = 3x + 10 \). Pick any two distinct numbers \( x_1 \) and \( x_2 \) and evaluate

   \[
   \frac{f(x_2) - f(x_1)}{x_2 - x_1}
   \]

   What have you evaluated?

25. Amy's will stipulates that \( \frac{1}{4} \) of her estate goes to charity, \( \frac{1}{8} \) goes to her nephew, and \( \frac{1}{2} \) goes to her son. Her college gets the rest.

   a. If her college gets $50,000, write a sentence to find the value of the estate.

   b. Find the value of the estate. (Lesson 1-7)

   In 26–28, translate each situation into a variation equation using \( k \) as a constant and graph the equation. (Lessons 2-1, 2-2)

26. \( P \) varies directly with the square of \( Q \).

27. \( V \) varies inversely with \( T \).

28. \( x \) is inversely proportional to the fifth power of \( y \).

29. Euler was one of the greatest mathematicians of all time. Find out some of his contributions to mathematics.
There are three ways to describe functions: by listing their ordered pairs; by giving a rule or equation; and by graphing them. Below are the graphs of the three functions studied in the previous lesson. Notice how Euler’s notation helps to distinguish the \( y \)-coordinate of the functions. One of the great advantages of Euler’s \( f(x) \) notation is apparent when more than one function is being studied.
Each point on the graph of a function \( f \) has coordinates of the form \((x, f(x))\). Below, to find the value of \( S(40) \) from the graph, start at 40 on the \( x \)-axis. Read up to the curve of the \( S \) function then across to find the value on the \( y \)-axis.

\[
S(40) = 120
\]

The **domain of a function** is the set of values that are allowable substitutions for the independent variable. The **range of a function** is the set of values that can result from the substitutions for the independent variable. The substitutions for the independent variable are often called *input*, and the resulting values of the dependent variable are often called *output*. For the graphs of functions \( T \), \( B \), and \( S \) on page 381, the domain is the same set, \( \{x: 0 \leq x \leq 60\} \), the allowable speeds in mph.

**Example 1**

Refer to the graphs of the functions \( T \) and \( B \) on the previous page.

a. What is the range of \( T \)?
b. What is the range of \( B \)?

**Solution**

The range is the set of \( y \)-values of the function.

a. \( T(x) = x \). The graph shows that if \( x \) is between 0 and 60, \( y \) is also between 0 and 60, so the range of \( T \) is \( \{y: 0 \leq y \leq 60\} \).
b. \( B(x) = \frac{x^2}{20} \). From the graph you can see that when \( x \) is between 0 and 60, \( y \) is between 0 and 180. Thus the range of \( B \) is \( \{ y : 0 \leq y \leq 180 \} \).

**Check**  You can check by calculations. Look at the equation for each function.

a. \( T(x) = x \) states that the \( y \)-values equal the \( x \)-values. Thus the domain \( \{ x : 0 \leq x \leq 60 \} \) should equal the range \( \{ y : 0 \leq y \leq 60 \} \).

b. \( B(x) = \frac{x^2}{20} \) is an equation for a parabola which opens upward and has a vertex at \( (0, 0) \). The range should extend from \( B(0) \) to \( B(60) \).

\[
B(0) = \frac{0^2}{20} = 0 \quad \text{and} \quad B(60) = \frac{60^2}{20} = 180,
\]
so it checks.

When you are not given the domain, assume that any real number possible can be substituted for the independent variable.

---

**Example 2**  For the function \( f(x) = \frac{12}{x^2} \) state: (a) the domain; (b) the range.

**Solution**  This is an inverse-square function.

a. Any real number except 0 can be substituted into \( \frac{12}{x^2} \). Thus the domain of \( f \) is the set of nonzero real numbers.

b. When \( x \neq 0 \), \( x^2 \) is always positive; so \( \frac{12}{x^2} \) is always positive. Thus the range of \( f \) is the set of positive real numbers.

**Check**  Graph the function. Observe that the graph has points for all \( x \)-values except 0, and for all positive \( y \)-values.
You have already seen examples of relations that are not functions. For example, consider the vertical line with equation \( x = -3 \), graphed below. Two different points on the graph have the same first coordinate, for instance: \((-3, -1)\) and \((-3, 3)\). By definition, in a function there cannot be two ordered pairs with the same first coordinate. This shows that \( x = -3 \) is not an equation for a function. Similarly, neither \( y \leq 3x + 6 \) nor \( y^2 = x \) describes a function, because each contains two points with the same first coordinate.

\[
\begin{align*}
\text{Relations that are not functions} \\
\text{When different points have the same first coordinate, they lie on the same vertical line. This simple idea shows how you can tell whether a relation is a function from its graph.}
\end{align*}
\]

**Theorem (Vertical-Line Test for Functions):**

No vertical line intersects the graph of a function in more than one point.

However, a function can have two ordered pairs with the same second coordinate. For instance, the function \( y = \frac{12}{x^2} \), graphed in Example 2, contains both \((2, 3)\) and \((-2, 3)\). This means that a horizontal line can intersect the graph of a function more than once.
Questions

Covering the Reading

In 1 and 2, define each term.
1. domain of a function
2. range of a function
3. True or false Every function has both a domain and a range.

In 4 and 5, refer to the graphs of functions T, B, and S at the beginning of this lesson.
4. Give their common domain.
5. Give the range of S.

6. In how many points does each line below intersect the graph of the parabola with equation \( x = y^2 \)?
   a. \( x = 4 \)
   b. \( x = 0 \)
   c. \( x = -1 \)

7. Is \( x = y^2 \) an equation for a function? Why or why not?

In 8–10, a relation is graphed. Is the relation a function?
8. 

9. 

10. 

Applying the Mathematics

In 11–15, (a) graph the relation; (b) tell whether the relation is a function; (c) if it is a function, state its domain and range.
11. \{ (2, 4), (3, 4), (5, 4) \}
12. \( 5d + 3c > 30 \), where \( c \) is the independent variable.
13. \( y = 3x^2 \)
14. \( y = \frac{12}{x} \)
15. \( x = -\frac{1}{2} y^2 \), where \( y \geq 0 \)
In 16–18, refer to the graph below. The oven was set for 350°, and the actual temperature $T$, was graphed as a function of time for 30 minutes.

![Graph of temperature over time](image)

16. What was the temperature of the oven when it was turned on?

17. If $F$ is the name of this function, estimate $F(20)$.

18. What is the maximum value of the temperature function?

In 19–22, refer to the graph below. Let $A(t)$ be Alice’s weight at age $t$ and $B(t)$ be Bill’s weight at age $t$. The domain of both these functions is $\{t: 0 \leq t \leq 25\}$.

![Graph of weight over age](image)

19. What is the range of $A$?

20. Estimate $B(5)$ from the graph.

21. For what values of $t$ is $A(t) = B(t)$?

22. During what age intervals did Bill weigh more than Alice?
23. Let \( f(x) = 1 + x^3 \). Find:
   a. \( f(2) \)
   b. \( f(-2) \)
   c. \( f\left(\frac{x}{2}\right) \) (Lesson 7-1)

24. Let \( g(x) = x^2 \). Find:
   a. \( g(x + h) \)
   b. \( g(x + h) - g(x) \)
   c. \( \frac{g(x + h) - g(x)}{h} \) (Lesson 7-1)

25. The function CTOF converts degrees Celsius to degrees Fahrenheit.
   CTOF: \( x \rightarrow 1.8x + 32 \). Then CTOF: \( 100 \rightarrow ? \). (Lesson 7-1)

26. A dress takes 4 yards of material, a skirt takes 2 yards, and a blouse takes 1.5 yards. Pearl Won wants to make 6 dresses, 3 skirts, and 7 blouses. (Lesson 4-2)
   a. Write a 1 x 3 matrix \( M \) for the materials needed for the various outfits.
   b. Write a 3 x 1 matrix \( P \) for the number of outfits wanted.
   c. Multiply the matrices in the correct order to find the total number of yards of material needed.

27. For what value(s) of \( c \) does \( \begin{cases} 3x - 4y = 9 \\ 12x - 16y = c \end{cases} \) have infinitely many solutions? (Lesson 5-6)

28. Pick any one of the following statements and determine whether it is true or false. If it is false, give a counterexample. It may help to make several sketches.
   a. If the domain of a function equals its range, the function is symmetric to the line with equation \( y = x \).
   b. The relation \( y = \begin{cases} 1, & \text{for } x = \text{a terminating or repeating decimal} \\ 0, & \text{for } x = \text{a nonterminating, nonrepeating decimal} \end{cases} \) is a function.
   c. A function may have discrete range and a continuous domain.
Composition of Functions

A car dealer offers a $1000 rebate and 15% discount off the price of a new car. If the sticker price of the car is $12,000, how much will you pay?

The answer to this question depends on which is applied first, the rebate or the discount. If you take the rebate first and then the discount, the selling price in dollars is

\[ .85(12,000 - 1000) = 9350. \]

(Recall that the price after a 15% discount is 85% of what it was before.) However, if you take the discount first, and then the rebate, the selling price in dollars is

\[ .85(12,000) - 1000 = 9200. \]

For a $12,000 car, taking the 15% discount before the $1000 rebate results in a lower selling price. Should you always take the discount first? To answer this question it helps to know about the operation called composition of functions. As with transformations, the composite of two functions \( f \) and \( g \), written \( g \circ f \), is the result of first applying \( f \), then applying \( g \) to the result.

In mapping notation: \( g \circ f : x \rightarrow g(f(x)) \) read "the composite of \( g \) and \( f \) maps \( x \) onto \( g \) of \( f \) of \( x \)."

In Euler’s notation: \((g \circ f)(x) = g(f(x)) \) read "the composite of \( g \) and \( f \) of \( x \) is \( g \) of \( f \) of \( x \)."

We call the rebate function \( r \). This function subtracts 1000.

\[ r(x) = x - 1000 \]

The discount function is \( d \). This function multiplies by \( .85 \).

\[ d(x) = .85x \]

Doing the rebate first means calculating \( d(r(x)) \).

\[ d(r(x)) = d(x - 1000) = .85(x - 1000) = .85x - 850 \]
Doing the discount first means calculating \( r(d(x)) \).
\[
r(d(x)) = r(0.85x) = 0.85x - 1000
\]

Thus \( r(d(x)) \) is always $150 less than \( d(r(x)) \). You should want the discount first.

**Example 1**

Let \( p(x) = x + 5 \) and \( s(x) = x^2 \). Find \((p \circ s)(7)\).

**Solution**
The composite says to square 7, then add 5 to the result.
\[
(p \circ s)(7) = p(s(7)) \\
= p(49) \\
= 49 + 5 \\
= 54
\]

Notice \((s \circ p)(7) = s(p(7)) = s(12) = 144\). So \((s \circ p)(7) \neq p \circ s(7)\). In general, *composition of functions is not commutative*.

**Example 2**
Consider \( g(x) = x^2 - 4 \) and \( h(x) = \frac{-x}{2} \). Find:

a. \((h \circ g)(x)\)

b. \((g \circ h)(x)\).

**Solution**
a. \((h \circ g)(x) = h(g(x))\)
First apply \( g \) to get \( h(x^2 - 4) \).
Now apply \( h \) to \( x^2 - 4 \) to get \( \frac{-(x^2 - 4)}{2} = \frac{-x^2}{2} + 2 \).

b. \((g \circ h)(x) = g(h(x))\)
First apply \( h \) to get \( g\left(\frac{-x}{2}\right)\).
Now apply \( g \) to \( \frac{-x}{2} \) to get \( \left(\frac{-x}{2}\right)^2 - 4 = \frac{x^2}{4} - 4 \).

**Check**
a. Does \((h \circ g)(x) = \frac{-x^2}{2} + 2\)? One way to check is to evaluate the composite for a specific value of \( x \), say 1. Does \((h \circ g)(1) = \frac{-1^2}{2} + 2\)?

\[
\text{Does } h(g(1)) = \frac{-1}{2} + 2 \ ? \\
h(1^2 - 4) = 1.5 \ ? \\
h(-3) = 1.5 \ ? \\
\frac{-(-3)}{2} = 1.5 \ ? \text{ Yes.}
\]

The answer checks for this particular case. This is not a fool-proof check.

b. This check is left for you to do.
In function composition, you must be careful of the domain and range of each function. The domain of the composite cannot have values that are not in the domain of the function that is evaluated first, and the range of the first function must be in the domain of the function that is evaluated second. Occasionally, the domain of the first function must be restricted by excluding certain values so that its output is in the domain of the second.

**Example 3**

Suppose \( r \) is the reciprocal function, \( r(x) = \frac{1}{x} \), and \( f \) is given by \( f(x) = x^2 - 4 \). Find a restriction on the domain of \( r \circ f \).

**Solution**

There are no restrictions for function \( f \) while \( x \) cannot be 0 for \( r(x) \). Now consider \( r(f(x)) \).

\[
r(f(x)) = r(x^2 - 4) = \frac{1}{x^2 - 4}
\]

Since the denominator cannot be zero, \( x^2 - 4 \neq 0 \) and \( x^2 \neq 4 \). So \( x \) cannot equal 2 or -2.

**Questions**

In 1–3, refer to the rebate and discount functions in this lesson.

1. If the sticker price is $13,500 and the rebate comes first, what is the cost of the car?

2. If the sticker price is $13,500 and the discount comes first, what is the cost of the car?

3. Explain why it is always better to take the discount first.

4. If \( f(x) = 3x^2 \) and \( g(x) = 4 - 5x \), calculate:
   a. \( f(g(10)) \)
   b. \( (f \circ g)(10) \)
   c. \( (f \circ g)(0) \)
   d. \( f(g(0)) \)

5. Let \( f(x) = x^2 + x - 9 \) and \( g(x) = 7x \). Calculate:
   a. \( f(g(5)) \)
   b. \( g(f(5)) \)
   c. \( g(f(x)) \)
   d. \( f(g(x)) \)

6. In Question 4, calculate \( f \circ f)(5) \).

7. **True or false** Composition of functions is commutative.

8. The domain for the composite must be in the ___ of the function that is evaluated first, and the ___ of the first function must be in the domain of the function that is evaluated second.
In 9 and 10, let \( r(x) = \frac{1}{x} \) and \( n(x) = x^2 - 9 \).

9. Find the restrictions on the domain of \( r \circ n \).

10. Find the restrictions on the domain of \( n \circ r \).

Let \( d(x) = \sqrt{x} \) and \( q(x) = x^3 - \pi \). Write a calculator key sequence to evaluate \( d(q(1.9)) \).

12. Let \( f(x) = \sqrt{x} \) and \( g(x) = x^2 \).
   a. Calculate \( f(g(-3)) \).
   b. Calculate \( g(f(-3)) \).

13. Consider \( r(x) = \frac{1}{x} \).
   a. Simplify \( r(r(x)) \).
   b. When is \( r(r(x)) \) undefined?

14. Composite functions can be used to describe relationships between functions of variation. Suppose \( w \) varies inversely as the square of \( z \) and \( z \) is proportional to the cube of \( x \).
   a. Give an equation for the function mapping \( z \) onto \( w \).
   b. Give an equation for the function mapping \( x \) onto \( z \).
   c. Give an equation for the function mapping \( x \) onto \( w \).
   d. Use words to describe the function of part c.

15. In the food chain, barracuda feed on bass and bass feed on shrimp. Suppose that the size of the barracuda population is estimated by the function \( r(x) = 1000 + \sqrt{2}x \), where \( x \) is the size of the bass population. Also suppose that the size of the bass population is estimated by the function \( s(x) = 2500 + \sqrt{x} \), where \( x \) is the size of the shrimp population.
   a. Find an equation of the composite which describes the size of the barracuda population in terms of the size of the shrimp population.
   b. About how many barracuda are there when the size of the shrimp population is 4,000,000?
16. Let \( w(x) = \sqrt{x^2 - 6} \). Find two functions \( f \) and \( g \) such that \( w(x) = f(g(x)) \).

17. A photograph is enlarged by a factor of \( K \) in each dimension. By what number is the area multiplied? (Lesson 2-3, Previous course)

18. Graph each function and state its domain and range.
   a. \( f(x) = \frac{1}{x} \)
   b. \( g(x) = \frac{1}{x + 2} \). (Lessons 2-7, 6-4, 7-1, 7-2)

19. Malone found the equation \( h = -25(t - 2)^2 + 100 \) gave the height \( h \) (in feet) of his model rocket \( t \) seconds after it was launched.
   a. How high did the rocket get?
   b. What is the domain of the function mapping \( t \) onto \( h \)?
   c. What is the range of the function mapping \( t \) onto \( h \)? (Lessons 6-4, 7-2)

20. What is the slope of any line that has the same nonzero number for its \( x \)- and \( y \)-intercepts? (Lessons 1-6, 2-4, 3-5)

21. Let \( f(x) = \) the father of \( x \), let \( m(x) = \) the mother of \( x \), let \( b(x) = \) the eldest brother of \( x \), and let \( h(x) = \) the husband of \( x \). Then \((m \circ h)(x)\) is the mother-in-law of \( x \). What relationship is defined by each of the following?
   a. \((h \circ m)(x)\)
   b. \((b \circ m)(x)\)
The daily charge to park a car at a city lot is 75¢ for the first half hour and 50¢ for each additional hour or portion of an hour. Thus the cost of parking is a function of time. The table below gives the cost if a car is parked at 7:00 P.M. and picked up before 1:30 A.M.

<table>
<thead>
<tr>
<th>time</th>
<th>number of minutes since 7:00</th>
<th>cost (in dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7:00- 7:30</td>
<td>$0 &lt; n \leq 30$</td>
<td>.75</td>
</tr>
<tr>
<td>7:30- 8:30</td>
<td>$30 &lt; n \leq 90$</td>
<td>$.75 + .50 = 1.25</td>
</tr>
<tr>
<td>8:30- 9:30</td>
<td>$90 &lt; n \leq 150$</td>
<td>$1.25 + .50 = 1.75$</td>
</tr>
<tr>
<td>9:30-10:30</td>
<td>$150 &lt; n \leq 210$</td>
<td>$1.75 + .50 = 2.25$</td>
</tr>
<tr>
<td>10:30-11:30</td>
<td>$210 &lt; n \leq 270$</td>
<td>$2.25 + .50 = 2.75$</td>
</tr>
<tr>
<td>11:30-12:30</td>
<td>$270 &lt; n \leq 330$</td>
<td>$2.75 + .50 = 3.25$</td>
</tr>
<tr>
<td>12:30-1:30</td>
<td>$330 &lt; n \leq 390$</td>
<td>$3.25 + .50 = 3.75$</td>
</tr>
</tbody>
</table>

The graph below shows the cost of parking a car for up to and including 390 minutes. Notice that the graph represents a function. You can verify this with the vertical-line test. However, the function is discontinuous because you must lift your pencil to draw the graph.
The domain consists of any positive real number of minutes less than or equal to the 1440 minutes in a day. The range is the discrete set of costs: \{$.75, $1.25, $1.75, $2.25, $2.75, $3.25, $3.75, \ldots\}.

The graph above looks like a series of steps and is called a \textbf{step function}. There is a special step function called the \textbf{greatest-integer} or \textbf{rounding-down function}, denoted \([x]\).

\begin{definition}
[x] = \text{the greatest integer less than or equal to } x.
\end{definition}

Applying the definition gives \([4\frac{1}{2}] = 4\) because 4 is the largest integer that is less than or equal to \(4\frac{1}{2}\). Likewise,

\([\pi] = 3, [7] = 7, \text{ and } [-5.2] = -6.\]

The last instance may seem wrong, but recall that \(-6 < -5.2 < -5\); so the largest integer less than or equal to \(-5.2\) is \(-6\). Notice that the domain of \(f: x \rightarrow [x]\) is the set of real numbers; the range is the set of integers.

Here is how to graph the function \(f\), where \(f(x) = [x]\).

For all \(x\) greater than or equal to 0 but less than 1, the greatest integer is 0. For all \(x\) greater than or equal to 1 but less than 2, the greatest integer is 1. In a similar manner you can get the other values in the table below. The graph is at the right below.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x) = [x])</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3 \leq x &lt; -2</td>
<td>-3</td>
</tr>
<tr>
<td>-2 \leq x &lt; -1</td>
<td>-2</td>
</tr>
<tr>
<td>-1 \leq x &lt; 0</td>
<td>-1</td>
</tr>
<tr>
<td>0 \leq x &lt; 1</td>
<td>0</td>
</tr>
<tr>
<td>1 \leq x &lt; 2</td>
<td>1</td>
</tr>
<tr>
<td>2 \leq x &lt; 3</td>
<td>2</td>
</tr>
<tr>
<td>3 \leq x &lt; 4</td>
<td>3</td>
</tr>
</tbody>
</table>

On the graph, there are open circles at \((1, 0), (2, 1), (3, 2), \text{ and so forth to indicate that these values are not solutions to } f(x) = [x].\) At these points the function value is jumping to the next higher step.
The greatest-integer function has many applications in formulas, for instance, those for determining tax and postal rates. It is often used when function values must be integers but formulas would give noninteger values. On many computers, INT(X) denotes the greatest-integer function.

**Example**

Describe what the following computer program does for any number N.

10 INPUT "A REAL NUMBER"; N
20 PRINT INT(N + .5)
30 END

**Solution**
The PRINT statement computes \([N + .5]\) for any value of \(N\) input into the program. We pick several values of \(N\) and evaluate this expression.

\[
\begin{align*}
\text{If } N = & \quad \text{The computer calculates} \\
4 & \quad [4 + .5] = 4 \\
33\frac{1}{2} & \quad [33\frac{1}{2} + .5] = [34] = 34 \\
-78.8 & \quad [-78.8 + .5] = [-78.3] = -79 \\
7.49 & \quad [7.49 + .5] = [7.99] = 7
\end{align*}
\]

You can see that this program takes any number, rounds it to the nearest integer, and prints the result.

To investigate how the function \(N \rightarrow [N + .5]\) works, consider each step. Adding 0.5 keeps numbers whose tenths place has a 0 to 4 (and should be rounded down) without changing the integer portion of the number. For instance, \(33.4\) becomes \(33.9\), which is still between 33 and 34. Adding 0.5 moves numbers whose tenths place has a 5 to 9 (and should be rounded up) beyond the next integer. For instance, \(7.6\) is mapped onto \(8.1\). Then, taking the greatest integer, the result is that \(N\) is rounded up.

**Questions**

In 1 and 2, refer to the parking example at the beginning of this lesson.

1. What would be the cost to park a car for 408 minutes?

2. What is the domain of the function?

3. a. \(f(x) = [x]\) is called the _?_ or _?_ function.
   b. The range of \(f\) \(x \rightarrow [x]\) is _?_.
   c. Why are there open circles at \((1, 0), (2, 1), (3, 2)\), and so forth in the graph of \(f\)?
Applying the Mathematics

4. \[ \left[ 3 \frac{3}{2} \right] \] 5. \[ 11.9 \] 6. \[ -11.7 \] 7. \[ 8 + .5 \]

In 8–10, refer to the Example.
8. What will the computer print when \( N = 64.39 \)?
9. When will the computer print 7?

10. If line 20 in the program is changed to

\[ 20 \text{ PRINT } 10 \times \text{INT}((N + 5)/10) \]

a different sort of rounding occurs. What kind of rounding is it?

11. Find a value of \( x \) for which \( [x + .5] \neq [x] + .5 \).

12. At present to mail a letter first class in the U.S. costs 25¢ for the first ounce or fraction thereof, and 17¢ for each additional ounce or fraction thereof. Let \( w \) = the weight in ounces of a letter and \( c \) = the cost in dollars of mailing it first class.
   a. Draw a graph of \( c \) as a function of \( w \) for \( 0 < w \leq 5 \).
   b. True or false The formula \( c = .25 + .17[w - 1] \) describes this function.

13. Multiple choice A stadium used for graduation has 25,000 seats. There are \( s \) seniors graduating. Which of the following represents the number of graduation tickets each senior may have?
   (a) \( \frac{25000}{s} \)  \hspace{1cm} (b) \( \left\lfloor \frac{25000}{s} \right\rfloor \)  \hspace{1cm} (c) \( 25000 \cdot s \)

14. Tyrone is paid a weekly salary of $175 plus an additional $45 for each $300 in sales.
   a. Find his salary during a week when he has $1000 in sales.
   b. Multiple choice When he has \( d \) dollars in sales, his weekly salary equals which of the following?
      (i) \( 175 + \left\lfloor \frac{45d}{300} \right\rfloor \)  \hspace{1cm} (ii) \( 175 + 45 \left\lfloor \frac{d}{300} \right\rfloor \)  \hspace{1cm} (iii) \( 175 + \left( \frac{45}{300} \right)d \)

15. The rounding-up function is denoted \( [x] \). This gives the smallest integer greater than or equal to \( x \). For example, \( [4.7] = 5 \).
   a. Find \( [2.3] \).
   b. Find \( [-1.8] \).
   c. For all \( 5 < x \leq 6 \), find \( [x] \).
   d. Graph \( r(x) = [x] \).

In 16–18, suppose \( f: x \rightarrow 3x - 4 \) and \( g: x \rightarrow x^2 \). (Lesson 7-3)
16. \( f \circ g: 8 \rightarrow ? \)  \hspace{1cm} 17. \( f \circ g: x \rightarrow ? \).
18. \( f(g(x)) = ? \)

19. Graph the function with equation \( x = y^2 + 1 \), where \( y \geq 0 \). (Lesson 7-2)
20. Define: function. (Lesson 7-1)

In 21 and 22, use the fact that a sequence is a special kind of function. It is a function whose domain is the set of positive integers. Find the range for the function determined by each formula. (Lessons 1-3, 1-4, 7-1)

21. \( t_n = 2n \)

22. \[
\begin{align*}
\begin{cases}
  t_1 = 1 \\
  t_n = \frac{t_{n-1}}{2}, \text{ for } n \geq 2
\end{cases}
\end{align*}
\]

23. Without using a calculator, simplify each root. (Previous course)
   a. \( \sqrt{20} \)  
   b. \( \sqrt{147} \)  
   c. \( \sqrt{\sqrt{16}} \)

24. Solve \( 2m^2 - 7 = 12m \). Round the solutions to the nearest hundredth. (Lesson 6-6)

25. The formula \( W = d + 2m + \left[ \frac{3(m + 1)}{5} \right] + y + \left[ \frac{y}{4} \right] - \left[ \frac{y}{100} \right] + \left[ \frac{y}{400} \right] + 2 \) gives the day of the week based on our current calendar where

   \( d = \) the day of the month of the given date

   \( m = \) the number of the month in the year with January and February regarded as the 13th and 14th months of the previous year; that is, 2/22/90 is 14/22/89. The other months are numbered 3 to 12 as usual.

   \( y = \) the year.

   Once \( W \) is computed, divide by 7 and the remainder is the day of the week, with Saturday = 0, Sunday = 1, ..., Friday = 6.

   a. Find the day of the week on which you were born.

   b. On what day of the week was the Declaration of Independence adopted?
In addition to the greatest-integer function, other functions are important enough to have special names. One of these is the absolute-value function and is denoted |x|. Recall the definition of absolute value:

\[ |x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ when } x < 0. \]

Because -x is the opposite of x, -x is positive when x is negative. For this reason, |x| is never negative. For example, |24| = 24 and |-24| = -(24) = 24. In general, |x| = |-x|. The function \( f(x) = |x| \) is a function whose domain is the set of real numbers and whose range is the nonnegative real numbers.

The graph of \( f(x) = |x| \) consists of two rays. When \( x \geq 0 \), the graph is a ray from (0, 0) with slope 1. This is shown on the graph in blue. When \( x < 0 \), the graph is a ray with slope -1. This is shown in green. The graph of \( f(x) = |x| \) is the union of these two rays, and so the graph of \( f(x) = |x| \) is an angle. The minimum point (0, 0) on the function is the vertex of the angle.

In many computer languages, the absolute-value function is denoted \texttt{ABS}. That is, \texttt{ABS}(X) = |X|.

**Example 1** Write a \texttt{LET} statement for each formula.

a. \( d = |a - b| \)

b. \( y = |3x| \)

**Solution**

a. \( \text{LET } D = \text{ABS}(A - B) \)

b. \( \text{LET } Y = \text{ABS}(3 \times X) \)

Some special functions are the powering functions. The simplest powering function is \( f(x) = x^1 \) which is called the **identity function**.
The simplest quadratic function is \( f(x) = x^2 \), which is called the **squaring function**, and the function \( f(x) = x^3 \) is called the **cubing function**. In general, the function \( f(x) = x^n \), where \( n \) is a positive integer, is called the **nth power function**. The following BASIC program gives the coordinates of points graphed below and on the next page. Recall that \( x^N \) means \( x^n \).

```basic
10 PRINT "VALUES FOR THE FUNCTION F(X) = X^N."
20 INPUT "FOR N = "; N
30 PRINT "F(X) = X^"; N
40 PRINT "X", "F(X)"
50 FOR X = -5 TO 5
60 PRINT X, X^N
70 NEXT X
80 END
```

If you have a function grapher, set the domain to \( \{x: -5 \leq x \leq 5\} \) and verify these graphs.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
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<tbody>
<tr>
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<td>5</td>
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</tbody>
</table>

The **Identity Function**

The **Squaring Function**

The **Cubing Function**
Several properties of the powering functions can be deduced.

1. The graph of every powering function \( f(x) = x^n \) passes through the origin, because \( 0^n = 0 \) for any positive integer value of \( n \).

2. For all powering functions the domain is the set of real numbers, because you can raise any real number to a positive integer power. (Note: many calculators will give an error message when a negative number is raised to a power with the \( y^x \) key. This means the calculator cannot do the problem, not that it cannot be done.)

3. To find the range, two cases must be considered for \( n \), a positive integer.
   (i) \( n \) is even:
   If \( x \geq 0 \), then \( x^n \geq 0 \) because any nonnegative number raised to a power is nonnegative. If \( x < 0 \), then raising \( x \) to an even power results in a positive number. Thus, when \( n \) is even, the range is \( \{y: y \geq 0\} \), and the graph of \( f(x) = x^n \) where \( n \) is even is in quadrants I and II. Check this by observing the graphs of \( f(x) = x^2 \) and \( f(x) = x^4 \).
   (ii) \( n \) is odd:
   If \( x \geq 0 \), then \( x^n \geq 0 \) because any nonnegative number raised to a power is nonnegative. If \( x < 0 \), then raising \( x \) to an odd power results in a negative number. Thus, the range is all real numbers, and the graph of \( f(x) = x^n \) where \( n \) is odd is in quadrants I and III. Check this by observing the graphs of \( f(x) = x \), \( f(x) = x^3 \), and \( f(x) = x^5 \).

4. The graph of every powering function has symmetry.
   (i) \( n \) is even:
   The even-powering functions have reflection symmetry. The graph of \( f(x) = x^n \) is symmetric to the \( y \)-axis when \( n \) is an even positive number.
(ii) \( n \) is odd:
The odd-powering functions have rotation symmetry. The graph of \( f(x) = x^n \) can be mapped onto itself under a \( 180^\circ \) rotation around the origin when \( n \) is an odd positive integer.

**Example 2** Which of the following graphs could represent the function with equation \( y = x^n \)?

![Graphs](image.png)

**Solution** This is an odd-powering function so the range is the set of all real numbers. Only (b) and (c) have the correct range. Also, the graph of \( y = x^n \) must have rotation symmetry about the origin; only graph (c) does. Thus, (c) could be the correct graph.

Another special function is the **square-root function** \( f: x \rightarrow \sqrt{x} \), where \( x \) is a nonnegative real number. In BASIC, \( \sqrt{x} \) is represented by SQR(X). The square root function that is on your calculator, \( \sqrt{\ } \), always gives the nonnegative value for the square root. The square-root function on your calculator is the **upper** half of the parabola with equation \( x = y^2 \).

![Graphs](image.png)

It would be possible to define a key on the calculator that would always give the nonpositive square root. This would be represented by the **lower** half of the parabola \( x = y^2 \), where \( y \leq 0 \). However, such a key is not needed. You can press \( \sqrt{\ } +/\ ) \) to get \(-\sqrt{x} \). Few calculators deal with complex numbers, so the domain of SQR is the set of nonnegative real numbers.
Questions

1. State the domain and range of the function \( f \) with \( f(x) = |x| \).

2. \( |117.8| = ? \) \( |1-11| = ? \)

3. Fill in the blanks. The graph of \( y = |x| \) can be thought of as the piecewise linear graph of \( y = \begin{cases} (a) & \text{if } x < 0, \\ (b) & \text{if } x \geq 0 \end{cases} \)

4. a. \( y = x^3 \) is called the \( ? \) function.
   b. \( y = x^n \) is called the \( ? \) function.

5. Quickly sketch the graphs indicated without plotting points or doing any calculations.
   a. \( f(x) = x^8 \)
   b. \( f(x) = x^9 \)
   c. the identity function
   d. \( f(x) = -\sqrt{x} \)

In 6–8, refer to the properties of the powering functions.

6. If \( n \) is even, the range of \( y = x^n \) is \( ? \) and the graph is in quadrants \( ? \) and \( ? \).

7. If \( n \) is odd, the range of \( y = x^n \) is \( ? \) and the graph is in quadrants \( ? \) and \( ? \).

8. What are acceptable values for \( n \)?

9. True or false The graphs of the odd powering functions have no minimum or maximum values.

In 10 and 11, write a LET statement in BASIC for each formula.

10. \( y = |x - 5| \)

11. \( c = \sqrt{a^2 + b^2} \)

12. In Chapter 2 you read that the power generated by a windmill is proportional to the cube of the wind speed \( w \).
   a. Write an equation using Euler’s notation to describe the variation.
   b. Sketch the graph over the appropriate domain.

In 13 and 14, a powering function is graphed. Write an equation for each function.

13. \( y \)

   \( (3,729) \)

14. \( y \)

   \( (2,32) \)
15. a. Graph \( d(x) = -|x| \).
   b. State the domain and range.

16. a. Graph \( f(x) = |2x| \), \( g(x) = 2|x| \), \( h(x) = |x| - 2|x| \), and \( i(x) = -2|x| \).
   b. Make a generalization about the effect of the value of \( a \) on the graphs of \( y = a|x| \) and \( y = |ax| \).

17. The formula \( e = |p - l| \) describes the allowable margin of error \( e \) for a given length of a product \( p \) where \( I \) stands for the ideal length. When constructing a swimming pool for international competitions the contractors aim for a length of 50.015 meters with an acceptable margin of error of .015 m.
   a. Write an equation satisfied by the possible lengths \( p \) of pools in this situation.
   b. What are the possible lengths?

18. a. Complete the program below so it accepts as input any two real numbers and prints the distance between them on the number line.
   ```plaintext
   10 INPUT "TWO REAL NUMBERS"; X1; X2
   20 PRINT "THE DISTANCE BETWEEN"; (i);
   "AND"; (ii); "IS"; (iii)
   30 END
   ```
   b. Modify the program above so it accepts as input the coordinates \((x_1, y_1), (x_2, y_2)\) of any two points, and prints the distance between them.
19. Graph $f(x) = -[x]$. (Lesson 7-4)

20. A school district has buses that hold 120 students each. Let $b$ be the number of buses needed to transport $s$ students. (Lesson 7-4)

**Multiple choice** Which formula describes this situation? (Remember: $\lceil \cdot \rceil$ indicates the rounding-up function.)

(i) $b = \left\lceil \frac{s}{120} \right\rceil$  
(ii) $b = \lceil 120s \rceil$  
(iii) $b = \left\lceil \frac{s}{120} \right\rceil$

21. If $f(x) = 3x^2$ and $g(x) = 2x^3$, does $f(g(x)) = g(f(x))$ for all values of $x$? (Lesson 7-3)

22. Give an equation for the line perpendicular to $3x + 2y = 5$ at the point $(1, 1)$. (Lesson 4-7)

23. Solve for $t$: $y = m + xt$. (Lesson 1-8)

24. Give an equation for a relation whose graph is a line but which is not a function. (Lesson 7-1)

25. What transformation changes

\[
\begin{bmatrix}
2 & -3 & 0 \\
1 & 4 & 6
\end{bmatrix}
\]
to

\[
\begin{bmatrix}
1 & 4 & 6 \\
-2 & 3 & 0
\end{bmatrix}
\]?
(Lesson 4-5)

26. You are at the hair stylist looking in a mirror at a clock. You see the figure at the right.

(Previous course)

a. In your mind, to tell the time, what transformation do you need to apply?

b. What time is it?

27. In the last two chapters you have seen several functions, some of which are listed below. Write an equation for the composite of two of these, or two of your own choosing, and then graph your composite. Try this with several pairs of functions. Some composites have surprising graphs.

\[
f(x) = [x] \\
g(x) = |x| \\
h(x) = x^2 \\
i(x) = x^3 \\
j(x) = \sqrt{x} \\
k(x) = \frac{6}{x}
\]

28. For this question, the symbol $\sqrt{1 - x^2}$ means $\sqrt{1 - x^2}$, for $-1 \leq x \leq 1$. Explore the graph of the function $f(x) = \sqrt{1 - x^2}$. (That is, what is the shape of the graph? What are its intercepts? Does the graph have any symmetry?)
Just as numbers and matrices can have inverses, so can functions. The inverse of a function has the following definition.

**Definition:**

The inverse of a function is the relation obtained by reversing the order of the coordinates of each ordered pair in the function.

That is, the inverse switches the $x$- and $y$-coordinates.

**Example 1**

Let $f = \{(1, 4), (2, 8), (4, 16), (-1, -4), (-2, -8), (-3, -12)\}$. Find the inverse of $f$.

**Solution**

The inverse is found by switching the coordinates of each ordered pair. If we call the inverse $g$, then

$$g = \{(4, 1), (8, 2), (16, 4), (-4, -1), (-8, -2), (-12, -3)\}.$$

**Check**

Notice that an equation for $f$ is $y = 4x$. An equation for $g$ is $x = 4y$. Thus $x$ and $y$ have been switched.

Recall that the domain of a function is the set of possible values for $x$ and the range is the set of possible values for $y$. Because the inverse is found by switching the $x$- and $y$-coordinates, the domain and range of the inverse are found by switching the domain and range of the function. That is, in general, if $f$ and $g$ are inverses, then

$$\text{domain of } f = \text{range of } g$$

$$\text{range of } f = \text{domain of } g.$$
This relationship is illustrated in the graph below of a function f and its inverse g. From the graph you can see that:

- domain of f = \{x: -2 \leq x \leq 2\}
- range of g = \{y: -2 \leq y \leq 2\}
- domain of g = \{x: 1 \leq x \leq 7\}
- range of f = \{y: 1 \leq x \leq 7\}

Recall that when the point \((x, y)\) is reflected over the line with equation \(y = x\), the image is \((y, x)\). That is, the reflection over the identity line switches the coordinates of the ordered pairs. Thus, the graphs of a function and its inverse are reflection images of each other about the line \(y = x\). This is easily seen in the above graph.

Caution: the word inverse, when used in the phrase inverse of a function, is different and unrelated to its use in the term inverse variation.

The inverse of a function is not always a function, as the next example shows.

**Example 2**
Consider the function with domain the set of all real numbers and equation \(y = x^2\).

a. What is an equation for the inverse?
b. Graph the function and its inverse on the same coordinate axes.
c. Show that the inverse is not a function.

**Solution**
a. To find an equation for the inverse, switch the coordinates \(x\) and \(y\). Given the function with equation \(y = x^2\), its inverse has equation \(x = y^2\).
b. The graphs of \( y = x^2 \) and \( x = y^2 \) are below. Notice again that the inverse is the reflection image of \( y = x^2 \) across the line with equation \( y = x \).

In Example 1, each \( y \) value was paired with just one \( x \) value. There the inverse is a function. However, in Example 2 some \( y \) values are paired with two different \( x \) values; that is, \((2, 4)\) and \((-2, 4)\) are both on the graph of \( y = x^2 \). When the coordinates are switched, the new first coordinate is paired with two different second coordinates; that is, \((4, 2)\) and \((4, -2)\) are both on the graph of the inverse \( x = y^2 \). Thus the inverse cannot represent a function. We conclude that to have an inverse that is a function, the original function cannot map two distinct members of the domain onto the same member of the range. We say there must be a one-to-one (abbreviated 1-1) correspondence between the domain and range of the function for its inverse to be a function.

You can also tell by looking at the graph of the original function whether or not its inverse represents a function. The horizontal-line test is useful in deciding if the situation of Example 2 exists.

**Theorem (Horizontal-Line Test for Inverses):**

The inverse of a function is itself a function if and only if no horizontal line intersects the graph of the function in more than one point.

To tell if an inverse is a function, you can either:

- **a.** apply the horizontal line test to the given function; or
- **b.** draw the graph of the inverse and apply the vertical-line test.
The graphs below show both strategies.

**Function**

1-1 function.
Passes horizontal-line test.

**Inverse**

Inverse is a function.
Passes vertical-line test.

Not a 1-1 function.
 Doesn’t pass horizontal-line test.

Inverse is not a function.
 Doesn’t pass vertical-line test.

In a function, the independent variable (usually \(x\)) determines the value of the dependent variable (usually \(y\)). In Example 2, because \(y = x^2\) is a function, \(x\) determines \(y\). When the function has an inverse, the process can be reversed. In Example 2, because the inverse is not a function, \(y\) does not determine \(x\).

### Questions

1. The inverse of a function can be found by __?__ the coordinates of the function.

2. Let \(f = \{(4, 8), (2, 4), (3, 6), (-1, -2), (-5, -10)\} \)
   a. Write an equation for the function and list the values of the domain.
   b. Write the coordinates of the inverse.
   c. Write an equation for the inverse function.
   d. Graph the function \(f\) and the inverse on the same set of axes.

3. The graphs of any function \(f\) and its inverse are reflection images over the line __?__.
4. Refer to Example 2.
   a. Is the function \( y = x^2 \) a 1-1 function?
   b. What is an equation for its inverse?
   c. Find two points other than those in the lesson that show the inverse
      is not a function.

In 5 and 6, give an equation for the inverse of the function.

5. \( y = 3x \)  
6. \( y = 9x^2 + 12x - 6 \)

7. To tell if the inverse of a function is a function:
   a. apply the \( f \) test if you have graphed the function.
   b. apply the \( f \) test if you have graphed the inverse.

In 8–10, (a) is the function 1-1? (b) Is the inverse a function?

8. 

9. 

10. 

11. The graph of \( y = 4x + 9 \) is shown at the right.
    a. Use the horizontal-line test to decide if the inverse is a function.
    b. Find an equation for the inverse.
    c. Graph the inverse.
    d. How are the slopes of the function and its inverse related?

12. a. Graph the inverse of the absolute-value function.
    b. Is the inverse a function? Explain why or why not.
    c. What rule describes the inverse of the absolute value function?

13. In 1987, a U.S. dollar was worth about 1.33 Canadian dollars. That
    means an item costing 1 U.S. dollar would cost about 1.33 Canadian
    dollars. Let \( x = \) cost in U.S. dollars and \( y = \) cost in Canadian
    dollars.
    a. Find an equation for \( f: x \rightarrow y \).
    b. Find an equation for \( g: y \rightarrow x \).
14. This question examines the inverse of an inverse variation. Consider the function with equation \( y = \frac{6}{x} \).
   a. Find an equation for the inverse.
   b. Is the inverse a function? Why or why not?
   c. If \( y \) varies inversely as \( x \), then \( x \) varies \( \frac{6}{y} \) as \( y \).

15. The graph at the right shows the height of a flag on a pole as a function of time. This would be a difficult function to write as a formula.
   (Lessons 7-4, 3-8)
   a. What is happening to the flag?
   b. Why are there some horizontal segments on the graph?

16. Given \( f(x) = x^2 + 1 \) and \( g(x) = x^2 - 1 \), find \( (f \circ g)(x) \).
   (Lesson 7-3)

17. Let \( f(x) = 3x + 4 \) and \( g(x) = \frac{1}{3}(x - 4) \).
   a. Find \( (g \circ f)(15) \).
   b. Find \( (g \circ f)(x) \).
   c. What is another name for the function \( g \circ f \)? (Lessons 7-3, 7-5)

18. Refer to the figure at the right. Lines \( \ell \) and \( m \) are parallel. Find the measures of the numbered angles. (Previous course E)

19. a. If \( f: x \rightarrow \frac{1}{2}x + 5 \), calculate \( \frac{f(10) - f(-10)}{10 - (-10)} \).
   b. What have you calculated?
   (Lesson 7-1)

20. Another way to think of an inverse is to do the opposite operations of the function in the reverse order.
   a. Pick an equation with 3 operations such as \( y = \frac{2x - 3}{7} \). List the steps in the correct order that you would do to evaluate a value for \( x \). Construct the inverse by doing the opposite operations in the reverse order to \( x \).
   b. Pick a routine that you do every day and see if you also do its inverse. Try these or one of your own.
   
   Is your drive home the inverse of your drive to school?
   Is your routine for getting ready to go to sleep the inverse of getting up in the morning?
   Is parallel parking a car the inverse of getting out of a parking spot by the curb?
The image seen in your rear-view mirror is the reflection image of the lettering on the ambulance.

In the last lesson, you learned that the inverse of a function is not always a function. When the inverse is a function, the symbol \( f^{-1} \), read "f inverse," is used to denote it. (Be careful: in this case the -1 is a symbol for inverse and is not an exponent.) Thus, for the function \( f(x) = 4x \), the inverse, which is also a function, can be denoted by \( f^{-1}(x) = \frac{1}{4}x \). This is read "the inverse of f of x is one fourth x."

**Example 1**

Use the function \( f(x) = 4x + 1 \).

a. Find a rule for the inverse and write the rule in Euler notation.

b. Find \( (f \circ f^{-1})(x) \).

c. Find \( (f^{-1} \circ f)(x) \).

**Solution**

a. To find a rule for the inverse, replace \( f(x) \) by \( y \) and then switch \( x \) and \( y \).

\[
f(x) = 4x + 1
\]

The inverse of

\[
y = 4x + 1
\]

is

\[
x = 4y + 1.
\]

Now solve for \( y \).

\[
x - 1 = 4y
\]

Subtract 1 from both sides.

\[
\frac{1}{4} (x - 1) = y
\]

Multiply both sides by \( \frac{1}{4} \).

The inverse function can be written \( f^{-1}(x) = \frac{1}{4} (x - 1) = \frac{x - 1}{4} \).
b. \( (f \circ f^{-1})(x) = f(f^{-1}(x)) \)
\[ = f\left(\frac{x - 1}{4}\right) \]
\[ = 4\left(\frac{x - 1}{4}\right) + 1 \]
\[ = x - 1 + 1 \]
\[ = x \]

c. \( (f^{-1} \circ f)(x) = f^{-1}(f(x)) \)
\[ = f^{-1}(4x + 1) \]
\[ = (4x + 1) - 1 \]
\[ = \frac{4x}{4} \]
\[ = x \]

The results of parts b and c above make sense, since \( f^{-1} \) undoes what \( f \) did to \( x \).

**Inverse Function Theorem:**

\( f \) and \( g \) are inverse functions if and only if
\[ (f \circ g)(x) = (g \circ f)(x) = x. \]

This theorem states that if you input a value into either composite, it will output the same value. You can use the composite of two functions to check whether the functions are indeed inverses of each other.

**Example 2** Are \( f(x) \to \frac{2}{3}x + 8 \) and \( g(x) \to \frac{3}{2}x - 8 \) inverses of each other?

**Solution** Find \( f \circ g \).

\[ (f \circ g)(x) = f(g(x)) \]
\[ = f\left(\frac{3}{2}x - 8\right) \]
\[ = \frac{2}{3} \left(\frac{3}{2}x - 8\right) + 8 \]
\[ = x - \frac{16}{3} + 8 = x + \frac{8}{3} \]

Since \( (f \circ g)(x) \neq x \), \( f \) and \( g \) are not inverses of each other.

It is necessary to check both composites \( f \circ g \) and \( g \circ f \) to show that two functions are inverses of each other. In some situations one composite may lead to the conclusion that two functions are inverses, while the other shows that they are not.
Example 3  Show that \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \) are not inverses.

Solution  Step 1: Find \((f \circ g)(x)\).

\[
(f \circ g)(x) = f(g(x)) \\
= f(\sqrt{x}) \\
= (\sqrt{x})^2 \\
= x
\]

It appears from this composite that \( f \) and \( g \) are inverses. However, notice that the domain is only the nonnegative real numbers, since the square-root function is done first.

Step 2: Find \((g \circ f)(x)\).

\[
(g \circ f)(x) = g(f(x)) \\
= g(x^2) \\
= \sqrt{x^2}
\]

The expression \( \sqrt{x^2} \neq x \); rather, \( \sqrt{x^2} = |x| \). You can see this by evaluating a negative number, say \(-3\).

\[
\sqrt{(-3)^2} = \sqrt{9} = 3
\]

The answer is not \(-3\), but \(3\), which is \(|-3|\). Thus, functions \( f \) and \( g \) are not inverses.

Example 3 shows that the squaring and square-root functions are not inverse functions. However, they can be inverses if the domain is restricted. Consider the graph of \( f(x) = x^2 \) at the right. It does not pass the horizontal-line test, so its inverse is not a function.

However, at the right are shown graphs of \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \) for \( x \geq 0 \). Both \( f \) and \( g \) pass the vertical-line test so they are each functions.
If then the inverse has equation
Solving for \( y \) gives

\[ y = x^2 \]
\[ x = y^2 \]
\[ \sqrt{x} = \sqrt{y^2} \]
\[ = |y| \]

Since \( y > 0 \), \( \sqrt{y^2} = y \) and we can write
Thus the inverse of
\( f(x) = x^2 \) is
\( g(x) = \sqrt{x} \).

Since \( g \) is a function, we can write \( g = f^{-1} \). That is, \( f^{-1}(x) = \sqrt{x} \), where \( x \geq 0 \).

In general, the odd-powering functions have inverses that are functions, but the inverses of even-powering functions are not functions. However, by restricting the domain of the even-powering functions so that these functions are 1-1, it is possible to make the inverse a function.

### Questions

#### Covering the Reading

1. If \( f \) is a function, then the symbol \( f^{-1} \) represents the ___ of the function and is read ___.

2. The function \( h(x) = 6x - 5 \) has an inverse which is a function.
   a. Find a rule for the inverse and write the rule in Euler notation.
   b. Check your answer to part a by finding \( h \circ h^{-1} \).

3. Refer to Example 1. Find \( (f \circ f^{-1})(2) \).

4. For any function \( f \) that has an inverse, \( (f \circ f^{-1})(x) = \) ___.

5. Refer to Example 2. Find \( (g \circ f)(x) \).

In 6 and 7, refer to Example 3.

6. Why is it necessary to check both \( f \circ f^{-1} \) and \( f^{-1} \circ f \) to see if two functions are inverses?

7. True or false. The square root function is the inverse of the squaring function.

8. a. Graph \( y = x^2 \) where \( x \geq 2 \).
   b. Is the relation in part (a) a function? Is its inverse a function?
9. Refer to the graph at the right of the cubing function \( y = x^3 \).
   a. What are the domain and range of this function?
   
   b. Graph the inverse function. (It is called the cube-root function.)
   
   c. Find the domain and range of the cube root function.

10. a. Find an equation for the inverse of the linear function \( y = mx + b \).
    b. How are the slopes of a linear function and its inverse related?
    c. When is the inverse not a function?

In 11–13, consider the absolute-value function graphed at the right. Which of the following ways of restricting the domain gives a function whose inverse is also a function?
11. \(-2 \leq x \leq 5\)
12. \(x \geq 1\)
13. \(x \geq -4.5\)

14. Alexis makes a commission of 40% of her ticket sales to a health and nutrition seminar plus a salary of $25,000 per year. An equation for her total income \( y \), in terms of ticket sales \( x \), is
   \[
   y = 25,000 + .4x.
   \]
   
   a. Find the inverse.
   
   b. What does the inverse find?

15. Let \( g(x) = 6x \).
   a. Find an equation for \( g^{-1} \).
   
   b. Find an equation for the inverse of \( g^{-1} \). This is denoted by \( (g^{-1})^{-1} \).
   
   c. If \( f \) is any function that has an inverse, what is \( (f^{-1})^{-1} \)?

16. Consider the function \( f: x \rightarrow x^2 + 8x + 16 \). Graph \( f \) and its inverse on the same set of axes. (Lessons 6-1, 7-6)

In 17–19, solve each equation. (Lessons 1-7, 6-5, 6-8)
17. \( 4x^2 = \frac{16}{289} \)
18. \( \frac{54}{y} = -144 \)
19. \( x^2 + 4 = 0 \)
20. The Hidawa Family Acrobatic Troupe perform an act shown below. Suppose Haimunda Hidawa leaves the teeter-totter at an initial velocity of 20 feet per second.

\[ \text{Before} \]

\[ \text{During} \]

a. Write an equation to describe Haimunda’s height after \( t \) seconds.

b. Assuming Haimunda has good aim, will she be launched high enough to land atop her two brothers? (Lesson 6-9)

In 21–24, match the graphs below to the correct function. (Lessons 7-4, 7-5, 7-6)

(a) \( y \) \hspace{1cm} (b) \( y \) \hspace{1cm} (c) \( y \) \hspace{1cm} (d) \( y \)

\[ \begin{align*}
\text{(e)} & \quad y \quad \text{\hspace{1cm} (f)} & \quad y \quad \text{\hspace{1cm} (g)} & \quad y
\end{align*} \]

21. \( f(x) = x^2 \) for \( x \geq 0 \)

22. \( g(x) = |x| + 1 \)

23. \( h(x) = x^3 \)

24. \( k(x) = [x - 2] \)
25. The graph below can be defined as:

\[ f(x) = \begin{cases} 
  x^2 & \text{if } -1 \leq x \leq 1 \\
  1 & \text{if } -3 \leq x \leq -1, \text{ or } 1 \leq x \leq 3 \\
  x - 2 & \text{if } x \geq 3 \\
  -x - 2 & \text{if } x \leq -3.
\end{cases} \]

a. Make up a function of your own choosing that has a pleasing shape. Give a piecewise definition for it.

b. Draw its inverse.
A function is a correspondence that maps elements from one set (its domain) onto elements of the same or another set (its range). Functions are often named by the single letter f. The mapping notation $f: x \rightarrow 3x^2$, Euler’s notation $f(x) = 3x^2$ and $y = 3x^2$ all describe the same quadratic function.

Some important functions have their own names.

**Greatest-integer function**

$$f(x)$$

INT: $x \rightarrow [x]$  

**Absolute-value function**

$$f(x)$$

ABS: $x \rightarrow |x|$  

**Identity function**

$$f(x)$$

$I: x \rightarrow x$  

**Squaring function**

$$f(x)$$

$y = x^2$  

**$n$th-Powering function, $n$ even**

$$f(x)$$

$y = x^n$, $n$ even  

**$n$th-Powering function, $n$ odd**

$$f(x)$$

$y = x^n$, $n$ odd  

Every function has an inverse, found by switching components of its ordered pairs. Some inverses are functions. A relation is a function if no vertical line intersects its graph in two points. A function $f$ has an inverse $f^{-1}$ if no horizontal line intersects its graph in two points. The graphs of $f$ and $f^{-1}$ are reflection images of each other over the line $y = x$. At the right, the square-root function is graphed; its graph is the reflection image of part of the squaring function.
Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description and a specific example.

Lesson 7-1
*relation
*function
value of a function
Euler's f(x) notation
arrow or mapping notation

Lesson 7-2
*domain of a function
*range of a function
Vertical-Line Test for Functions

Lesson 7-3
composition of functions
composite of f and g, g ° f

Lesson 7-4
step function
*greatest-integer function, rounding-down function
[x], INT (X)
rounding-up function

Lesson 7-5
*absolute-value function
*identity function
*squaring, cubing, nth-powering function
*square-root function

Lesson 7-6
*inverse of a function f, f⁻¹
one-to-one correspondence, 1-1 correspondence
Horizontal-Line Test for Inverses

Lesson 7-7
Inverse-Function Theorem
*cube-root function
CHAPTER 7

Progress Self-Test

Take this test as you would take a test in class. Use graph paper and a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. If \( f(x) = 9x^2 - 11x \), find \( f(3) \).

2. Suppose \( T: n \rightarrow \frac{n(n + 1)}{2} \). Then \( T(12) = \) ?

3. If \( f(n) = n^2 \), find \( f(n + 1) \).

In 4 and 5, determine whether the relation is a function.

4. \( \{ (95, -5), (4, 4), (5, 5) \} \)

5. \( \begin{array}{c|cccc} x & 1 & 2 & 3 & -2 & -1 \\ \hline y & 2 & 5 & 16 & 5 & 2 \end{array} \)

In 6–8, consider the functions \( f(x) = x^2 \) and \( g(x) = -8x \).

6. Find \( f(g(7)) \).

7. Determine \( g(f(x)) \).

8. Are \( f \) and \( g \) inverses of each other? Justify your answer.

9. The graph of the powering function \( p: x \rightarrow x^6 \) lies in which quadrants?

In 10 and 11, a function contains only the points \((1, 2), (3, 4), \) and \((5, 6)\).

10. What is its domain?

11. What is its inverse?

12. A function has equation \( g(x) = 5x + 10 \). Find an equation for \( g^{-1} \).

In 13 and 14, use the graphs below.

13. Which are graphs of functions?

14. Of those graphs that are functions, which have inverses that are also functions?

15. The cost of making a phone call from an airplane is \$7.50 \) for the first three minutes and \$1.75 \) for each additional minute or portion of a minute.

   a. Graph the function for any call lasting up to 8 minutes.

   b. How much would it cost Isaiah Rich to make a call for \( 6\frac{1}{3} \) minutes?

In 16–19, refer to the function graphed below.

16. Is the function 1-1?

17. What is the range of the function?

18. Graph the inverse of the function.

19. How can you restrict the domain of the original function so that the inverse is also a function?

20. For the function \( f(x) = x^2 + 2 \), determine

   a. the domain;

   b. the range.

21. Consider \( g(x) = -[x + 1] \). Find \( g(-4.3) \).

22. Graph \( f \), where \( f(x) = |x| - 3 \).

23. In a computer program, \( \text{ABS}(X) \) denotes the absolute value function.

   \[ 10 \text{ INPUT "TWO NUMBERS"; M,N} \]
   \[ 20 \text{ PRINT ABS(M-N)} \]
   \[ 30 \text{ END} \]

   a. If 3.2 is input for \( M \) and 7 for \( N \), what will be printed when the program is run?

   b. In general, what will the output from this program represent?
**SPUR** stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers.

**Objective A:** Find the value of a function. (Lessons 7-1, 7-4, 7-5)

1. If \( f(x) = 2x - 3 \), what is \( f(4) \)?
2. Suppose \( t: n \rightarrow 5 - 4n^2 \). Then \( t(-2) = {?} \).
3. Let \( h: a \rightarrow a^5 \). Find \( h(-3) \).
4. If \( g(x) = 4x^2 \), find \( g(x + 2) \).
5. If \( h(x) = 8x - 20 \), find \( h(x + 5) - h(x) \).
6. If \( M: x \rightarrow 3x^2 + 4x \), then \( M(2k) = {?} \).
7. \([-2.5] = {?} \)
8. Find \( f(x) \) for \(-2 \leq x < -1\), if \( f(x) = [x] \).
9. \(|-5 - 9| = {?} \)
10. If \( g(x) = |x| - 12 \), find \( g(4.5) \).

**Objective B:** Find the composite of functions. (Lesson 7-3)

11. In the symbolism \( f \circ g \), which function is applied first?

In 12–15, let \( t(x) = x^2 + 1 \) and \( m(x) = x - 6 \).
12. Find \( t(m(5)) \).
13. Find \( t(m(x)) \).
14. What rule describes \( m \circ t \)?
15. The function \( t \circ m \) maps \(-10\) onto what number?
16. If \( a: x \rightarrow -\frac{3}{2}x \) and \( b: x \rightarrow -\frac{7}{2}x \), what is \((b \circ a)(x)\)?

**Objective C:** Read simple BASIC programs with special functions. (Lessons 7-4, 7-5)

In 17–20, use this program where \( N > 0 \).

```plaintext
10 INPUT N
20 PRINT 1000 * INT((N + 500)/1000)
30 END
```

17. a. What will line 20 print if you run the program and input 25,962?
   b. If \( N = 1.2 \cdot 10^3 \), what will line 20 print when the program is run?
18. Describe what this program does when run.
19. Modify this program so it rounds positive numbers to the nearest hundred.
20. If line 20 reads
   PRINT 3 + ABS(-8 * (N + 2))
what will be printed when \(-4\) is input for \( N \)?

**Objective D:** Obtain rules for inverses of functions. (Lessons 7-6, 7-7)

21. If \( f: x \rightarrow 2x + 7 \), then \( f^{-1}: x \rightarrow {?} \).
22. If \( g(x) = -x^2 \) for \( x \leq 0 \), then \( g^{-1}(x) = {?} \).
23. A function has equation \( y = 4x - 2 \). In slope-intercept form, what is an equation for its inverse?
24. A function has equation \( y = |x| \). What is an equation for its inverse?
**Objective E:** Determine whether a given relation is a function. (Lessons 7-1, 7-2, 7-4, 7-5)

25. Is the relation described by the table below a function? If not, why not?

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-1</th>
<th>-4</th>
<th>0</th>
<th>-16</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>y</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>-4</td>
</tr>
</tbody>
</table>

In 26–31, tell whether the set or rule describes a function.

26. \{(1, 2), (2, 3), (3, 4), (4, 1)\}

27. \(x = y^2\)

28. \(y = [x + 1]\)

29. \(xy = 15\)

30. \(x = -5\)

31. \(x = |y|\)

32. A relation has the points (1, 2), (2, 1), (1, 3), and (4, 1). Take out one point so that this relation is a function.

**Objective F:** Determine the domain and range of simple functions given the function rule. (Lessons 7-2, 7-4, 7-5)

33. What real number is not in the domain of \(f\), where \(f(x) = \frac{1}{x}\)?

In 34–39, give the domain and range of the function described.

34. \{(2, -2), (3, -3), (-9, 9), (-\frac{1}{2}, \frac{1}{2})\}

35. \(y = x^4\)

36. \(f(x) = \sqrt{x}\)

37. \(f(x) = [x]\)

38. \(y = x^2 + 1\)

39. \(y + 5 = 3(x - 4)^2\)

**Objective G:** Determine relationships between a function and its inverse. (Lessons 7-6, 7-7)

40. For what positive integer values of \(n\) does the function with equation \(y = x^n\) have an inverse that is also a function?

41. Suppose the domain of a linear function is \(\{x: x \geq 0\}\) and the range is \(\{y: y = 6\}\). What are the domain and range of the inverse?

In 42 and 43, true or false.

42. If a function is 1-1, then its inverse is also a function.

43. If functions \(f\) and \(g\) are inverses of each other, then \((f \circ g)(x) = (g \circ f)(x)\).

44. Show that \(f: x \to 2x + 3\) and \(g: x \to \frac{1}{2}x - 3\) are not inverses of each other.

**Uses**

**Objective H:** Find values of functions involving real data. (Lessons 7-1, 7-4, 7-5)

In 45–47, let \(P(x)\) and \(B(x)\) be the populations of Philadelphia and Baltimore in year \(x\).

<table>
<thead>
<tr>
<th>Year</th>
<th>Philadelphia</th>
<th>1900</th>
<th>1950</th>
<th>1980</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1,290,000</td>
<td>2,070,000</td>
<td>1,690,000</td>
</tr>
<tr>
<td></td>
<td>Baltimore</td>
<td>509,000</td>
<td>950,000</td>
<td>787,000</td>
</tr>
</tbody>
</table>

45. What does \(B(1950)\) represent?


   b. What does part a represent?

47. a. Calculate \(\frac{P(1950) - P(1900)}{1950 - 1900}\).

   b. What does the answer to part a represent?

48. The famous scientist Galileo found a relationship between the distance \(d(t)\) a dropped object falls (in feet) in the time \(t\) (in seconds). \(d(t) = 32t^2\). (Of course, he used different units.) What was the approximate distance fallen when \(t = 1.5\)?

49. Multiple choice You earn $5 per hour. The time you work is rounded down to the nearest hour. What is a rule for the function that relates time \(t\) in hours to wages \(w\) in dollars?

   (a) \(w = 5t\)
   (b) \(w = 5\lceil t \rceil\)
   (c) \(w = 5\lfloor t \rfloor\)
   (d) \(w = 5(t - \frac{1}{2})\)
50. Evening phone rates (not including taxes) between Chicago and New York City are 47¢ for the first minute and 33¢ for each additional minute or portion of a minute. How much will it cost to make a 12 minute 10 second phone call between these cities?

REPRESENTATIONS deal with pictures, graphs, or objects that illustrate concepts.

- **Objective I:** Graph a function, given its rule. *(Lessons 7-2, 7-4, 7-5)*
  
  51. Graph \( f(x) = x^2 + 6x + 9 \).
  
  52. Graph \( g \) if \( g: x \rightarrow \frac{10}{x} \).
  
  53. Graph \( y = |x| + 4 \).
  
  54. If \( h(x) = |x| - 11 \), graph \( h \).
  
  55. Let \( y = -2x^3 \). Graph this function.
  
  56. Refer to Question 50 above. Graph the relation between time \( t \) in minutes and cost \( c \) in dollars for the domain \( 0 \leq t \leq 10 \).
  
  57. The graph of \( y = x^9 \) lies in which quadrants?
  
  58. True or false The graph of the identity function lies in quadrants II and IV.

- **Objective J:** Determine the domain and range of a relation from its graph. *(Lesson 7-2)*

  In 59–62, find the domain and range of the relation graphed below.
  
  59. 
  
  60. 
  
  61. 
  
  62. 

- **Objective K:** Apply the vertical and horizontal line tests for a function and its inverse. *(Lessons 7-2, 7-6)*

In 63 and 64, use the graphs below.

(a) \[ y = -x^2 \]

(b) \[ x = |y| \]

(c) \[ y = \frac{1}{x} \] (1, 1)

(d) \[ xy = 12 \]

63. Multiple choice Which of the above is not a graph of a function?

64. **a.** Multiple choice Which of the above is a function whose inverse is not a function?
  
  **b.** How can you restrict the domain of the function in your answer to part a so that the inverse is a function?

65. Draw a graph of a function with domain \( \{ -1 < x < 1 \} \) that has an inverse which is not a function.

66. True or false The horizontal-line test fails if a function is not 1-1.

- **Objective L:** Graph the inverse of a function. *(Lessons 7-6, 7-7)*

67. Suppose \( f \) and \( f^{-1} \) are graphed on the same set of coordinate axes. How are the graphs related?

68. Graph the inverse of the function at the right.

69. Graph the inverse of the function with equation \( y = |x| \).
The ratio of \( F \), the amount of food a mammal must eat per day, to \( m \), its body mass, is not constant across species. For a mouse, \( \frac{F}{m} \approx 0.6 \); that is, it must eat three-fifths of its mass per day. At the other extreme, for the elephant \( \frac{F}{m} \approx 0.02 \), so an elephant needs to eat only .02, or \( \frac{1}{50} \), of its mass per day.

It can be shown that the ratio of amount of food eaten daily to body mass varies as the negative one-third power of the mass. This can be written as

\[
\frac{F}{m} = km^{-1/3}.
\]

In this chapter you will learn how to interpret negative powers and fractional powers and how to solve equations involving powers and roots. You will also learn about other applications of powers and roots in the real world.
<table>
<thead>
<tr>
<th>Mammal</th>
<th>Weight (lb)</th>
<th>Food per day (lb)</th>
<th>Food per day weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>elephant</td>
<td>14000</td>
<td>320</td>
<td>.023</td>
</tr>
<tr>
<td>moose</td>
<td>800</td>
<td>48</td>
<td>.060</td>
</tr>
<tr>
<td>raccoon</td>
<td>21</td>
<td>4</td>
<td>.19</td>
</tr>
<tr>
<td>guinea pig</td>
<td>1.5</td>
<td>.72</td>
<td>.48</td>
</tr>
<tr>
<td>mouse</td>
<td>.80</td>
<td>.48</td>
<td>.60</td>
</tr>
</tbody>
</table>
Recall that the expression $b^n$, read "$b$ to the $n$th power" or "the $n$th power of $b$," is the result of an operation called powering or exponentiation. The variable $b$ is called the base, $n$ is called the exponent, and the expression $b^n$ is called a power. In your earlier study of algebra and on several occasions in this book you have worked with powers. You know, for example, that

\[
10^2 = 10 \cdot 10, \\
(-\frac{2}{3})^3 = (-\frac{2}{3})(-\frac{2}{3})(-\frac{2}{3}), \\
\text{and } x^7 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \text{ for all numbers } x.
\]

These sentences are based on the following meaning of powering when the exponent is a positive integer.

**Repeated Multiplication Model for Powering:**

If $b$ is a real number and $n$ is a positive integer, then

\[
b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}.
\]

In this chapter you will study powers with exponents that are not positive integers. The properties of the familiar powers will be used to determine the meaning of these new powers.

Recall how repeated multiplication can be used to work with powers.

**Product of Powers:**

\[
10^2 \cdot 10^3 = (10 \cdot 10) \cdot (10 \cdot 10 \cdot 10) = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5
\]

\[
x^4 \cdot x^2 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6
\]

**Power of a Power:**

\[
(10^2)^3 = 10^2 \cdot 10^2 \cdot 10^2 = (10 \cdot 10) \cdot (10 \cdot 10) \cdot (10 \cdot 10) = 10^6
\]

\[
(x^4)^2 = x^4 \cdot x^4 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^8
\]

**Power of a Product:**

\[
(3 \cdot 10)^4 = (3 \cdot 10) \cdot (3 \cdot 10) \cdot (3 \cdot 10) \cdot (3 \cdot 10) = 3^4 \cdot 10^4
\]

\[
(8x^5)^2 = (8x^5) \cdot (8x^5) = (8\cdot x\cdot x\cdot x\cdot x) \cdot (8\cdot x\cdot x\cdot x\cdot x) = 8^2x^{10} = 64x^{10}
\]

Each situation above is an instance of a general pattern which we assume.
Postulate 4: Properties of Powers

For any nonnegative bases and real exponents, or any non-zero bases and integer exponents:

Product of Powers Property: \( b^m \cdot b^n = b^{m+n} \)
Power of a Power Property: \((b^m)^n = b^{mn}\)
Power of a Product Property: \((ab)^m = a^m b^m\)
Quotient of Powers Property: \(\frac{b^m}{b^n} = b^{m-n}\)
Power of a Quotient Property: \(\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\)

Applying these postulates enables you to rewrite expressions and calculate with powers.

Example 1  Write \(4^5 \cdot 4^8\) as a power of 4.

Solution  Apply the Product of Powers Postulate.
\[
4^5 \cdot 4^8 = 4^{5+8} = 4^{13}
\]

Check  Use your calculator.
\[
4^5 = 1024; \quad 4^8 = 65,536; \quad 4^{13} = 67,108,864
\]

Note that
\[
\]

Example 2  Simplify \((x^2)^5\).

Solution  Use the Power of a Power Postulate.
\[
(x^2)^5 = x^{2\cdot5} = x^{10}
\]

Check  Test a special case, say \(x = 3\). Does \((3^2)^5 = 3^{10}\)?
\[
(3^2)^5 = 9^5 = 59,049 \\
3^{10} = 59,049
\]

This case checks.
Example 3 Verify that \( \frac{2^{11}}{2^8} = 2^3 \).

Solution 1 Rewrite the numerator and denominator using the repeated multiplication meaning of \( b^n \).

\[
\frac{2^{11}}{2^8} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{2 \cdot 2 \cdot 2}{1} = 2^3
\]

Solution 2 Use a calculator. A possible key sequence is

\[2 \downarrow \uparrow 11 \div \uparrow \uparrow 2 \downarrow \uparrow 8 \div \uparrow \uparrow \]

The display shows that \( 2048 \div 256 = 8 \), and \( 8 = 2^3 \).

Properties of powers are often used when working with large numbers.

Example 4 The earth is about \( 93 \cdot 10^6 \) miles from the sun. Light travels at about \( 1.86 \cdot 10^5 \text{ mi/sec} \). About how long does it take light from the sun to reach the earth?

Solution Use the formula \( d = rt \).

\[
t = \frac{93 \cdot 10^6 \text{ mi}}{1.86 \cdot 10^5 \text{ mi/sec}} = \frac{93}{1.86} \cdot 10^1 \text{ sec} = 50 \cdot 10 \text{ sec} = 500 \text{ sec}
\]

It takes about 500 seconds, or 8.3 minutes, for light to travel from the sun to the earth.

When the Quotient of Powers Property is applied to equal powers of the same base, the result is surprising to some people. For instance,

\[
\frac{2^8}{2^8} = 2^{8-8} = 2^0.
\]

It is also true that

\[
\frac{2^8}{2^8} = \frac{256}{256} = 1.
\]

The above statements and the transitive property of equality prove that \( 2^0 = 1 \).
In fact, whenever $b$ is a nonzero real number,

$$\frac{b^n}{b^n} = b^{n-n} = b^0.$$ 

Also, $\frac{b^n}{b^n} = 1$ because any nonzero number divided by itself equals one. Thus we have proved the following theorem.

**Zero Exponent Theorem:**

If $b$ is a nonzero real number,

$$b^0 = 1.$$ 

Properties of exponents are used to simplify many expressions.

---

**Example 5**

Three tennis balls stacked tightly as shown at the right just fill a cylindrical can. What is the ratio of the volume of the balls to the volume of the can?

**Solution** The balls may be considered as congruent spheres of radius $r$. The volume of each sphere is $\frac{4}{3}\pi r^3$. Let $V_B$ equal the total volume of the three tennis balls. Then

$$V_B = 3\left(\frac{4}{3}\pi r^3\right) = 4\pi r^3.$$ 

The can is a cylinder with radius $r$ and height $6r$. Let $V_C$ equal the volume of the can. Then

$$V_C = \pi r^2 h = \pi r^2 (6r) = 6\pi r^3.$$ 

Therefore,

$$\frac{V_B}{V_C} = \frac{4\pi r^3}{6\pi r^3} = \frac{2}{3}.$$ 

---

**Questions**

In 1–3, (a) evaluate the expression using a postulate or theorem from this lesson. (b) Check your answer by applying the repeated multiplication model for powering.

1. $6^2 \cdot 6^3$  
2. $\frac{10^3}{10^2}$  
3. $(4^2)^5$  
4. $2^3$ raised to what power is $2^{12}$?  
5. Verify that $(2 \cdot 5)^4 = 2^4 \cdot 5^4$. 

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In 6–11, name the property that justifies the statement.
6. \( x^2 \cdot x^7 = x^9 \)  
7. \( (3a)^5 = 3^5a^5 \)
8. \( \frac{y^{12}}{y^3} = y^9 \)  
9. \( \left( \frac{x}{2} \right)^{10} = \frac{x^{10}}{2^{10}} \)
10. \( (b^3)^{13} = b^{39} \)  
11. For \( x \neq 0 \), \( x^0 = 1 \).

In 12–17, simplify.
12. \( (x^4)^3 \)  
13. \( (6x^7)^2 \)  
14. \( 10x^7 \cdot 3x \)
15. \( \frac{n^{18}}{n^6} \)  
16. \( \frac{n^{15}}{(n^3)^5} \)  
17. \( \frac{z^{100}}{z^0} \)

18. Refer to Example 4. Pluto is about \( 4.6 \cdot 10^9 \) mi from the sun. About how long does it take light to travel from the sun to Pluto?

In 19 and 20, refer to Example 5.
19. Which two properties of exponents are used in the solution?
20. Suppose a tennis can could hold four balls stacked tightly on top of each other. What would be the ratio of the volume of the balls to the volume of the can?

21. \( x^2 \) and \( x^6 \) are powers of \( x \) whose product is \( x^8 \). Find four more pairs of powers of \( x \) whose product is \( x^8 \).

In 22 and 23, solve.
22. \( (6 \cdot 10)^x = 216000 \)  
23. \( 0 < 10^y < 2 \) if \( y \) is a nonnegative integer.

In 24–26, simplify.
24. \( \frac{w^5 \cdot w^6}{w^3 \cdot w^8} \)  
25. \( \frac{(-8x^2)^3}{2^x} \)  
26. \( \left( \frac{4}{x} \right)^2 \left( \frac{x}{2} \right)^4 \)

27. a. Evaluate \( F(r) = 4\pi r^2 \), when \( r = 4 \cdot 10^3 \). Write your answer as a constant times a power of 10.
   b. If the value of \( r \) given in part a is the approximate radius of the earth in miles, what does \( F(4 \cdot 10^3) \) represent?

28. a. For which figure is the ratio of volume to surface area greater: a sphere or a cube?
   b. Justify your answer.

In 29 and 30, the population of the U.S. in 1980 was about \( 227 \cdot 10^6 \).
29. If the land of the U.S. was about \( 3.5 \cdot 10^6 \) mi\(^2\), what was the average number of people per square mile of land?
30. In 1980 people in the U.S. consumed about 86.7 lbs of fresh fruit per person. What was the total weight of fresh fruit consumed in the U.S. that year?
31. What is a postulate? (Lesson 1-5)
32. What is a theorem? (Lesson 1-6)

In 33 and 34, solve the system. (Lessons 5-2, 5-3, 6-6)

33. \(3A + B = 7\)
\[B = 9A + 5\]
34. \(y = x^2\)
\[y = 3x + 4\]

35. Write the first four terms of the sequence generated by the formula \(t_n = 500(1.1)^n\). (Lesson 1-3)

36. a. Below is a table of some powers of 2.

\[
\begin{align*}
2^0 &= 1 \\
2^1 &= 2 \\
2^2 &= 4 \\
2^3 &= 8 \\
2^4 &= 16 \\
2^5 &= 32 \\
2^6 &= 64 \\
2^7 &= 128
\end{align*}
\]

Look carefully at the last (units) digit of each numeral. Predict the last digit of \(2^{13}\). Check your prediction by calculation.

b. What should be the last digit of \(2^{20}\)? Justify your answer.

c. Explain how to find the last digit of any positive integral power of 2.

d. Explore powers of 3. Describe the patterns that occur in the last digits of these powers.
Suppose a person deposits $2000 in a bank that pays interest at an annual rate of 6%. If no money is added or withdrawn, after one year the bank account will have the original amount invested plus 6% interest.

Amount after 1 year: \(2000 + 0.06(2000) = 2000(1 + 0.06) = 2000(1.06) = 2120\)

Notice that to find the amount after 1 year, you do not have to add the interest; rather you just multiply the amount invested by 1.06. Similarly, at the end of the second year there will be 1.06 times the balance from the first year.

Amount after 2 years: \(2000(1.06)(1.06) = 2000(1.06)^2 = 2247.20\)
Amount after 3 years: \(2000(1.06)^2(1.06) = 2000(1.06)^3 \approx 2382.03\)

Because the interest earns interest each year, the process is called compounding. Notice the general pattern.

Amount after \(t\) years: \(2000(1.06)^t\)

For example, after 12 years there will be \(2000(1.06)^{12} \approx 4024.39\).

The amount of money will be more than double the original deposit.

There is a more general formula. Replace 6% by \(r\), the annual interest rate, and 2000 by \(P\), the principal, or original amount invested. Repeat the process shown above to find \(A\), the amount the investment is worth after \(t\) years.

**Compound Interest Formula:**

Let \(P\) be the amount of money invested at an annual interest rate of \(r\) compounded annually. Let \(A\) be the total amount after \(t\) years. Then

\[ A = P(1 + r)^t. \]

In the Compound Interest Formula, notice that \(A\) is directly proportional to \(P\); for example, doubling the principal doubles the amount at the end. However, \(A\) is not directly proportional to \(r\); doubling the rate does not necessarily double the amount earned.
Example 1 Emilio invests $1000 in an account compounded at an annual rate of 6% and another $1000 in a second account compounded at an annual rate of 12%. How much money will Emilio have earned in each account after 3 years?

Solution In Emilio's first account, \( P \) is $1000, \( r \) is .06, and \( t = 3 \).

\[
P(1 + r)^t = A
\]

\[
1000(1.06)^3 \approx 1191.02
\]

In his second account, he will have

\[
1000(1.12)^3 \approx 1404.93.
\]

Emilio will have earned $191.02 interest in the 6% account and $404.93 in the 12% account.

It may surprise you that the 12% account earned more than twice as much interest as did the 6% account. In accounts where one interest rate is double the other, the interest will be double for the first compounding only. Thereafter, the interest earns interest and the larger account will earn more than double the interest of the other. That is why the 12% account will have earned more than double the interest of the 6% account.

Example 2 In the situation of Example 1, how much money will Emilio earn in the 4th year in the account paying 12% compounded annually?

Solution 1 One way to find the interest of a particular year is to find the balance from the previous year, then multiply by the rate \( r \). The balance after 3 years in the second account was found to be $1404.93. The fourth year's interest is \((1404.93)(.12) \approx 168.59\).

Solution 2 Another way to find the interest of the fourth year is to subtract the total in the third year from the total in the fourth. Let \( F(t) \) be the amount in Emilio's second account at the end of \( t \) years.

\[
F(4) = 1000(1.12)^4 \approx 1573.51
\]

\[
F(3) = 1000(1.12)^3 \approx 1404.93
\]

\[
\text{difference} = 168.58
\]

Few people today do calculations of compound interest by hand. Banks require their employees to look in tables or use calculators or computers. A calculator sequence for evaluating \( P(1 + r)^t \) is

\[
P \times (1 + r)^t = .
\]

Before calculators, people used tables of logarithms, numbers which you will study in Chapter 9.
Most banks compound interest more than once a year. If a bank compounds *semi-annually*, the interest rate at each compounding is *half of the annual interest rate* but there are *two compoundings each year* instead of just one. Therefore, the compound interest formula becomes

\[ A = P \left( 1 + \frac{r}{2} \right)^{2t}. \]

A bank that compounds *quarterly* uses the compound interest formula

\[ A = P \left( 1 + \frac{r}{4} \right)^{4t}. \]

In this way, the compound interest formula can be generalized.

**General Compound Interest Formula:**

Let \( P \) be the amount invested at an annual interest rate \( r \) and compounded \( n \) times per year. Let \( A \) be the amount after \( t \) years. Then

\[ A = P \left( 1 + \frac{r}{n} \right)^{nt}. \]

The compound interest formula in either form is found in other situations besides bank accounts.

**Example 3**

Glenda’s parents bought an airline ticket for $400 with a credit card which charges an annual rate of 18% and compounds the interest monthly. If they do not pay for the ticket for five months, what will the ticket have cost them?

**Solution**

Because the compounding is monthly, \( n = 12 \) in the compound interest formula.

\[ A = P \left( 1 + \frac{r}{12} \right)^{12t} \]

Substitute \( P = 400, r = .18, \) and for five months, \( t = \frac{5}{12}. \)

\[ A = 400 \left( 1 + \frac{.18}{12} \right)^{12 \cdot \frac{5}{12}} \]

Do some arithmetic before using a calculator.

\[ A = 400(1 + .015)^5 \]

\[ A = 430.92 \]

Glenda’s parents will owe $430.92. It has cost them $30.92 to delay payment for 5 months.
In 1 and 2, Lucy invests $3000 in a CD (certificate of deposit) that pays interest at a rate of 8% compounded annually. The interest is left in the account and she makes no deposits or withdrawals.

1. To find next year's balance, you can multiply this year’s balance by \( 1 + \frac{0.08}{1} \).

2. How much will be in the account after four years?

3. A person deposits $2000 in a bank account that pays interest at an annual rate of 6%. If no money is added or withdrawn, tell how much will be in the account after 1, 2, 3, 4, and 5 years.

In 4 and 5, refer to Examples 1 and 2.

4. True or false Emilio earned more than $500 interest in each of his accounts in the first four years.

5. In two ways find the interest earned in the fourth year in the account paying 6% compounded annually.

6. In Example 2, why do the solutions show different answers?

7. True or false Noel invests $1000 compounded annually at 4%. Chris invests $1000 compounded annually at 8%.
   a. In the first year, Chris’s account will earn twice as much as Noel’s.
   b. In the second year, Chris’s account will earn twice as much as Noel’s.

8. Refer to Example 3. Glenda’s parents charged $600 worth of clothes and didn’t pay for seven months. How much did they wind up paying for the clothes?

9. Multiple choice In a compound interest situation, the total amount \( A \) is directly proportional to:
   (a) the rate \( r \).
   (b) the time \( t \).
   (c) the number \( n \) of compoundings in a year.
   (d) the initial amount \( P \).

10. Write the compound interest formula for an account that compounds interest: (a) quarterly; (b) monthly; (c) daily.

11. Katie puts $10,000 in a 6-year 8.625% savings certificate where interest is compounded daily.
    a. How much will she earn during the entire six year period?
    b. How much will she earn in the sixth year?

12. In 1987 the U.S. national debt was about $2.35 trillion. If none of this old debt is paid off and the interest on the debt is 9.5% compounded annually, what would the debt be eight years later in 1995?
In 13 and 14, recall that simple interest \( I \) is found by the formula \( I = Prt \), where \( P \) is principal, \( r \) is the rate, and \( t \) is the time.

13. Suppose $1000 is invested at 6%.
   a. How much simple interest is earned in 5 years?
   b. How much interest would be earned in 5 years if the $1000 was compounded annually at 6% interest?
   c. How much more does compound interest yield than simple interest for the problems in parts a and b?

14. Jody’s rich uncle sends her $2000 to help her meet expenses for college. She insists that he charge her interest, so he makes the following list of how much she needs to pay according to when she can pay it back.

<table>
<thead>
<tr>
<th>After:</th>
<th>Pay:</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 years</td>
<td>$2360</td>
</tr>
<tr>
<td>4 years</td>
<td>$2480</td>
</tr>
<tr>
<td>5 years</td>
<td>$2600</td>
</tr>
<tr>
<td>6 years</td>
<td>$2720</td>
</tr>
<tr>
<td>7 years</td>
<td>$2840</td>
</tr>
</tbody>
</table>

a. Is her uncle charging Jody simple or compound interest? Justify your answer.
b. What is the interest rate?

15. Refer to the BASIC program below.

```
10 PRINT "A PROGRAM TO CALCULATE BANK BALANCE"
20 INPUT "PRINCIPAL, ANNUAL RATE, NO. OF YEARS"; P, R, Y
30 PRINT "YEAR", "AMOUNT"
40 FOR C = 1 TO Y
  50 A = P * (1 + R)
60 PRINT C, A
70 P = A
80 NEXT C
90 END
```

a. Lines 40 through 80 calculate \( A \) recursively. Modify the program so it calculates \( A \) explicitly from an interest formula.
b. Use the given program or your modification to print the amount that will be in an account at the end of each of the first 10 years, if $250 is invested at a rate of 6% compounded annually.

16. Banks are required to advertise the effective annual yield on an account. This is the rate of interest earned after all the compoundings that take place within a year. Find the effective annual yield in a 6% account which is compounded monthly. (Hint: Use \( P = \$1 \).)
Review

In 17–22, simplify. (Lesson 8-1)

17. \(3x^2 \cdot 2x^3\)
18. \(y^5 \cdot y^0\)
19. \((4z^2)^5\)

20. \((ab)^2 \cdot a\)
21. \(v^3 \cdot v^6 \cdot v^9\)
22. \(\frac{b^{n+1}}{b^n}\)

In 23–25, give all positive integer solutions. (Lesson 8-1)

23. \(\frac{9^8}{9^2} = 9^x\)
24. \((7.3)^4 = x^4\)
25. \(0 < 2^n < 100\)

26. Skill sequence. Solve for \(x\). (Lessons 1-7, 6-6)
   a. \(90 = \frac{1}{4}x + 20\)
   b. \(90 - x = \frac{1}{4}x + 20\)
   c. \(90 - x^2 = \frac{1}{4}x + 20\)

27. Gretta tied a ball to a string and twirled it around. She wanted to know how the tension \(T\) in the string is related to the speed \(s\) of her action. She started by twirling a 2 ft string and obtained the following data.

<table>
<thead>
<tr>
<th>Speed (ft/sec)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension (lb)</td>
<td>300</td>
<td>1200</td>
<td>2700</td>
<td>4800</td>
<td>7500</td>
<td>10800</td>
</tr>
</tbody>
</table>

   a. Graph the above data points.
   b. Write an equation relating \(T\) and \(s\). (You do not have to find the constant of variation.) (Lesson 2-6)

Exploration

28. a. Find out the interest rate on a passbook savings account at a local savings institution. Find out also how often the interest rate is compounded, and how the institution calculates the interest it pays.
   b. Conduct a survey of several local savings institutions. Which offers the highest rate of interest? Which institution do you recommend? Why?

29. a. Use either the computer program given in Question 15 or one of your modifications to find out how long it will take to double your money if it is invested annually at a rate of
   (i) 4%
   (ii) 6%
   (iii) 8%
   (iv) 10%
   b. Generalize your results from part a.
Arithmetic or linear sequences are formed by beginning with some number and adding a constant to each term to get the next term. If, instead, each term is multiplied by a constant to get the next term, then a geometric or exponential sequence is formed.

**Definition:**

A geometric or exponential sequence is a sequence in which \( g_n = r \cdot g_{n-1} \), and \( r \neq 0 \). The number \( r \) is the constant multiplier.

Notice that the definition above gives a recursive formula for the \( n \)th term of a geometric sequence whose first term is \( g_1 \).

**Example 1** Write the first six terms of the geometric sequence in which \( g_1 = 3 \) and \( r = 5 \).

**Solution** The values for \( g_1 \) and \( r \) are given. Use the definition.

\[
\begin{align*}
g_2 &= 5 \cdot g_1 = 5 \cdot 3 = 15 \\
g_3 &= 5 \cdot g_2 = 5 \cdot 15 = 75 \\
g_4 &= 5 \cdot g_3 = 5 \cdot 75 = 375 \\
g_5 &= 5 \cdot g_4 = 5 \cdot 375 = 1875 \\
g_6 &= 5 \cdot g_5 = 5 \cdot 1875 = 9375
\end{align*}
\]

From the first term and the constant multiplier of a geometric sequence, you can find an explicit formula for the sequence. In Lesson 8-2 a $2000 investment in an account at 6% interest compounded annually was discussed. The amounts in the account after successive years exemplify a geometric sequence. The sequence begins with 2000. The constant multiplier is 1.06.

\[
\begin{array}{c|c|c|c}
\text{principal} & \text{after 1 yr} & \text{after 2 yr} & \text{after 3 yr} \\
2000 & 2000(1.06)^1 & 2000(1.06)^2 & 2000(1.06)^3 \\
2000 & 2120 & 2247.20 & 2382.03
\end{array}
\]

An expression for the amount of money at the end of the \( n \)th year is \( 2000(1.06)^n \). However, a formula for \( g_n \), which is the amount the investment is worth at the end of \( n - 1 \) years, is

\[ g_n = 2000(1.06)^{n-1}. \]

This formula can be generalized to any geometric sequence.
Explicit Formula for a Geometric Sequence:

In the geometric sequence with first term \( g_1 \) and constant ratio \( r \),

\[
g_n = g_1 r^{n-1}.
\]

Notice that in the explicit formula, the exponent is \( n-1 \). When you substitute 1 for \( n \) to find the first term, the constant multiplier has an exponent of zero.

\[
g_1 = g_1 r^{1-1} = g_1 r^0
\]

This is consistent with the property that if \( r \neq 0 \), \( r^0 = 1 \).

**Example 2**  Write the first five terms of the sequence defined by \( g_n = 8(-5)^{n-1} \).

**Solution**  Substitute \( n = 1, 2, 3, 4, \) and 5 into the formula.

\[
\begin{align*}
g_1 &= 8 \cdot (-5)^0 = 8 \cdot 1 = 8 \\
g_2 &= 8 \cdot (-5)^1 = 8 \cdot (-5) = -40 \\
g_3 &= 8 \cdot (-5)^2 = 8 \cdot 25 = 200 \\
g_4 &= 8 \cdot (-5)^3 = 8 \cdot (-125) = -1000 \\
g_5 &= 8 \cdot (-5)^4 = 8 \cdot 625 = 5000
\end{align*}
\]

Notice that in each case, the ratio of successive terms is the same. For instance

\[
\frac{-40}{8} = \frac{200}{-40} = \frac{-1000}{200} = \frac{5000}{-1000} = -5.
\]

For this reason the constant multiplier \( r \) is also called the constant ratio of successive terms.

**Example 3**  A ball is dropped from a height of 50 feet, and it bounces to 90% of its previous height after each bounce. Let \( h_n \) be the height after \( n \) bounces.

a. Find an explicit formula for \( h_n \).

b. Find the height of the ball on the tenth bounce.
Solution
a. Because each term is the constant value .9 times the previous term, the sequence is geometric. On the first bounce, the ball will bounce to $50(.9) \text{ ft} = 45 \text{ ft}$ so $h_1 = 45$. Also, $r = .9$. Thus,

$$h_n = 45(.9)^{n-1}.$$  

b. On the tenth bounce, $n = 10$ and

$$h_{10} = 45(.9)^{10-1} \approx 17.43.$$  

So on the tenth bounce the ball will rise between 17 and 18 feet.

Questions

1. Arithmetic sequences are formed by beginning with some number and __ a constant to each term to get the next term; geometric sequences are formed by __ each term by a constant to get the next term.

2. Could the sequence 5, 15, 45, 135, ... be an exponential sequence? Justify your answer.

3. In a geometric sequence, the __ of successive terms is constant.

In 4 and 5, give the first five terms of the sequence defined by the given formula.

4. $g_n = 2 \cdot 3^{n-1}$  
5. $t_1 = 6, t_n = \frac{2}{3}t_{n-1}$

6. a. Write the first six terms of the geometric sequence whose first term is -3 and whose constant ratio is 4.  
   b. Give a recursive formula for the $n$th term.  
   c. Give an explicit formula for the $n$th term.

7. Refer to Example 3. If the ball bounces to 75% of its previous height each time, and the ball was dropped from a height of 50 ft, find the height of the tenth bounce.

In 8–10, find a. the next term, and b. an explicit formula for the $n$th term of the geometric sequence.

8. 2, 6, 18, 54, 162, ...  
9. 100, 20, 4, .8, ...  
10. 40, -40, 40, -40, ...

11. The fifth term of a geometric sequence is 140. The constant multiplier is 2.  
   a. What is the sixth term?  
   b. What is the first term?

12. Your little brother agreed to pay you some money because you helped him stay out of trouble. He agreed to show his gratitude by paying you 1¢ on July 1st, 2¢ on July 2nd, 4¢ on July 3rd, 8¢ on July 4th, and doubling the amount each day for the entire month. How much will he have to pay you on July 31?
13. A diamond was purchased for $2500. If its value increases 6% each year, give the value of the diamond after ten years.

14. Use the formula for simple interest $I = Prt$ where
   
   $P =$ the principal sum invested
   $r =$ the annual interest rate
   $t =$ the time in years.
   
   a. If you invest $1000 at 8% interest, write the sequence of your balances over the next 6 years.
   
   b. Use your answer to part a. Does simple interest create a geometric sequence? If so, what is the common ratio? If not, what kind of sequence does it lead to?

15. $S_1$ is a side of a square with length 8 cm. $S_2$ is the side of the square formed by joining the midpoints of the sides of $S_1$. $S_2$ is found using the Pythagorean theorem.

   \[(S_2)^2 = \left(\frac{1}{2}S_1\right)^2 + \left(\frac{1}{2}S_1\right)^2\]
   
   \[= 4^2 + 4^2\]
   
   \[= 32\]
   
   \[S_2 = 4\sqrt{2}\]

   a. Let $S_n$ be a side of the square formed by joining the midpoints of the sides of $S_{n-1}$. Find the lengths of $S_3$, $S_4$, $S_5$.
   
   b. Is the sequence $S_n$ geometric? If so, what is the common ratio? If not, explain why not.
   
   c. Let $A_n$ be the area of the square with side $S_n$. Find $A_1$ and $A_2$.
   
   d. Is the sequence $A_n$, the areas of the squares with sides $S_n$, geometric? If so, what is the common ratio? If not, explain why not.

In 16–18, consider the BASIC program below.

10 REM GENERATING A GEOMETRIC SEQUENCE
20 INPUT "FIRST TERM, COMMON RATIO, NO. OF TERMS";
   G1, R, N
30 PRINT "POSITION","TERM"
40 FOR I = 1 TO N
50   GI = G1 * R ^ (I - 1)
60   PRINT I, GI
70 NEXT I
80 END

16. What is printed when this program is run and the user inputs 10 for G1, 2 for R, and 7 for N?

17. What values should you input for G1, R, and N to print the table at the right.

<table>
<thead>
<tr>
<th>POSITION</th>
<th>TERM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>.25</td>
</tr>
<tr>
<td>5</td>
<td>.0625</td>
</tr>
</tbody>
</table>
18. a. Use the program to print the first 50 terms of the geometric sequence that begins .001, .003, .009 ....
   b. Write the last term of the sequence in part a without scientific notation.

19. An account pays 7.75% interest compounded quarterly. How much would be in the account if $500 were left untouched for 10 years? 
   (Lesson 8-2)

20. Multiple choice Which of these accounts will have twice as much in it as an account where $1000 is compounded at an annual rate of 5% for 3 years? 
   (Lesson 8-2)
   (a) an account where $2000 is compounded at 5% for 3 years 
   (b) an account where $1000 is compounded at 10% for 3 years 
   (c) an account where $1000 is compounded at 5% for 6 years

In 21–24, simplify. (Previous course, Lesson 8-1)
21. $4x \cdot 5x^2$
22. $(y^m)^n + (y^n)^m$
23. $\left(\frac{x}{3}\right)^4$
24. $\frac{a^5}{a^3}$

In 25 and 26, rewrite using the Distributive Postulate. (Lessons 1-5, 6-1)
25. $2m^6 + m^3$
26. $4\left(\frac{1}{6} + t^3\right)^2$

27. a. Graph the function with equation $y = \frac{3}{x}$.
   b. On the same coordinate axes as part a, sketch the image of $y = \frac{3}{x}$ translated 2 units to the left.
   c. Write an equation for the image. (Lesson 6-4)

In 28 and 29, find the slope:
28. of the line of symmetry of $f(x) = \frac{3}{x}$. (Lessons 2-4, 2-7)
29. of the line through $(8, 9)$ and $(-5, 9)$.
30. If $h: x \rightarrow 2(x - 1)^2$, then $h(3) - h(0) = {?}$ . (Lesson 7-1)

31. In 1987 the world population passed 5 billion and was estimated to be increasing at the rate of 1.7% per year. Using this model, in what year would the world population be 10 billion?
You have seen negative exponents when writing numbers in scientific notation. For example,

\[ 10^6 = 1,000,000 = \text{one million} \]
\[ 10^{-6} = .000001 = \text{one millionth}. \]

This suggests that, in general, \( x^n \) and \( x^{-n} \) are reciprocals. This is easily proved.

**Negative Exponent Theorem:**

If \( x > 0 \), then \( x^{-n} = \frac{1}{x^n} \).

**Proof:**

By the Product of Powers Postulate, \( x^n \cdot x^{-n} = x^{n+n} = x^0 = 1 \).

Dividing both sides by \( x^n \) (which can always be done because \( x \neq 0 \)),

\[ x^{-n} = \frac{1}{x^n}. \]

Thus the negative sign in an exponent means reciprocal. The Negative Exponent Theorem states that an expression with a negative exponent is the reciprocal of the expression without the negative sign.

In fact, \( x^{-1} \) equals \( \frac{1}{x^1} \), so \( x^{-1} \) is the reciprocal of \( x \).

**Example 1** Write \( 5^{-3} \) as a decimal.

**Solution** \( 5^{-3} \) is the reciprocal of \( 5^3 \).

Thus \( 5^{-3} = \frac{1}{5^3} = \frac{1}{125} = .008 \).

**Check** Use a calculator to find \( 5^{-3} \). Press

\[ 5 \boxed{\text{y}^\text{x}} 3 \boxed{\pm} = \] or \( 5 \boxed{\text{y}^\text{x}} 3 = \boxed{\text{x}^{-3}} \).

The Negative Exponent Theorem allows formulas to be rewritten without fractions. This may be desirable if the formula appears in a typed prose paragraph.
Example 2  Rewrite Newton’s Law of Universal Gravitation

\[ W = \frac{k}{r^2} \]

using negative exponents.

Solution  

\[ W = k \cdot \frac{1}{r^2} \]  

definition of division

\[ W = kr^{-2} \]  

Negative Exponent Theorem

Caution: a negative sign in an exponent does not make the expression negative. *All* the powers of a positive number are positive. Here are some powers of 9.

\[
\begin{align*}
9^0 &= 1 \\
9^1 &= 9 \\
9^2 &= 81 \\
9^3 &= 729 \\
9^4 &= 6561 \\
9^{-1} &= \frac{1}{9} \\
9^{-2} &= \frac{1}{81} \\
9^{-3} &= \frac{1}{729} \\
9^{-4} &= \frac{1}{6561}
\end{align*}
\]

When a positive exponent signifies time in the future, the corresponding negative exponent stands for time in the past.

Example 3  In an account with a rate of 7% compounded annually, how much money did Soren invest 5 years ago if he has $9817.86 now?

Solution  Here \( A = $9817.86 \) and \( r = .07 \). Soren wants to know how much was invested 5 years ago, so \( t = -5 \).

Then  

\[
\begin{align*}
P &= 9817.86(1 + .07)^5 \\
P &= 9817.86(1.07)^5 \\
P &= \frac{9817.86}{1.07^5} \\
P &= 6999.9985
\end{align*}
\]

Soren originally invested $7000.

Check  If $7000 is invested at 7% for five years, then

\[
A = 7000(1.07)^5 = 9817.8621,
\]

which checks with the given information.
Negative exponents satisfy the postulates about powers stated in Lesson 8-1.

Example 4  Simplify (a) \( \frac{10^3}{10^7} \) and (b) \( x^5 \cdot x^{-1} \). (Assume \( x > 0 \).)

Solution
(a) This is a quotient of powers of the same base.
\[
\frac{10^3}{10^7} = 10^{3-7} = 10^{-4}
\]
(b) Use the Product of Powers Postulate.
\[
x^5 \cdot x^{-1} = x^{5+(-1)} = x^4
\]

Check
(a) Use the repeated multiplication meaning of \(10^7\).
\[
\frac{10^3}{10^7} = \frac{10 \cdot 10 \cdot 10}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10} = \frac{1}{10 \cdot 10 \cdot 10 \cdot 10} = \frac{1}{10^4}
\]
(b) From the Negative Exponent Theorem you know that
\[
x^{-1} = \frac{1}{x}
\]
Use the definition of \(b^0\) to rewrite \( x^5 \cdot x^{-1} = x \cdot x \cdot x \cdot \frac{1}{x} = x^4 \).

Questions

1. Simplify.
   a. \( b^x \cdot b^y \)
   b. \( b^x \cdot b^0 \)
   c. \( b^x \cdot b^{-x} \)

2. Multiple choice \( a^x \) and \( a^{-y} \) are:
   (a) reciprocals
   (b) opposites
   (c) neither (a) nor (b).

3. Write without an exponent. Do not use a calculator.
   a. \( 8^0 \)
   b. \( 8^{-1} \)
   c. \( 8^{-2} \)

4. Write \( 7^x \) without an exponent when \( x \) is:
   a. -3
   b. -2
   c. -1

5. Multiple choice Assume \( b > 0 \). For what values of \( n \) is \( b^n < 0 \)?
   (a) \( n < 0 \)
   (b) \( 0 < n < 1 \)
   (c) all values of \( n \)
   (d) no values of \( n \)

6. Rewrite \( \frac{k}{r^2} \) using negative exponents and without a fraction.
7. Interpret the formula \( P = A(1 + r)^t \) when \( t = 2 \).

8. Alex has $20,000 now in an account. The interest rate compounded annually has been 8%. How much was in the account 4 years ago?

In 9–11, write without an exponent.

9. \( 10^7 \cdot 10^6 \)  
10. \( 2^{-3} \cdot 13^0 \)  
11. \( (10^{-3})^2 \)

In 12–14, simplify.

12. \( \frac{8x^6}{6x^8} \)  
13. \( \frac{4y^{-1}}{y^2} \)  
14. \( \frac{z^5}{10z^{-6}} \)

15. If \( x^3 = 5 \), what is the value of \( x^{-3} \)?

16. Suppose \( 0 < x < 1 \). Is \( x^2 \) smaller or larger than \( x \)?

In 17–22, write without an exponent.

17. \( (3^{-3})^{-4} \)  
18. \( 3.78 \cdot 10^{-4} \)  
19. \( \frac{1}{2^{-5}} \)

20. \( \frac{3^{-3}}{3^{-2}} \)  
21. \( (\frac{1}{2})^5 \)  
22. \( \frac{1}{4^3} \)

23. Write \((2.5 \cdot 10^{-2})^3\) in scientific notation.

24. The intensity \( I \) of light varies inversely as the square of the distance \( d \) from the observer. Let \( k \) be the constant of variation.
   a. Write this inverse variation formula with positive exponents.
   b. Rewrite this inverse variation formula with negative exponents.

25. Benjamin Franklin was one of the most famous scientists of his day and conducted many science experiments. In one he noticed that a given amount of oil dropped on the surface of a lake would not spread out beyond a certain area. In units we use today, he found that 0.1 cm\(^3\) of oil spread to cover about 40 m\(^2\) of the lake. About how thick is such an oil layer? Express your answer in scientific notation. (Although in Franklin’s time no one knew about molecules, Franklin’s experiment resulted in the first estimate of a molecule’s size. Nowadays we know that the layer of oil stops spreading when it is one molecule thick.)

26. Let \( f(x) = 2^x \).
   a. Evaluate the function for integers between -3 and 3.
   b. Graph \( y = f(x) \)
   c. Use your graph to estimate the value of \( 2^{1/2} \). Check your estimate with a calculator.

27. Consider the sequence 6561, 2187, 729, ....
   a. Can it be geometric?
   b. Find an explicit rule for the \( nth \) term.
   c. Find the 15th term. \((Lesson \ 8-3)\)
28. Suppose \( g_n = g_1 r^{n-1} \) where \( g_1 \neq 0 \) and \( r \neq 0 \). (Lesson 8-3)
   a. Write an explicit formula for \( g_{n+1} \).
   b. Prove that \( \frac{g_{n+1}}{g_n} = r \).
   c. State in words the theorem you proved in part b.

29. Bonnie invests $1000 for 5 years at 10% interest compounded annually. (Lessons 2-6, 8-3)
   a. How much money will Bonnie have at the end of each year?
   b. How much interest does she earn each year?
   c. Calculate how much more interest Bonnie earned in the second year than in the first, and again in the third year than in the second, and so on.
   d. If the differences you found in part c are thought of as the values of a function, would it be increasing, decreasing, or neither?

30. Find the matrix for \( R_{90} \) and \( R_{90}^{-1} \). (Lesson 4-7)

31. If \( f: x \rightarrow 3x + 6 \), then \( f^{-1}: x \rightarrow \frac{x-6}{3} \). (Lesson 7-7)

32. This question causes some students trouble: Simplify \( \frac{1}{10^9} \). (Previous course)

33. Examine these columns closely. Describe two patterns relating the powers of 5 at the left to the powers of 2 at the right.

<table>
<thead>
<tr>
<th>( 5^n )</th>
<th>( 2^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5^6</td>
<td>15,625</td>
</tr>
<tr>
<td>5^5</td>
<td>3,125</td>
</tr>
<tr>
<td>5^4</td>
<td>625</td>
</tr>
<tr>
<td>5^3</td>
<td>125</td>
</tr>
<tr>
<td>5^2</td>
<td>25</td>
</tr>
<tr>
<td>5^1</td>
<td>5</td>
</tr>
<tr>
<td>5^0</td>
<td>1</td>
</tr>
<tr>
<td>5^{-1}</td>
<td>0.2</td>
</tr>
<tr>
<td>5^{-2}</td>
<td>0.04</td>
</tr>
<tr>
<td>5^{-3}</td>
<td>0.008</td>
</tr>
<tr>
<td>5^{-4}</td>
<td>0.0016</td>
</tr>
<tr>
<td>5^{-5}</td>
<td>0.00032</td>
</tr>
<tr>
<td>5^{-6}</td>
<td>0.000064</td>
</tr>
<tr>
<td>2^6</td>
<td>64</td>
</tr>
<tr>
<td>2^5</td>
<td>32</td>
</tr>
<tr>
<td>2^4</td>
<td>16</td>
</tr>
<tr>
<td>2^3</td>
<td>8</td>
</tr>
<tr>
<td>2^2</td>
<td>4</td>
</tr>
<tr>
<td>2^1</td>
<td>2</td>
</tr>
<tr>
<td>2^0</td>
<td>1</td>
</tr>
<tr>
<td>2^{-1}</td>
<td>0.5</td>
</tr>
<tr>
<td>2^{-2}</td>
<td>0.25</td>
</tr>
<tr>
<td>2^{-3}</td>
<td>0.125</td>
</tr>
<tr>
<td>2^{-4}</td>
<td>0.0625</td>
</tr>
<tr>
<td>2^{-5}</td>
<td>0.03125</td>
</tr>
<tr>
<td>2^{-6}</td>
<td>0.015625</td>
</tr>
</tbody>
</table>

34. Make a chart similar to the one in Question 33 using the powers of 4 and 2.5.
   a. Describe how the patterns in these charts are similar to the patterns in Question 33.
   b. Find another pair of numbers with the same properties.
The island of Delos today

According to Greek history, about 400 B.C., inhabitants of the island Delos (called Delians) were suffering from a serious epidemic. As was common in those times, the community leaders sought advice from an oracle, a wise person through whom a god was believed to speak. The Delians were told that if they exactly doubled the size of their cubical altar to Apollo, the epidemic would end. "Double the size" meant, in this case, "double the volume."

If each side of the original altar was one unit, the Delians knew that the length of the new altar had to be a number that when cubed gave 2; that is, they needed to solve $x^3 = 2$.

Recall that $x$ is a square root of $t$ when $x^2 = t$.

Similarly, $x$ is a cube root of $t$ when $x^3 = t$.

As examples:

-12 is a square root of 144 because $(-12)^2 = 144$.
4 is a cube root of 64 because $4^3 = 64$.

In effect, the Delians were told by their oracle to construct a length equal to the cube root of 2. They could not do this, and the epidemic continued, so the story goes.

Square roots and cube roots are special cases of the following, more general, idea.
Definition:

Let \( n \) be an integer greater than 1. Then \( b \) is an \( n \)th root of \( x \) if and only if \( b^n = x \).

There are no special names for \( n \)th roots other than square roots (when \( n = 2 \)) and cube roots (when \( n = 3 \)). We call them fourth roots, fifth roots, and so on. The \( n \)th roots of a real number may be real or complex.

**Example 1**

Show that 2, -2, 2i, and -2i are fourth roots of 16.

**Solution**

Each of 2, -2, 2i, and -2i is a fourth root of 16 because each satisfies \( b^4 = 16 \):

\[
\begin{align*}
2^4 &= 16 \\
(-2)^4 &= 16 \\
(2i)^4 &= 2^4 \cdot i^4 = 16 \cdot 1 = 16 \\
(-2i)^4 &= (-2)(-2)(-2)(-2) = (-2)^4 i^4 = 16 \cdot 1 = 16
\end{align*}
\]

It can be proved that every nonzero real number has \( n \) distinct \( n \)th roots. The real \( n \)th roots can be determined from a graph. For instance, the real \( n \)th roots of 16 are the first coordinates of the points where the horizontal line \( y = 16 \) intersects the curve \( y = x^n \), as shown below at the left.

The number 7.6 has only one real cube root; it is the \( x \)-coordinate of the point where \( y = 7.6 \) intersects \( y = x^3 \) as shown above at the right. An accurate drawing might show \( x \approx 2 \); a calculator displays:

\[ 1.9660951 \]
The graph of \( y = x^n \), for \( n \) even, looks much like the graph of \( y = x^4 \). It verifies:

- When \( n \) is even, a positive real number has 2 real \( n \)th roots.
- When \( n \) is even, a negative real number has no real \( n \)th roots.

When \( n \) is odd, every horizontal line intersects \( y = x^n \) in exactly one point. Thus:

- When \( n \) is odd, every real number has exactly 1 real \( n \)th root.

Not until 2000 years after the Greeks was it recognized that \( n \)th roots could be represented as powers. However, to ensure that every power has exactly one value, it is necessary to restrict the domain of the base and the power to the positive reals.

\[ \frac{1}{n} \] Exponent Theorem:

When \( x \geq 0 \) and \( n \) is an integer greater than 1, \( x^{1/n} \) is an \( n \)th root of \( x \).

Proof:

By the definition of \( n \)th root, \( b \) is an \( n \)th root of \( x \) if and only if \( b^n = x \). Suppose \( b = x^{1/n} \).

\[
(x^{1/n})^n = x^{(\frac{1}{n} \cdot n)} \quad \text{Power of a Power Property}
\]
\[
= x^1 = x.
\]

Thus \( x^{1/n} \) is an \( n \)th root.

In order for the other properties of powers to hold for the \( \frac{1}{n} \)th power, \( x^{1/n} \) must be the nonnegative root of \( x \). Specifically, \( x^{1/2} \) is the nonnegative square root of \( x \) and \( 2^{1/3} \) represents the positive cube root of 2.

In the \( \frac{1}{n} \) Exponent Theorem, \( x \) is assumed to be nonnegative to avoid complex roots and multiple solutions. Thus,

\[ 81^{1/4} = 3 \quad \text{and} \quad \left(\frac{64}{27}\right)^{1/3} = \frac{4}{3}. \]
Pay close attention to parentheses. While \((-4)^{1/2}\) and \((-8)^{1/3}\) are not defined,
\[
-4^{1/2} = -(4^{1/2}) = -2.
\]

Because roots can be written as powers, you can use a calculator’s power key to evaluate \(x^{1/n}\).

**Example 2**
Approximate \(70^{1/7}\) to the nearest hundredth.

**Solution 1**
Key in \(70 \left( \sqrt[7]{1 \div 7} \right) = \). The calculator will display 1.8347861 or something close. \(70^{1/7} \approx 1.83\).

**Solution 2**
You can shorten the key sequence by using the reciprocal key \(\frac{1}{x}\). Key in \(70 \left( \sqrt[7]{1 \div x} \right) = \).

**Check**
Calculate \((1.8347861)^7\). You should get 70 or very close to it.

**Questions**

1. Let \(n\) be an integer greater than 1. Then \(x\) is an \(n\)th root of \(t\) if and only if \(____\).

2. Which are 6th roots of 64: \(2, -2, 2i, -2i\)?

In 3 and 4, without using a calculator find the positive

3. 4th roots of 81

4. cube roots of 8.

In 5 and 6, state the real root or roots of the equation.

5. \(x^3 = 125\)

6. \(x^4 = 10,000\)

7. a. Who were the Delians?

   b. What length did the Delians have to construct to satisfy the oracle?

8. What is the meaning of \(x^{1/n}\)?

9. a. Write a calculator key sequence to evaluate \(80^{1/3}\).

   b. Estimate \(80^{1/3}\) to the nearest hundredth.

10. **True or false**

    a. Both 5 and \(-5\) are 6th roots of 15,625.

    b. \(15,625^{1/6} = 5\)

    c. \(15,625^{1/6} = -5\)

    d. \(5i\) is a 6th root of 15,625.

11. For what values of \(t\) and \(n\) is the symbol \(t^{1/n}\) defined?
In 12–14, evaluate without a calculator.
12. \(125^{1/3}\)  
13. \(144^{1/2}\)  
14. \((\frac{16}{81})^{1/4}\)

In 15–17, use a calculator to approximate to the nearest thousandth.
15. \(24^{1/2}\)  
16. \((1,419,857)^{1/5}\)  
17. \(1000^{1/10}\)

Applying the Mathematics

18. **Multiple choice** Try the key sequence

\[
\begin{align*}
&\text{for some positive number } t. \text{ Repeat for other positive numbers } t \text{ until you see a pattern. What generalization about positive numbers does this illustrate?} \\
&(a) \frac{3t}{3} = t \\
&(b) (t^{1/3})^3 = t \\
&(c) (3^t)^3 = t \\
&(d) 3(t^{1/3}) = t
\end{align*}
\]

In 19–26, tell which symbol, <, =, or >, will give a true statement.
19. \(100^{1/2} \ ?\ 100^{1/5}\)  
20. \(-7 \ ?\ 2401^{1/4}\)  
21. \(-18^{1/3} \ ?\ 18^{1/3}\)  
22. \((8^{1/3})^3 \ ?\ 8\)  
23. \(9^{1/2} \ ?\ \left(\frac{1}{2}\right)^9\)  
24. \(9^{1/2} \cdot 4^{1/2} \ ?\ (9 \cdot 4)^{1/2}\)  
25. \((z^6)^{1/3} \ ?\ z^2\)  
26. \(64^{1/2} \cdot 64^{1/2} \ ?\ \left(64^1\right)\)

27. Use \(t = \frac{1}{2}\) in the compound interest formula \(A = P(1 + r)^t\) to calculate the amount you would have after 6 months if you invested \$2000 at a yearly effective yield of 6%. Does the answer seem reasonable?

28. a. Verify that \(i\sqrt{2}\) is a 4th root of 4.
   b. Why then is \(4^{1/4} \neq i\sqrt{2}\)?

29. Let the frequency of the musical note A above middle C be \(F_1\), which is often set to 440 vibrations per second. Let \(n\) be the number of notes above that A. Then the frequencies that determine the other notes found on a piano can be found by the recursive formula

\[
\begin{align*}
F_1 &= 440 \\
F_n &= F_{n-1} \cdot 2^{1/12}
\end{align*}
\]

a. Rewrite the formula explicitly.

b. What is the frequency of the note one octave above \(F_1\) (one octave is 12 notes)?

Itzhak Perlman, world-renowned violinist, in concert performance at Carnegie Hall, New York City. Mr. Perlman's pianist for this 1988 concert was Samuel Sanders, noted collaborative artist for major concert stars worldwide.
In 30–32, write without negative exponents. (Lesson 8-4)

30. \( \frac{3x^{-3}}{y^2} \)  
31. \( 4m^{-1} \)  
32. \( \frac{2x}{9x^{-2}} + \frac{-7}{9x^{-3}} \)

In 33–35, evaluate. (Lesson 8-4)

33. \( (\frac{3}{2})^2 \)  
34. \( 3^{-1} + (-1)^3 \)  
35. \( ((14^{-1})^0)^1)^2 \)

In 36 and 37, (a) give the next term of the geometric sequence and (b) give an explicit expression for the \( n \)th term. (Lesson 8-3)

36. 10, 20, 40, 80, …  
37. 6, 3, …

38. State the Power of a Product Postulate. (Lesson 8-1)

39. If the expenses of a $1000 vacation are charged for four months on a credit card with an annual rate of 16% with interest compounded monthly, how much will the vacation cost? (Lesson 8-2)

40. If \( \sqrt{32} = x\sqrt{2} \), what is \( x \)? (Previous course, Lesson 6-8)

41. a. Draw an isosceles right triangle with legs 3 cm long. 
   b. Find the length of the hypotenuse in part a without measuring. (Check your answer by measuring.)
   c. Draw a second triangle adjacent to the first as shown at the right. (One leg is 3 cm; the other is the hypotenuse of the first right triangle.)
   d. Find the length of the hypotenuse in part c without measuring. (Previous course)

42. Consider the sequence \( 2^{1/2}, 2^{1/3}, 2^{1/4}, 2^{1/5}, \ldots \) .
   a. As the sequence keeps going, what number do the terms approach?
   b. Suppose 2 is replaced by 4 in the sequence. What happens then?
   c. Generalize parts a and b.
You know that $16^{1/4}$ is the positive 4th root of 16. What does $16^{3/4}$ signify? The answer depends on both the meaning of the fraction $\frac{3}{4}$ and on the Power of a Power Property.

$$16^{3/4} = 16^{1/4 \cdot 3}$$

Thus $16^{3/4}$ is the 3rd power of the positive 4th root of 16. We can calculate this.

$$\left(16^{1/4}\right)^3 = 2^3 = 8$$

Notice also that $16^{3/4} = 16^{3 \cdot 1/4} = (16^3)^{1/4}$. So $16^{3/4}$ is the 4th root of the 3rd power of 16. In general, with fraction exponents the numerator is the power and the denominator the root. This generalizes to the following theorem.

**Rational Exponent Theorem:**

For any positive real number $x$ and positive integers $m$ and $n$,

$$x^{m/n} = (x^{1/n})^m$$

the $m$th power of the positive $n$th root of $x$, and

$$= (x^m)^{1/n}$$

the positive $n$th root of the $m$th power of $x$.

Because $x^{1/n}$ is defined only when $x > 0$, the Rational Exponent Theorem applies only to a positive base $x$.

**Proof:**

The proof generalizes the special case considered above.

$$x^{m/n} = x^{(1/n)m} \quad m/n = \frac{1}{n} \cdot m$$

Power of a Power Property

Also,

$$x^{m/n} = x^{m \cdot (1/n)} \quad m/n = m \cdot \frac{1}{n}$$

Power of a Power Property

An expression with a rational exponent can be simplified by finding either powers first or roots first. Usually it is easier to find the root first, because you end up working with fewer digits.

**Example 1** Simplify $25^{3/2}$.

**Solution** Find the square root of 25 first, then cube it.

$$25^{3/2} = (25^{1/2})^3 = 5^3 = 125$$
Check 1 Find the power first: \(25^{3/2} = (25^3)^{1/2} = 15625^{1/2} = 125\).

As expected, the result is the same in both cases.

Check 2 Use your calculator. Many machines require parentheses for the fraction exponent. Here is a possible key sequence:

\[
25 \left( \frac{3}{2} \right) =
\]

With practice, you should be able to simplify many expressions with fraction exponents mentally. There is a property of fraction exponents that will help you do this. Notice in Example 1 that the exponent \(\frac{3}{2}\) is greater than 1 and subsequently the answer 125 is larger than the base 25. In general:

when \(x > 1\) and \(\frac{m}{n} > 1\),

\(x^{m/n}\) will be larger than the base \(x\), and

when \(x > 1\) and \(0 < \frac{m}{n} < 1\),

\(x^{m/n}\) will be smaller than the base \(x\).

This can be verified with other rational powers of 25.

\[
\begin{align*}
25^0 &= 1 \\
25^{1/4} &= 2.236\ldots \\
25^{1/2} &= 5 \\
25^{3/4} &= 11.180\ldots \\
25^1 &= 25 \\
25^{5/4} &= 55.901\ldots \\
25^{3/2} &= 125 \\
25^{7/4} &= 279.508\ldots
\end{align*}
\]

Example 2 Simplify 12358\(^{3/5}\).

Solution Use the following key sequence on your calculator. Use \(\frac{3}{5} = .6\).

\[
12358 \left( \frac{3}{5} \right) =
\]

The answer is approximately 285.21.

Check 1 Because \(\frac{3}{5} < 1\) the answer should be smaller than the original base; it is. This is a rough check.

Check 2 If \(12358^{3/5} \approx 285.21\), then 285.21 to the \(\frac{5}{3}\) power should be about 12358. We find

\[
285.21^{5/3} \approx 12357.8,
\]

so it checks.
You can check that the Properties of Powers Postulate given in Lesson 8-1 holds with rational powers. For instance, \(25^{1/2} \cdot 25^1 = 25^{3/2}\). The properties of powers can be used to solve equations with positive rational exponents. The strategy in solving equations of this kind is to raise both sides of the equation to a power. This can be done because any number can be substituted for its equal in an algebraic expression. Thus if \(x = y\), then equation substituted \(x^n\) and \(y^n\) must give the same value, so \(x^n = y^n\).

**Example 3** Solve \(x^{5/4} = 243\).

**Solution**  Recall that any number times its reciprocal equals 1. Thus to solve for \(x\), raise both sides of the equation to the \(\frac{4}{5}\) power.

\[
(x^{5/4})^{4/5} = 243^{4/5}
\]

Apply the Power of a Power Property.

\[
x^1 = 243^{4/5}
\]

Simplify and calculate.

\[
x = 81
\]

**Check**  Does \(81^{5/4} = 243\)? Yes. It is reasonable that \(x\) should be less than 243 because \(\frac{5}{4}\) is more than 1.

Rational exponents have many applications, including growth situations, investments, radioactive decay, and change-of-dimension situations (for example, area to volume and back). The following application involves a dimensional relation.

**Example 4** The volume \(V\) of a soap bubble is related to its surface area by the formula \(V = 0.094A^{3/2}\). What is the surface area of a bubble with volume 5 cm\(^3\)?

**Solution**  Here \(V = 5\), so

\[
0.094A^{3/2} = 5.
\]

Divide both sides by 0.094.

\[
A^{3/2} = 53.19
\]

Raise each side to the \(\frac{2}{3}\) power.

\[
(A^{3/2})^{2/3} = 53.19^{2/3}
\]

Finally, simplify and calculate.

\[
A \approx 14.14
\]

A 5 cm\(^3\) bubble has a little more than 14 cm\(^2\) of surface area.

**Check**  This is left for you to do in the questions.
Questions

In 1 and 2, write as a power of $x$.
1. the 4th power of the 9th root of $x$
2. the seventh root of the cube of $x$

3. a. Rewrite $100,000^{4/5}$ in two ways as a power of a power of $100,000$.
   b. Which way would be easier to calculate mentally?
   c. Calculate $100,000^{4/5}$.

In 4–6, simplify without a calculator.
4. $27^{2/3}$
5. $32^{3/5}$
6. $36^{3/2}$

In 7–9, evaluate with a calculator.
7. $729^{3/2}$
8. $729^{2/3}$
9. $1331^{5/3}$

In 10–12, $x > 1$. Complete with $<, >$, or $=.$
10. $x^{7/8} \ ? x$
11. $x^{5/3} \ ? x^{3/5}$
12. $x^{3/4} \ ? x^{3/5}$

In 13–15, solve and check.
13. $\sqrt{5/2} = 100$
14. $k^{2/3} = 64$
15. $x^{4/9} = 12$

In 16 and 17, refer to Example 4.
16. Check the answer found.
17. A bubble with volume $10 \text{ cm}^3$ does not have twice the surface area as a bubble with volume $5 \text{ cm}^3$. How much surface area does a $10 \text{ cm}^3$ bubble have?

   b. Simplify $16^{n/4}$, where $n$ is a positive integer.

19. This exercise shows why rational exponents are used only with positive bases.
   a. If $(-8)^{1/3}$ were to equal the real cube root of $-8$, then $(-8)^{1/3} = \ ?$.
   b. If $(-8)^{2/6}$ follows the Rational Exponent Theorem, then $(-8)^{2/6} = ((-8)^2)^{1/6} = \ ?$.
   c. In this question, does $(-8)^{1/3} = (-8)^{2/6}$?

In 20–22, apply the property of exponents $(\frac{x}{y})^n = \frac{x^n}{y^n}$ to calculate.
20. $(\frac{64}{27})^{2/3}$
21. $(\frac{1000}{343})^{4/3}$
22. $(\frac{16}{25})^{3/4}$

In 23–25, use properties of powers to simplify.
23. $(x^8)^{1/4}$
24. $B^{2/3} \cdot B$
25. $\frac{3}{2}y^{2/3} \cdot \frac{3}{2}y^{3/2}$

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26. The diameter $D$ of the base of a tree of a given species roughly varies directly with the $\frac{3}{2}$ power of its height $h$.
   a. Suppose a young sequoia 500 cm tall has a base diameter of 14.5 cm. Find the constant of variation.
   b. The largest known living tree is a California sequoia called "General Sherman." It has a base diameter of about 985 cm. Approximately how tall is General Sherman?
   c. One story on a modern office building is about 3 m high. General Sherman is about as tall as a story office building.

In 27 and 28, use this information about similar figures: If $A_1$ and $A_2$ are the areas of two similar figures and $V_1$ and $V_2$ are their volumes, then

$$\frac{A_1}{A_2} = \left(\frac{V_1}{V_2}\right)^{2/3}.$$

27. Two similar figurines have volumes 20 cm$^3$ and 25 cm$^3$. What is the ratio of the amounts of paint (surface area) they need?

28. Solve the formula for $\frac{V_1}{V_2}$.

29. If $a^b = c$ then __ is a __ root of __. (Lesson 8-5)

30. If $x^t$ is the reciprocal of $x$, what is the value of $t$? (Lesson 8-4)

31. Simplify: $\frac{2^{100}}{2^{99}}$. (Lesson 8-4)

32. State the Power of a Power Property. (Lesson 8-1)

33. Each year, the Bruised Beauties Car Lot marks down a certain model of car to 75% of its value in the preceding year. If the car is worth $7000 at the end of the first year, what will be its value at the end of the fifth year? (Lesson 8-3)

34. Find the lateral area and surface area of the right circular cylinder shown at the right. (Previous course)

35. a. Consider the numbers $25^{1/2}$, $25^{2/3}$, $25^{3/4}$, $25^{4/5}$, and $25^{5/6}$. Approximate these with decimals rounded to the nearest thousandth.
   b. Let $f(n) = 25^{n/(n+1)}$. Calculate $f(100)$ and $f(1000)$.
   c. What is the approximate value of $25^{n/(n+1)}$ when $n$ is a very large number, say 1 billion?
You have now learned meanings for many rational exponents. For any positive number \( x \) and any positive integer values of \( m \) and \( n \) (except \( n = 0 \)):

\[
\begin{align*}
    x^0 &= 1 & \text{(Zero Exponent Theorem, Lesson 8-1)} \\
    x^{-n} &= \frac{1}{x^n} & \text{(Negative Exponent Theorem, Lesson 8-4)} \\
    x^{1/n} &= \text{positive \( n \)th root of \( x \)} & \text{(\( \frac{1}{n} \) Exponent Theorem, Lesson 8-5)} \\
    x^{m/n} &= (x^{1/n})^m = (x^m)^{1/n} & \text{(Rational Exponent Theorem, Lesson 8-6).}
\end{align*}
\]

Now we apply these properties to evaluate expressions in which the exponent is a negative rational number. Since \( x^{-m/n} = \left(\left(x^{-1}\right)^{m}\right)^{1/n} = \left(\left(x^m\right)^{1/n}\right)^{-1} \), and these exponents can be in any order, you have the choice of first taking the reciprocal, the \( m \)th power, or the \( n \)th root.

**Example 1** Evaluate \( 81^{-1/4} \)

**Solution** Here we take the reciprocal and then the 4th root.

\[
81^{-1/4} = \frac{1}{81^{1/4}} = \frac{1}{3}
\]

**Check** You can evaluate \( 81^{-1/4} \) on a calculator. One possible key sequence is

\[
81 \div (1 \pm 4) =
\]

Your calculator should display 0.333333, or \( \frac{1}{3} \), which checks.

**Example 2** Simplify \( \left(\frac{27}{1000}\right)^{-2/3} \)

**Solution 1** Think \( \left(\left(\left(\frac{27}{1000}\right)^{-1}\right)^{1/3}\right)^2 \) . This applies the reciprocal, cube root, and square in that order. Remember to work with the innermost parentheses first.

\[
\left(\left(\left(\frac{27}{1000}\right)^{-1}\right)^{1/3}\right)^2 = \left(\left(\frac{1000}{27}\right)^{1/3}\right)^2 = \left(\frac{10}{3}\right)^2 = \frac{100}{9}
\]

**Solution 2** Think \( \left(\left(\left(\frac{27}{1000}\right)^{1/3}\right)^2\right)^{-1} \) . This does the cube root, square, and reciprocal in that order.

\[
\left(\left(\left(\frac{27}{1000}\right)^{1/3}\right)^2\right)^{-1} = \left(\left(\frac{3}{10}\right)^2\right)^{-1} = \left(\frac{9}{100}\right)^{-1} = \frac{100}{9}
\]
Check  Change everything to decimals. \( \frac{27}{1000} = .027 \) and \(-\frac{2}{3} = -.6\). Key in \(.027 \sqrt[5]{.66666667} \) on a calculator. We get 11.11111124, which is very close to 11.1 or \( \frac{100}{9} \).

The ideas used in Lesson 8-6 to solve equations with positive rational exponents can be employed with negative rational exponents as well.

**Example 3** Solve \( x^{2/5} = 9 \).

**Solution** The reciprocal of \(-\frac{2}{5}\) is \(-\frac{5}{2}\), so take each side to the \(-\frac{5}{2}\) power.

\[
(x^{2/5})^{-5/2} = 9^{-5/2}
\]
\[
x = 9^{-5/2} = \left((9^{1/2})^5\right)^{-1} = (3^5)^{-1} = 243^{-1} = \frac{1}{243}
\]

**Check** Does \( \left(\frac{1}{243}\right)^{-2/5} = 9? \left(\left(\frac{1}{243}\right)^{-1}\right)^{1/5}\) \(= (243^{1/5})^2 = 3^2 = 9\).

You should not need a calculator to evaluate many expressions involving simple fractional exponents, such as \(\frac{1}{4}\) or \(-\frac{2}{3}\), when the answer is a simple fraction. However, when the powers you must evaluate are not simple fractions, you will usually need a calculator.

**Example 4** The number of hours \( h \) that milk stays fresh is a function of the surrounding temperature \( t \). Use the formula

\[
h(t) = 180 \cdot 10^{-0.04t}
\]

to predict how long newly pasteurized milk will stay fresh when stored at temperature \( 8^\circ C \).

**Solution** Substitute \( t = 8 \) in the formula.

\[
h(8) = 180 \cdot 10^{-0.04(8)}
\]
\[
h(8) = 180 \cdot 10^{-0.32}
\]

Use a calculator.

\[
h(8) \approx 86
\]

When left at \( 8^\circ C \), newly pasteurized milk will stay fresh about 86 hours, or a little more than 3\( \frac{1}{2} \) days.
Questions

In 1–3, evaluate without using a calculator.
1. $125^{-1/3}$
2. $81^{-3/4}$
3. $(\frac{9}{4})^{-5/2}$

In 4–6, estimate to the nearest thousandth with a calculator.
4. $10^{-1/2}$
5. $10^{-0.004}$
6. $50 \cdot 2.79^{-3/5}$

In 7–9, solve.
7. $s^{1/4} = 3$
8. $t^{-2/3} = 36$
9. $x^{-3/2} = \frac{1}{8}$

10. Refer to Example 4.
   a. How long will newly pasteurized milk stay fresh if it is left out at 24°C?
   b. When milk is stored at 8°C it stays fresh about $\frac{1}{2}$ times as long as it will at 24°C.

11. Tell whether or not the expression equals $b^{-3/4}$ for $b > 0$.
   a. $\frac{1}{b^{3/4}}$
   b. $\frac{1}{(b^3)^{1/4}}$
   c. $\left(\frac{b^{-1}}{3}\right)^{1/4}$
   d. $-b^{3/4}$
   e. $\left(b^{1/4}\right)^{-3}$
   f. $\left(b^{1/4}\right)^{1/3}$

In 12–14, tell whether the number is positive, negative, or zero. Do not use a calculator.
12. $(.98956)^{-3/4}$
13. $(1.0825)^0$
14. $(-.07)(3)^{-4}$

15. Find $n$ if $\left(\frac{99}{100}\right)^{-3/4} = \left(\frac{100}{99}\right)^n$.

In 16 and 17, simplify each expression into the form $ax^n$. Check your answer by substituting values.
16. $\frac{x}{3x^{2/3}} \cdot 6x^{1/2}$
17. $\frac{-3x^{-3/4}}{4x^{1/4}}$

18. a. Evaluate $64^x$, where $x$ increases by sixths from -1 to 1. (There are 13 values to evaluate: $64^{-1}$, $64^{-5/6}$, $64^{-4/6}$ = $64^{-2/3}$, and so on until $64^1$.)
   b. Explain the pattern of answers to part a.

19. Let $F$ be the amount of food a mammal with body mass $m$ must eat daily to maintain its mass. In the chapter opener you read that $\frac{F}{m} = km^{-1/3}$. Is $F$ directly proportional to $m$? Justify your answer.

20. The product of $x^2$ and $x^3$ is $x^5$. Find six more pairs of integer powers of $x$ whose product is $x^5$. (Lessons 8-1, 8-4)
21. The Galapagos Islands are a chain of islands in the Pacific Ocean that belong to Ecuador. They are famous for their variety of plant and animal life. A biologist has shown that \( S \), the number of different plant species on an island, varies with the area \( A \) of the island according to the formula:

\[
S = 28.6A^{0.32}
\]

Estimate \( S \) (round to the nearest whole number) for:

a. the smallest island in the Galapagos chain, which has an area of about 2 square miles; and

b. the largest island, Albemarle, which has an area of about 2250 square miles. (Lesson 8-6)

22. Solve \( \left( \frac{16}{625} \right)^n = \frac{2}{5} \). (Lesson 8-6)

23. Solve this system \[
\begin{cases}
3x^{-1} + 2y^{-1} = 27 \\
2x^{-1} - y^{-1} = 4
\end{cases}
\]
(Hint: Let \( a = x^{-1} \) and \( b = y^{-1} \).) (Lesson 5-3, 8-4)

24. Use the formula \( P = 14.7 \cdot 10^{-0.09h} \), in which \( P \) is the atmospheric pressure in pounds per square inch at altitude \( h \) in miles above sea level. Find the atmospheric pressure:

a. in Albuquerque, NM, where \( h = .9 \);

b. in Miami, FL, which is approximately at sea level;

c. on top of Mt. McKinley, AK, which is about 20,320 feet above sea level.

d. Graph this relation, using the three points determined in parts a–c and two points of your choice.

e. As \( h \) increases, does \( P \) increase or decrease?

Mt. McKinley, Alaska
The symbol $\sqrt[n]{x}$ is called the radical sign, or just a radical. Its origin is the Latin word radix, which means root. When $x$ is positive, $\sqrt[n]{x}$ stands for the positive square root of $x$. Since $x^{1/2}$ also stands for this square root, $\sqrt[n]{x} = x^{1/2}$. There is a natural generalization from square roots to $n$th roots of positive numbers.

**Definition:**

When $x$ is positive and $n$ is an integer $\geq 2$, $\sqrt[n]{x} = x^{1/n}$.

**Example 1** Evaluate $\sqrt[4]{81}$.

**Solution** $\sqrt[4]{81} = 81^{1/4}$, the positive number whose 4th power is 81.

You probably know that $3^4 = 81$. If so, you have found $\sqrt[4]{81} = 3$ without a calculator.

The symbol $\sqrt[n]{x}$ was first used by Albert Girard around 1633. Note that $\sqrt[n]{x}$, like $x^{1/n}$, does not represent all $n$th roots of $x$. When $x$ is positive and $n$ is even, $x$ has two real $n$th roots. But only the positive real root is denoted by $\sqrt[n]{x}$. Thus 2, -2, 2i, and -2i are fourth roots of 16, but $\sqrt[4]{16} = 2$ only. The negative fourth root can be denoted by $-\sqrt[4]{16}$, or -2.

Scientific calculators sometimes have a key for $n$th roots, $\sqrt[n]{\hspace{1em}}$ or $\sqrt[n]{\hspace{1em}}$ or $\sqrt[n]{\hspace{1em}}$. You may also be able to find the $n$th root of a number by using the $\text{INV}$ or $\text{2nd}$ key before the powering key $\text{y}^x$.
Example 2  Estimate $\sqrt[3]{2}$, the length of the altar the Delians (Lesson 8-5) were asked to make.

Solution 1  Use the following key sequence:

$$2 \sqrt[3]{2} \ 3 =$$

Our calculator displays 1.2599211, which is about 1.26. The Delians had to make an altar with an edge about 1.26 times as long as the edge of the original altar.

Solution 2  Key in $2 \ \text{INV} \ \sqrt[3]{2} \ 3 =$. You should get 1.259911, the same answer as in Solution 1.

Check  Key in $1.259921 \ 3 = \text{INV} \ \sqrt[3]{2}$ to see whether $(1.259921)^3 \approx 2$. It is.

The Delians did not have calculators, of course. They also did not have decimals, which were invented only in 1585. They could work with fractions, but no simple fraction cubed equals 2; that is, $\sqrt[3]{2}$ is irrational. The Delians did not have the mathematical tools to follow the oracle’s advice.

Since $\sqrt[n]{x} = x^{1/n}$, the $m$th powers of these numbers are equal; that is, $(\sqrt[n]{x})^m = (x^{1/n})^m = x^{m/n}$. Also, if $x$ is replaced by $x^m$ in the definition, the result is $\sqrt[n]{x^m} = (x^m)^{1/n}$, which also equals $x^{m/n}$. Thus there are two radical expressions equal to $x^{m/n}$.

**Root of a Power Theorem:**

When $x > 0$, $m$ and $n$ are integers and $n \geq 2$,

$$\sqrt[n]{x^m} = (\sqrt[n]{x})^m = x^{m/n}.$$

Example 3  Simplify $\sqrt[3]{x^{12}}$.

Solution 1  Use the definition of $\sqrt{}$: $\sqrt[3]{x^{12}} = (x^{12})^{1/3} = x^4$.

Solution 2  Use the Root of a Power Theorem: $\sqrt[3]{x^{12}} = x^{12/3} = x^4$.

When products or quotients are under the radical sign, the Power of a Product Property can help to simplify them. You are familiar with this property. If $x \geq 0$ and $y \geq 0$, then for any value of $m$,

$$ (xy)^m = x^m \cdot y^m. $$

If $m$ is replaced by $\frac{1}{2}$, the result looks like a new property.

$$ (xy)^{1/2} = x^{1/2} \cdot y^{1/2} $$
However, rewriting the $\frac{1}{2}$ powers as square roots, the property becomes familiar.

$$\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$$

You used this property to simplify radicals in Chapter 6.

If $m = \frac{1}{n}$ in the Power of a Product Property, then

$$(xy)^{1/n} = x^{1/n} \cdot y^{1/n}.$$ 

Rewriting these $n$th roots with radical signs results in the following theorem.

Root of a Product Theorem:

For any positive real numbers $x$ and $y$, and any integer $n > 1$,

$$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}.$$ 

The Root of a Product Theorem allows you to rewrite $n$th roots. The idea is to find perfect $n$th powers under the radical sign. For instance,

$$\sqrt[3]{80} = \sqrt[3]{8} \cdot \sqrt[3]{10} = 2\sqrt[3]{10}.$$ 

**Example 4** Suppose $x > 0$. Rewrite $\sqrt[3]{875x^7}$.

**Solution** Factor the expression inside the root into as many perfect cubes as possible.

$$\sqrt[3]{875x^7} = \sqrt[3]{125 \cdot 7 \cdot x^6} \cdot x$$

Use the Root of a Product Theorem.

$$= \sqrt[3]{125} \cdot \sqrt[3]{7} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x}$$

Apply the definition of the $n$th root and the Root of a Power Theorem.

$$= 5 \cdot \sqrt[3]{7} \cdot x^2 \cdot \sqrt[3]{x}$$

Use the Root of a Product Theorem to multiply $\sqrt[3]{7}$ and $\sqrt[3]{x}$. Thus,

$$\sqrt[3]{875x^7} = 5x^2 \sqrt[3]{7x}.$$
The inverse of taking the cube root is cubing. To check, cube the answer.

\[
(5x^2\sqrt[3]{7x})^3 = 5^3(x^2)^3(\sqrt[3]{7x})^3 = 125x^6 \cdot 7x = 875x^7
\]

This checks with the original expression under the cube root symbol.

Check 2  Substitute some number for \(x\) and use a calculator. We let \(x = 2\).

Does \(\sqrt[3]{875} \cdot 2^7 = 5 \cdot 2^2\sqrt[3]{7} \cdot 2^?\)

Does \(\sqrt[3]{112000} = 20\sqrt[3]{14}\)?

Yes, both equal approximately 48.20.

Questions

1. The radical expression \(\sqrt[n]{x}\) equals what power of \(x\)?

In 2–4, evaluate without a calculator.

2. \(\sqrt[3]{16}\)

3. \(\sqrt[3]{216}\)

4. \(\sqrt[5]{125}\)

5. State the Root of a Power Theorem.

In 6–8, simplify. Assume all variables are positive.

6. \(\sqrt[3]{x^{15}}\)

7. \(\sqrt[3]{x^6}\)

8. \((\sqrt[3]{x})^{14}\)

9. Who first used the “\(\sqrt{\text{~}}\)” symbol, and in what century?

10. a. Write a calculator key sequence to evaluate \(\sqrt[3]{38.720}\).

   b. Estimate \(\sqrt[3]{38.720}\) to the nearest tenth.

In 11–13, use a calculator to approximate to the nearest hundredth.

11. \(\sqrt[3]{10}\)

12. \(\sqrt[4]{4}\)

13. \(\sqrt[5]{314892}\)


In 15–17, simplify. Assume \(x > 0\).

15. \(\sqrt{10} \cdot \sqrt{40}\)

16. \(\sqrt{900} \cdot \sqrt{30}\)

17. \(\sqrt{x} \cdot \sqrt{x^7}\)

In 18–20, rewrite with a smaller number or exponent under the radical sign. Assume all variables are positive.

18. \(\sqrt{121x^5}\)

19. \(\sqrt[3]{54}\)

20. \(\sqrt[3]{125p^9q^{12}}\)
Applying the Mathematics

**Multiple choice.** In 21 and 22, which of (a) to (c) is not equal to the others?

21. (a) \(3^{1/2}\)  
   (b) \(\sqrt[3]{3}\)  
   (c) \((\sqrt[3]{3})^3\)  
   (d) All are equal.

22. (a) \(\sqrt[4]{25}\)  
   (b) \(\sqrt{5}\)  
   (c) \(5^{0.5}\)  
   (d) All are equal.

23. Which is greater, \(\sqrt{2} + \sqrt{3}\) or \(\sqrt{5}\)?

24. Solve for \(x\): \(\sqrt[4]{432} = x\sqrt{2}\).

25. a. Write the next three terms in this geometric sequence: 4, 6, 9, 
   b. Write an explicit formula for the \(n\)th term. *(Lesson 8-2)*

26. Suppose that a copy machine can reduce to 74% of its original size in linear dimensions. If you made copies of copies, what would be the new dimensions of an \(8\frac{1}{2} \times 11\) in. original which was copied
   a. two times?  
   b. four times?  
   c. \(n\) times? *(Lesson 8-3)*

In 27–29, write without exponents. *(Previous course, Lesson 8-7)*

27. \(10^{-6}\)  
28. \(4^{3/2}\)  
29. \((\frac{1}{3})^{-3}\)

30. Simplify a. \(\frac{7}{35}\)  
   b. \(\frac{x}{xy}\) *(Lesson 1-5)*

31. If \(f(x) = \frac{(x + 1)}{(x - 2)} / \frac{(x + 3)}{(x - 4)}\), what values of \(x\) are not in the domain of \(f\)? *(Lessons 1-5, 7-2)*

In 32 and 33, solve. *(Lessons 8-6, 8-7)*

32. \(m^{3/2} = 27\)  
33. \(157 = x^{5/3}\)

34. If a savings bond is to grow to $10,000 after earning 6.5% interest compounded annually for 10 years, how much money should the bond cost originally? *(Lesson 8-4)*

35. Right triangles are built on each other as pictured at the right.
   a. Find \(h_1, h_2, h_3,\) and \(h_4\).
   b. Make a conjecture about \(h_5\) and \(h_6\).
   c. What would happen if the outside legs were all of length 5? 
   d. Suppose the outside legs are all of length \(x\). Find a formula for \(h_n\), the length of the \(n\)th hypotenuse.
**Lessons**

**Powers and Roots of Negative Numbers**

**Integer powers:** You have for many years calculated positive integer powers of negative numbers using repeated multiplication. Here are the first few positive integer powers of \(-8\).

\[
\begin{align*}
(-8)^1 &= -8 \\
(-8)^2 &= (-8)(-8) = 64 \\
(-8)^3 &= (-8)(-8)(-8) = -512 \\
(-8)^4 &= (-8)(-8)(-8)(-8) = 4096
\end{align*}
\]

Notice that these integer powers alternate between positive and negative. The same is true if zero and negative powers of \(-8\) are considered. As you might expect, \((-8)^n\) is the reciprocal of \((-8)^n\).

\[
\begin{align*}
(-8)^0 &= 1 \\
(-8)^{-1} &= -\frac{1}{8} \\
(-8)^2 &= \frac{1}{64} \\
(-8)^{-3} &= -\frac{1}{512} \\
(-8)^{-4} &= \frac{1}{4096}
\end{align*}
\]

Again, even powers are positive, odd powers are negative. All the power properties still work with integer powers. However, be aware of the order of operations. Whereas \((-8)^4 = 4096\), \(-8^4 = -4096\) because the power is calculated before taking the opposite.

**n-th roots:** Even the simplest roots of negative numbers cause problems. Recall what happens when square roots of negative numbers are multiplied. For example, the property \(\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}\) does not hold.

\[
\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2 \cdot \sqrt{6} = -\sqrt{6}
\]

In general, 4th roots, 6th roots, and other even roots present so much trouble that neither the symbol \(\sqrt[n]{x}\) nor the symbol \(x^{1/n}\) is defined when \(n\) is even and \(x\) is negative.

If a number is negative, then it has exactly one real odd root. For instance, \(-8\) has one real cube root, namely \(-2\). It is customary to write

\[
\sqrt[3]{-8} = -2
\]

just as you would for a positive base. Similarly,

\[
\sqrt[3]{-27} = -3.
\]

Now multiply these numbers by each other. The product is 6, which is the cube root of 216, so it is true that \(\sqrt[3]{-8} \cdot \sqrt[3]{-27} = \sqrt[3]{216} = 6\). Fifth roots, seventh roots, and all other odd roots have the same property. Thus it is possible to use radical signs to represent these roots and have the Product of Roots property hold.
Definition:

When \( x \) is negative and \( n \) is an odd integer \( \geq 3 \), \( \sqrt[n]{x} \) stands for the real \( n \)th root of \( x \).

For example, \( \sqrt[5]{-32} = \) the real 5th root of -32, which is -2. However, -17 has no real 4th roots, so the symbol \( \sqrt[4]{-17} \) is not defined.

**Rational powers:** The properties of rational exponents do not work even for cube roots of negative numbers. Again consider -2, the cube root of -8. If we would write \( (-8)^{1/3} = -2 \) then it would seem necessary that \( (-8)^{2/6} = -2 \).

But \( (-8)^{2/6} \) would have to equal \( ((-8)^2)^{1/6} = (64)^{1/6} = 2 \). Since we could not substitute even \( \frac{2}{6} \) for \( \frac{1}{3} \) when they are exponents with negative numbers, and keep the properties of powers, we do not define rational powers of negative numbers.

To summarize: When \( x \) is negative,

\[ x^n \] is defined if and only if \( n \) is an integer.

\( \sqrt[n]{x} \) is defined if and only if \( n \) is 2 (resulting in an imaginary number) or \( n \) is an odd integer greater than 2 (resulting in a negative number).

When \( n \) is even, \( x^n \) is positive, so \( \sqrt[n]{x^n} \) is defined for all integers greater than 2. In Lesson 6-1 you learned that for all real numbers \( x \), \( \sqrt[n]{x^2} = |x| \). Can \( \sqrt[n]{x^n} \) be simplified in a similar way?

Consider some specific cases:

\[
\begin{align*}
\sqrt[3]{10^3} &= 10 \\
\sqrt[3]{(-10)^3} &= -10 \\
\sqrt[5]{2^5} &= 2 \\
\sqrt[5]{(-2)^5} &= -2 \\
\sqrt[4]{10^4} &= 10 \\
\sqrt[4]{(-10)^4} &= 10 \\
\sqrt[6]{3^6} &= 3 \\
\sqrt[6]{(-3)^6} &= 3
\end{align*}
\]

The sentences above are instances of a general pattern for simplifying \( \sqrt[n]{x^n} \).

**\( n \)th Root of \( n \)th Power Theorem:**

For all real numbers \( x \), and integers \( n \geq 2 \):

- if \( n \) is odd \( \sqrt[n]{x^n} = x \);
- if \( n \) is even \( \sqrt[n]{x^n} = |x| \).

(Note that the statement \( \sqrt{x^2} = |x| \) is a special case of the Root of a Power Theorem when \( n = 2 \).)
**Example 1:** Simplify:

a. \( \sqrt[4]{(-4)^{11}} \)

b. \( \sqrt[3]{c^{8}} \)

**Solution**  
Apply the Root of a Power Theorem.  
a. 11 is odd, so \( \sqrt[4]{(-4)^{11}} = -4 \).
b. 8 is even, so \( \sqrt[3]{c^{8}} = |c| \).

**Example 2** Simplify \( \sqrt[3]{16x^{12}} \), (a) for \( x \geq 0 \); (b) for any real number \( x \).

**Solution**  
(a) When \( x \geq 0 \), \( \sqrt[3]{16x^{12}} = \sqrt[3]{16} \sqrt[3]{x^{12}} = 2x^{3} \)  
(b) If \( x \) may be positive or negative, then the answer to (a) is incorrect because \( \sqrt[3]{16x^{12}} \) is always positive and \( 2x^{3} \) could be negative. This can be rectified by using the \( |\cdot| \) sign.  
\( \sqrt[3]{16x^{12}} = 2 |x^{3}| = 2 |x|^{3} \)

**Questions**

1. Calculate \((-6)^{n}\) for all integer values of \( n \) from 3 to -3.
2. Tell whether the number is positive or negative.  
   a. \((-2)^{3}\)  
   b. \((2)^{3}\)  
   c. \((-2)^{3}\)

In 3–8, simplify.

3. \( \sqrt[3]{-27} \cdot \sqrt[3]{-1} \)  
4. \( \sqrt[3]{-64} + \sqrt[3]{-8} \)  
5. \( \sqrt{-16} \cdot \sqrt{-1} \)  
6. \( 5 \sqrt[5]{32} \)  
7. \( \sqrt[3]{7^{3}} \)  
8. \( \sqrt[6]{x^{6}} \)
9. **True or false** \( \sqrt[3]{(-6)^{3}} = -6 \)

In 10–12, tell whether the symbol is defined or not. If defined, tell whether the number is real or imaginary. If real, tell whether the number is positive or negative.

10. \( \sqrt[4]{-16} \)  
11. \( \sqrt[5]{-16} \)  
12. \( \sqrt{-16} \)

In 13–15, simplify the expression (a) when \( x \geq 0 \); (b) for any real number \( x \).

13. \( \sqrt[3]{-8x^{3}} \)  
14. \( \sqrt[4]{x^{20}} \)  
15. \( \sqrt[3]{x^{3}} \)
In 16–19, rewrite each root. Assume variables may stand for any real numbers.

16. $\sqrt[4]{32x^{12}}$  
17. $\sqrt[5]{3125x^{10}y^{17}}$  
18. $\sqrt[3]{-m^9p^{15}}$  
19. $\sqrt[8]{(-10)^8a^{11}b^{18}}$

20. a. Write a calculator key sequence to evaluate $3\sqrt[3]{2000}$.
   b. Use a calculator to evaluate the expression in part a. (Lesson 8-8)

21. a. If $\sqrt[5]{x} = 7$, then $\ $? is a $\ ?$ root of $\ ?$.
   b. Rewrite the equation in part a using rational exponents.
   c. Solve this equation. (Lesson 8-8)

22. A fast ship’s speed $s$ (in knots) varies directly as the seventh root of the power $p$ (in horsepower) being generated by the engine. (Lessons 2-1, 8-5, 8-8)
   a. Write an equation expressing this relation.
   b. By how much is the speed multiplied when the horsepower is tripled?

In 23 and 24, the maximum distance $d$ you can see from a building of height $h$ is given by the formula

$$d \approx k\sqrt{h}.$$  

23. The CN Tower in Toronto is about 4 times as tall as the Los Angeles City Hall. About how many times farther can you see from the top of the CN Tower than from the top of L.A. City Hall? (Lesson 8-8)

24. About how many times farther can you see from the 108th floor of the World Trade Center than from the sixth floor? (Assume floors have the same height.) (Lesson 8-8)

25. A ball is thrown upwards and its height $h$ in feet after $t$ seconds is described by the equation

$$h = -16t^2 + 48t + 6.$$  
a. From what height was the ball thrown upwards?
   b. What is the maximum height the ball attains?
   c. When will the ball hit the ground? (Lessons 6-2, 6-5, 6-6)

26. Many students memorize the approximations $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$. If you know these, you can estimate $\sqrt{8}$ without a calculator because $\sqrt{8} = 2\sqrt{2} \approx 2 \cdot 1.414 = 2.828$. Name some other irrational square roots of integers between 1 and 100 that you could estimate from knowing approximate values of $\sqrt{2}$ or $\sqrt{3}$.
Solving $ax^n = b$

Acrobats carefully time the pendulum swing of a trapeze. See Example 2.

The general strategy for solving an equation with $n$th powers or $n$th roots is to raise both sides to the reciprocal power as that will make the exponent of the variable one. For example, if an equation involves the 6th power, take the 6th root.

**Example 1** Find all real solutions to $3x^6 = 46,875$.

**Solution** Divide both sides by 3 to get $x^6 = 15,625$. By definition, $x$ is a 6th root of 15,625. Take the 6th root of each side.

$$\sqrt[6]{x^6} = \sqrt[6]{15,625}$$

Thus $x = 5$ or $x = -5$.

**Check** Substitute each solution into the original equation.

$$3(5)^6 = 3(15,625) = 46,875$$
$$3(-5)^6 = 3(15,625) = 46,875$$

If an equation involves an $n$th root, take the $n$th power. In Example 2, the equation involves square roots, so square both sides.

**Example 2** The time $t$ (in seconds) that it takes a pendulum to complete one full swing is given by the formula

$$t = 2\pi \sqrt{\frac{L}{g}}$$

where $L$ is the length of the arm of the pendulum (in cm) and $g$ is a constant due to gravity. Suppose a ball on a string, swinging like a pendulum, takes 2 seconds to complete one swing back and forth. If $g = 980$ cm/sec$^2$, find the length of the string.
Solution Here \( t = 2 \text{ sec}, g = 980 \text{ cm/sec}^2 \), and 
we wish to find \( L \).

Divide both sides by \( 2\pi \).

\[
\frac{1}{\pi} = \sqrt{\frac{L}{980}}
\]

Square both sides of the equation.

\[
\left(\frac{1}{\pi}\right)^2 = \left(\sqrt{\frac{L}{980}}\right)^2
\]

\[
\frac{1}{\pi^2} = \frac{L}{980}
\]

\[
\frac{980}{\pi^2} = L
\]

Use a calculator.

\[99.29 = L\]

The string is about 99 cm long, a little short of one meter.

Check

\[2\pi \sqrt{\frac{99.29}{980}} \approx 1.99995 \approx 2.\] The answer checks.

When you take the \( n \)th power or root of both sides of an equation, you may gain or lose solutions. Consequently, every answer that you find must be checked. If an answer does not check, it is called extraneous and is not a solution to the original equation. The following example shows this.

Example 3 Solve \( 3 - \sqrt[4]{y} = 10 \).

Solution Add -3.

Raise both sides to 4th power.

Simplify.

\[-\sqrt[4]{y} = 7 \]

\[(-\sqrt[4]{y})^4 = 7^4 \]

\[y = 2401\]

Check Does \( 3 - \sqrt[4]{2401} = 10? \)

\[3 - 7 = 10?\]

No, so 2401 is not a solution. It is extraneous. The original sentence has no solution.

Notice that, in the solution of Example 3, as soon as you write the equation

\[-\sqrt[4]{y} = 7,\]

you might see there is no solution. The left side represents a negative number, so it cannot equal the positive number 7.
Questions

In 1–3, determine the number of real solutions.

1. \( x^2 = -100 \)  
2. \( x^3 = -100 \)  
3. \( x^4 = 16 \)

In 4–9, find all real solutions.

4. \( \sqrt[3]{v} = 9 \)  
5. \( w^{1/3} = 4 \)  
6. \( 4\sqrt{x} = 2 \)  
7. \( y^4 = 14641 \)  
8. \( 27 = z^3 \)  
9. \( 3x^5 = 96 \)

10. In Example 2, if the pendulum takes 4 sec to complete one full swing, how long is the pendulum?

11. What is an extraneous solution?

12. Explain why you don’t need to solve \( \sqrt{m - 3} = -10 \) in order to know that it has no solutions.

In 13–16, find all real solutions.

13. \( -3x^3 = 13824 \)  
14. \( \sqrt[3]{3x - 4} = 10 \)  
15. \( s^{1/3} + 2.4 = 8 \)  
16. \( \sqrt[3]{z} + 9 = 10\sqrt[3]{z} \)

17. Recall that when traveling at a fast rate, a ship’s speed \( s \) (in knots) varies directly as the seventh root of the power \( p \) (in horsepower) being generated by the engine. The equation \( s = 6.492p^{1/7} \) describes the situation. If a ship is traveling at a speed of 25 knots, about how much horsepower is the engine generating?

18. In a geometric sequence \( g_1 = 2000 \) and \( g_5 = 125 \).
   a. Find all possible real values for \( r \), the common multiplier.
   b. List the possible values for \( g_2, g_3, \) and \( g_4 \).

19. A formula that police use for finding the speed \( s \) (in mph) that a car was going from the length \( L \) (in feet) of its skid mark is \( s = 2\sqrt{5L} \).
   a. The Guinness Book of World Records (1985 Edition) reports that in 1960, a Jaguar in England had the longest skid mark recorded, 950 feet. About how fast was the Jaguar going?
   b. About how far does an auto travel if it skids from 50 mph to a stop?

20. Solve for \( m \): \( \sqrt[6]{m^6} = 64 \). (Lesson 8-9)

In 21–23, rewrite each root. Assume all variables are nonnegative. (Lesson 8-9)

21. \( \sqrt[5]{-125x^6} \)  
22. \( \sqrt[6]{32x^5y^{11}} \)  
23. \( \sqrt[5]{50a^3} \cdot \sqrt[4]{8b^4} \)
24. Let \( f(x) = (x - 5)^0 \), \( g(x) = (x - 5)^1 \), and \( h(x) = (x - 5)^2 \). (*Lessons 3-2, 6-4, 7-1*)
   a. Graph \( f \), \( g \), and \( h \) on the same axes.
   b. For what value(s) of \( x \) does \( f(x) = g(x) \)?
   c. For what value(s) of \( x \) does \( g(x) = h(x) \)?

25. In 1980 the world was using petroleum at a rate of \( 1.35 \cdot 10^{20} \) J/yr, where \( J \) is a unit of energy called a joule. At that time the world’s supply of petroleum was estimated to be about \( 10^{22} \) J. If the global rate of consumption remains constant, about how long will the world’s supply of petroleum last? (*Lesson 8-1*)

26. Consider the function \( f \) described by \( f(x) = 3x - 4 \). (*Lesson 7-6*)
   a. Write an algebraic expression for \( f^{-1}(x) \).
   b. Describe the relationship between the slopes of \( f \) and \( f^{-1} \).

27. The length of the skid mark in Question 19 is the same as would be found by using the braking distance formula of Lesson 7-1.
   a. Explore other speeds, calculating skid mark lengths and braking distances. Are they the same for any other speeds?
   b. Should they be the same?
Solving \(a(x-h)^n = b\)

You have solved equations of the form \(x^n = b\). Simply take the \(n\)th root of each side or raise each side to the \(\frac{1}{n}\) power. It takes one step. Solving \(ax^n = b\) takes two steps. First divide both sides by \(a\); then take each side to the \(\frac{1}{n}\) power. The equation \(a(x-h)^n = b\) can be solved in three steps.

Step 1. Divide each side by \(a\).

Step 2. Raise each side to the \(\frac{1}{n}\) power. Remember that it is possible to gain or lose solutions.

Step 3. Add \(h\) to each side.

The solutions to \(a(x - h)^n = b\) are \(h\) larger than the solutions to \(ax^n = b\).

---

**Example 1**

Michael has $500 he would like to save for college. His goal is to find an investment that would allow his money to double to $1000 in four years. What compound annual interest rate \(r\) would make this happen?

**Solution**

Use the formula from Lesson 8-2.

\[ A = P(1 + r)^t \]

Here \(P\) is the original principal, \(r\) the annual rate, \(t\) the number of years, and \(A\) is the total value of the investment after \(t\) years.

Substitute in the formula.

\[ 1000 = 500(1 + r)^4 \]

Now use the steps. First divide both sides by 500.

\[ 2 = (1 + r)^4 \]

Since \(1 + r\) is positive, take the \(\frac{1}{4}\) power of each side.

\[ 2^{1/4} = ((1 + r)^4)^{1/4} \]

\[ 2^{1/4} = 1 + r \]

A calculator shows that \(2^{1/4} \approx 1.189207\). Rounding to the nearest hundredth gives \(2^{1/4} = 1.19\). Thus

\[ 1.19 \approx 1 + r \]

Add \(-1\) to each side.

\[ 0.19 \approx r \]

Thus, for $500 to double to $1000 in four years, Michael needs to find an investment yielding at least 19% interest. A savings account will not do the job. His goal is probably unrealistic.
Example 2  A piece of furniture loses a fixed part of its value each year. If in five years it is worth 60% of what it was originally, what percent of value is it losing each year?

Solution  Think of its original value as 100%. Let \( r \) be the rate of loss. The value is multiplied by \( 1 - r \) five times. The result is 60% of its value.

The equation is

\[
.60 = (1 - r)^5
\]

or

\[
.60 = (1 - r)^5.
\]

Raise both sides to the \( \frac{1}{5} \) power.

\[
(.60)^{1/5} = ((1 - r)^5)^{1/5}
\]

\[
(.60)^{1/5} = 1 - r
\]

\[
-0.097 \approx -r
\]

The furniture is losing about 9.7% of its value each year.

Check  If the furniture loses 9.7% of its value each year, then each year its value is 90.3% of the previous year. Is \((90.3\%)^5 = .60\)? Yes, \((0.903)^5 = 0.600397\)... Since the furniture loses 40% of its original value after 5 years, it is reasonable that the yearly depreciation should be between 8% and 10%.

The algorithm for solving \( a(x - h)^n = k \) applies to any real exponent \( n \), including fractions.

Example 3  Given \( x + 9 > 0 \), solve for \( x \): \( 3(x + 9)^{2/5} = 48 \).

Solution  Divide each side by 3.

Raise each side to the \( \frac{5}{2} \) power.

\[
(x + 9)^{2/5} = 16
\]

\[
((x + 9)^{2/5})^{5/2} = 16^{5/2}
\]

\[
x + 9 = 1024
\]

\[
x = 1015
\]

Check  Does \( 3(1015 + 9)^{2/5} = 48 \)?

\[
3 \cdot (1024)^{2/5} = 48?
\]

\[
3 \cdot 16 = 48? \text{ Yes.}
\]

Questions

1. You wish to solve \( a(x - h)^n = k \), where \( a \) and \( k \) are positive.
   a. What is a reasonable first step?
   b. What is a second step?
   c. What is a third step?

2. Multiple choice  The equation \( y^2 = 17 \) has solutions of \( y = \sqrt{17} \) or \( y = -\sqrt{17} \). Compared to these, the solutions to \((y - 3)^2 = 17\) are:
   (a) 3 units smaller  (b) \( \sqrt{3} \) units smaller  (c) 3 units larger  (d) -\( \sqrt{3} \) units larger.
3. Refer to Example 1. Suppose that Michael could leave his $500 in an investment for 7 years instead of 4.
   a. What equation can be used to find the annual interest rate Michael must get to double his money?
   b. What compound annual interest rate would allow his money to double in 7 years?

4. Sherri wants to invest $200 for 6 years. What compound annual interest rate would allow her investment to triple in that time?

In 5 and 6, solve and check.

5. \(8(t - 5)^3 = 27\)
6. \(5(x + 2)^{1/4} = 405\)

7. Refer to Example 2. Suppose a piece of furniture is worth 70% of its original cost after 4 years. If its depreciation has been at a constant rate, find the percent of value it has been losing each year.

In 8–11, solve and check.

8. \(\left(\frac{m - 3}{16}\right)^2 = 16\)
9. \(110(r + 1)^{1/2} = 1870\)
10. \(5 + \sqrt{2x - 5} = 1\)
11. \((3x + 5)^{2/3} = 4\)

12. **Multiple choice** Which of these is a solution to \(a(x - h)^n = k\)?
   (a) \(\left(\frac{k}{a-h}\right)^{1/n}\)
   (b) \(\left(\frac{k}{a}\right)^{1/n} + h\)
   (c) \(\left(\frac{k-a}{n}\right)^{1/n}\)
   (d) \(\sqrt[n]{k}\)

13. Joyce was left $5000 by her great aunt. It took 4½ years for the will to go through probate, and she was told the amount had grown to $6247.12 through annual compounding.
   a. At what rate was the amount growing?
   b. Is this a fair return?

14. A cube has sides of length \(s\) millimeters. Then 0.5 millimeter is shaved off each dimension, so the volume \(V\) of the shaved cube is \(V = (s - 0.5)^3\). What was the original length of a side of a cube which ended up with a volume of 1000 cubic millimeters?

15. Refer to Michael’s situation at the beginning of the lesson. Suppose Michael invested \(d\) dollars. What rate would it take to double in 4 years if the account were compounded
   a. monthly?
   b. daily?

16. **Multiple choice** A health spa installed a circular tub with a 3' radius. A contractor was hired to build a circular deck of width \(w\) around the tub. The total surface area of the deck and the area under the tub should be 100 square feet. Which equation can be solved to find \(w\) in this situation?
   (a) \(100 = \pi(w + 3)^2\)
   (b) \(100\pi = (w + 3)^2\)
   (c) \(100 = \pi(w - 3)^2\)
   (d) \(100\pi = (w - 3)^2\)
In 17–19, simplify each expression without a calculator. \(\text{Lessons 8-6, 8-8}\)

17. \((64)^{\frac{7}{6}}\) 
18. \((\frac{25}{3})^{\frac{3}{2}}\) 
19. \(\sqrt[3]{125^2}\)

20. Solve \(2z^7 = \frac{1}{8192}\). \(\text{Lesson 8-10}\)

21. a. Give a counterexample to the statement \(\sqrt[4]{x^4} = x\) for all real numbers \(x\).

b. For what values of \(x\) is it true that \(\sqrt[4]{x^4} = x\)? \(\text{Lesson 8-9}\)

22. \(\textbf{Multiple choice}\) Which is not a 4th root of 625? \(\text{Lesson 8-4}\)
   (a) 5 
   (b) -5 
   (c) 5i 
   (d) \(\sqrt[4]{5}\)

23. Write the system of three inequalities whose solutions comprise the shaded region graphed at the right. \(\text{Lesson 5-7}\)

24. Write in scientific notation. \(\text{Previous course}\)
   a. 30,000,000 
   b. 300 
   c. .03 
   d. .0000003

25. The square of the cube of the \(n\)th power of \(x\) is what power of \(x\)? \(\text{Lesson 8-1}\)

In 26–27, consider the BASIC program below.

10 REM T = NO. OF YRS TO DOUBLE MONEY
20 REM R = RATE OF INTEREST AS DECIMAL
30 PRINT “HOW LONG DOES IT TAKE TO DOUBLE YOUR MONEY?”
40 PRINT “TIME IN YRS.”,”“ANNUAL RATE”
50 FOR T = 1 TO 15
60 LET R = 2 *(1/T) - 1
70 PRINT T,R
80 NEXT T
90 END

26. a. Solve the formula \(2 = 1(1 + R)^T\) for \(R\) to derive the formula in line 60.

b. Is \(R\) a linear function of \(T\)? Why or why not?

27. Run the program and study the output. At what rate of interest must you invest to double your money in 5 years compounded annually? 10 years? 15 years?

28. Given the interest rates at the banks in your neighborhood, what is a reasonable range of doubling times for an investment?
When $x > 0$, the expression $x^m$ is defined for any real number $m$. This chapter has covered the meanings of $x^m$ when $m$ is a positive or negative rational number.

In previous courses, you have learned basic properties of powers, which we assume as postulates:

- **Product of Powers Property** \( x^m \cdot x^n = x^{m+n} \)
- **Power of a Power Property** \( (x^m)^n = x^{mn} \)
- **Power of a Product Property** \( (xy)^n = x^ny^n \)
- **Quotient of Powers Property** \( \frac{x^m}{x^n} = x^{m-n} \)
- **Power of a Quotient Property** \( \left( \frac{x}{y} \right)^m = \frac{x^m}{y^m} \)

These properties hold for all values of $x$ when $m$ and $n$ are positive integers. If we restrict $x$ to be positive, then they hold for all values of $m$ and $n$.

These values can be calculated using theorems:

- **Zero Exponent Theorem** \( x^0 = 1 \)
- **Negative Exponent Theorem** \( x^{-m} = \frac{1}{x^m} \)
- **$\frac{1}{n}$ Exponent Theorem** \( x^{1/n} = \text{positive solution to } b^n = x \)
- **Root of a Power Theorem** \( x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m \)

When $x$ is positive, \( \sqrt[n]{x} = x^{1/n} \). When $x$ is negative and $n$ is odd, then \( \sqrt[n]{x} \) is the real $n$th root of $x$. If $n = \frac{1}{m}$ in the Power of a Product Property, then a new property results:

- **Root of a Product Property** \( \sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y} \)

These properties are of assistance in solving equations of the form $x^n = b$, $ax^n = b$ or $a(x - h)^n = b$. These equations can arise from compound interest situations, from relations between lengths, areas, and volumes, and a variety of situations leading to geometric sequences. Three formulas are:

- **Compound Interest Formula:** \( A = P(1 + r)^t \)
- **Explicit Formula for a Geometric Sequence:** \( g_n = g_1r^{n-1} \)
- **Recursive Formula for a Geometric Sequence:** \( g_n = rg_{n-1}, n > 1 \).

### Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition or statement for those terms marked with an * *. For all other terms you should be able to give a general description and a specific example of each.

**Lesson 8-1**
- powering, exponentiation
- base, exponent, power
- *Repeated Multiplication Model for Powering
- Product of Powers Property
- Power of a Power Property
- Power of a Product Property
- Quotient of Powers Property
- Power of a Quotient Property
- Zero Exponent Theorem

**Lesson 8-2**
- principal
- compounding
- *Compound Interest Formula

**Lesson 8-3**
- geometric sequence, exponential sequence
- *explicit formula for a geometric sequence
- *recursive formula for a geometric sequence

**Lesson 8-4**
- Negative Exponent Theorem

**Lesson 8-5**
- *square root, cube root, $n$th root
- $\frac{1}{n}$ Exponent Theorem

**Lesson 8-6**
- Rational Exponent Theorem

**Lesson 8-7**
- \( \sqrt[n]{x} \)
- Root of a Power Theorem
- Root of a Product Theorem

**Lesson 8-9**
- $n$th Root of $n$th Power Theorem

**Lesson 8-10**
- extraneous solution
Take this test as you would take a test in class. You will need a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. Order from largest to smallest: $3^{-4}$, $-3^4$, $(-3)^4$, $(-3)^4$.

In 2–4, write as a decimal or simple fraction.

2. $(625)^{1/2}$
3. $\sqrt[5]{11,390,625}$
4. $(\frac{1}{32})^{6/5}$

In 5 and 6, simplify. Assume $x > 0$ and $y > 0$.

5. $\sqrt[4]{625x^4y^8}$
6. $\sqrt[5]{-96x^{15}y^5}$

In 7–9, solve.

7. $9x^4 = 144$
8. $c^{3/2} = 64$
9. $5^n \cdot 5^{21} = 5^{29}$

10. Recall the formula $T = 2\pi \sqrt{\frac{L}{g}}$ for the time $T$ (in seconds) it takes a pendulum to complete one full swing, where $L$ is length (in centimeters) and $g$ is acceleration due to gravity. How long (to the nearest cm) is a pendulum that takes 1 second to swing? (Use 980 cm/sec$^2$ for $g$.)

11. An original set of architect's sketches is increasing in value at an annual rate of 17%. If it is valued at $13,500 now, what will it be worth (to the nearest hundred dollars) in three years?

12. A bank account pays 5.75% compounded daily. If you deposit $200 in the account and leave it untouched for 5 years, how much will be in the account then?

13. $400 is to be invested. What rate of interest should be paid to allow the money to double to $800 in 4 years if the account is compounded annually?

In 14–16, solve.

14. $100(A - 5)^4 = 1600$
15. $\frac{1}{5} (20 - P)^{1/2} = 5$
16. $\frac{\sqrt[5]{125}}{\sqrt[3]{343}} = \frac{5}{7}$

17. Write without an exponent: $\frac{2.1 \cdot 10^2}{10^{-3}}$.

18. Suppose each of a certain type of bacterium splits into two every half hour. If there are 5 bacteria initially, about how many will there be after 24 hours?

19. Find an explicit formula for the $n$th term in the geometric sequence: 2, 8, 32, 128,….

20. Multiple choice Which expression equals $a^{4/5}$ for all $a > 0$?
   (a) $\frac{1}{\sqrt[4]{a}}$    (b) $\frac{1}{\sqrt[5]{a^4}}$    (c) $a^{5/4}$    (d) $(-a)^{4/5}$

21. Evaluate without a calculator: $216^{1/3}$.

22. Recall the formula $h(t) = 180 \cdot 10^{-0.04t}$ for the number of hours $h$ that milk stays fresh in a surrounding temperature $t°C$. How long will milk stay fresh when stored at 15°C?

23. Identify all real 4th roots of 81.

24. True or false $\sqrt[5]{64} = -2$.

25. Radioactive material will change (decay) to different material over a period of time. The time it takes half of the mass of the radioactive material to decay is called the half-life. Radioactive carbon$^{14}$ has a half-life of 5600 years. If one starts with 40 grams of carbon$^{14}$, how many grams are left after 3 half-lives?
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers.

- **Objective A.** Evaluate \( x^n \) when \( n \) is an integer. (Lessons 8-1, 8-4, 8-9)
  In 1–6, write as a decimal or simple fraction.
  1. \( .2^6 \)
  2. \( 12^{-5} \)
  3. \( 3.4 \cdot 10^{-3} \)
  4. \( \left( \frac{2}{3} \right)^{-1} \)
  5. \( \left( \frac{1}{5} \right)^{-4} \)
  6. \( (-2)^2 \)

- **Objective B.** Evaluate \( x^b \) when \( b \) is a rational number. (Lessons 8-5, 8-6, 8-7)
  In 7–12, write as a decimal or simple fraction. Estimate decimals to the nearest hundredth.
  7. \( 1000^{1/3} \)
  8. \( \sqrt[3]{27} + 64 \)
  9. \( 3 \cdot 27^{1/8} \)
  10. \( 16^{3/4} \)
  11. \( 16^{-1/2} \)
  12. \( \left( \frac{27}{16} \right)^{-2/3} \)
  13. \( \left( \frac{1}{64} \right)^{3/2} \)
  14. \( 2^{1.5} \)
  15. \( 80^{2/3} \)
  In 16 and 17, true or false.
  16. \( -7 = \sqrt[3]{117.649} \)
  17. \( 3^{6.4} < 3^{6.5} \)

- **Objective C.** Simplify radicals. (Lessons 8-8, 8-9)
  In 18–20, write as a decimal or simple fraction.
  18. \( \sqrt[3]{625} \)
  19. \( \sqrt{8} \)
  20. \( \sqrt[3]{\left( \frac{8}{125} \right)^2} \)
  In 21–23, estimate to the nearest hundredth.
  21. \( \sqrt{4} \)
  22. \( \sqrt[3]{-80} \)
  23. \( \sqrt[5]{346} \)
  In 24–29, simplify. Assume variables under the radical sign are positive.
  24. \( \sqrt{a^6} \)
  25. \( \sqrt[3]{54x^3} \)
  26. \( \sqrt[6]{128x^8y^7} \)
  27. \( \sqrt[3]{-80a^9} \)
  28. \( \sqrt[5]{-64c^{10}} \)
  29. \( \sqrt[7]{7a^3} \cdot \sqrt[14]{14a} \)

- **Objective D.** Solve equations of the form \( ax^n = b \) or their equivalent radical forms. (Lessons 8-6, 8-7, 8-10)
  In 30–39, solve.
  30. \( 3x^2 = 192 \)
  31. \( -27 = a^4 \)
  32. \( x^3 = 12 \)
  33. \( x^2 = 9 \)
  34. \( m^{3/2} = \frac{1}{27} \)
  35. \( 4a^{-2/5} = 9 \)
  36. \( \sqrt[3]{a} = 2 \)
  37. \( 4\sqrt[4]{b} = 3 \)
  38. \( \sqrt[5]{c} + 4 = 3 \)
  39. \( \sqrt[3]{c} - 4 = 3 \)
Objective E. Solve equations of the form $a(x - h)^t = k$. (Lesson 8-11)

40. Multiple choice From the equation $200(r + 1)^4 = 3200$, you can conclude that $|r + 1| = 2$ if, to both sides, you:
   (a) divide by 200 and then take the fourth power;
   (b) take the 4th power and then divide by 200;
   (c) divide by 200 and then take the $\frac{1}{4}$ power;
   (d) subtract 1, then divide by 200, and then take the $\frac{1}{4}$ power.

In 41–43, solve. Round answers to the nearest tenth.

41. $\sqrt[3]{x + 1} - 9 = 16$
42. $\frac{1}{4} (9 + y)^{1/2} = 14$
43. $.2(r - 1)^5 = 3.4$

Objective F. Solve equations or simplify expressions using properties of exponents. (Lessons 8-1, 8-4, 8-6, 8-7)

In 44–47, solve.

44. $(9^5 \cdot 9^3) = 9^x$
45. $\frac{2^5}{2^{-1}} = 2^x$
46. $(7^{1/2})^3 = 7^n$
47. $(2 \cdot 5)^{-3} = y^{-3}$

In 48–51, simplify.

48. $(-4x^2)^3$
49. $\frac{-8x^{10}y^{3/2}}{2xy^{1/2}}$
50. $\left(\frac{a}{b}\right)^3 \left(\frac{2b}{3a}\right)^4$
51. $\frac{15c}{(3c^{-6})(20c^6)}$

Objective G. Find terms or the rule for a geometric sequence. (Lesson 8-3)

In 52–54, give the first five terms of the geometric sequence described.

52. constant ratio 4, first term 5
53. first term $\frac{1}{2}$, second term $\frac{3}{4}$
54. $a_1 = 10$, $a_n = -2a_{n-1}$

55. Multiple choice Which of the following contains the first three terms of a geometric sequence?
   (a) 16, 4, -8, ...
   (b) $\frac{4}{5}, \frac{9}{5}, \frac{14}{5}$...
   (c) $3\frac{1}{3}, 33\frac{1}{3}, 333\frac{1}{3}$, ...
   (d) -0.04, 0.16, 0.36, ...

56. True or false The first four terms of the geometric sequence described by the formula $a_n = \frac{3}{16} (-2)^{n-1}$ are $\frac{3}{16}, \frac{3}{4}, \frac{3}{2}, 3$.

In 57 and 58, find an explicit rule for the $n$th term of the geometric sequence.

57. 2, 1, 5, ...
58. 10, 30, 90, ...

59. Find the 50th term of a geometric sequence whose first term is 6 and whose constant multiplier is 1.05.

60. True or false. If $0 < x < 1$, $\sqrt[3]{x} > x$.

61. Suppose $x > 1$. Arrange from smallest to largest: $x, \sqrt{x}, x^2, x^{5/4}, x^{-2/3}$

Properties deal with the principles behind the mathematics.
In 62–67, use the properties listed below. Assume \( Q > 0, x \neq 0, \) and \( n \) is an integer greater than 1. Identify all properties which apply to the simplification.

I. \( Q^0 = 1 \quad \text{II. } Q^x = \left( \frac{1}{Q} \right)^x \)

III. \( Q^{1/n} = \sqrt[n]{Q} \quad \text{IV. } \sqrt[n]{PQ} = \sqrt[n]{P} \sqrt[n]{Q} \)

V. \( Q^{\nu n} = (\sqrt[n]{Q})^\nu = \sqrt[n]{Q^\nu} \)

62. \((6.789)^{5-5} = 1\)
63. \((82^{2/3}) = 2\sqrt[3]{2}^2\)
64. \((y^{1/7})^7 = y\)
65. \((25)^{-1/2} = \frac{1}{5}\)
66. \((\frac{1}{y})^{3/4} = (\sqrt[3]{y})^3\)
67. \(\sqrt[3]{32} = 2\sqrt[3]{2}\)

**Uses** deal with applications of mathematics in real situations.

**Objective J.** Apply the compound interest formula. (Lessons 8-2, 8-4, 8-11)

75. Sue invests $150 in a savings account which pays 5.75% interest, compounded annually. How much money will be in the account if the $150 is left untouched for 6 years?

76. Investment A offers an annual interest rate of 8%, compounded daily. Investment B offers an annual interest rate of 6%, compounded daily. Leo is considering investing $200 in one of these accounts. Which will yield a higher amount: investment A for 3 years, or investment B for 4 years?

In 77 and 78, Caryn now has $6000 in an account earning interest at a rate of 9%, compounded quarterly.

77. Assuming she made no deposits or withdrawals in the past four years, how much money did she have four years ago?

78. How much interest will she earn in the eighth year?

79. Melvin has $4000 and wants to find an investment which will allow the amount to double in 10 years. What interest rate compounded annually will accomplish this?

**Objective K.** Solve real world problems which can be modeled by powers and roots. (Lessons 8-1, 8-4, 8-5, 8-6, 8-7, 8-10, 8-11)

80. The Pentagon, one of the world’s largest office buildings, occupies about \(1.2 \cdot 10^5\) m\(^2\). The Pentagon occupies what percent of the area \(1.9 \cdot 10^6\) m\(^2\) which is the area of Monaco, a small country in the South coast of France?

81. The power \(P\) of a radio signal varies inversely as the square of the distance \(d\), from the transmitter. Write a formula for \(P\) as a function of \(d\) using:
   a. a positive exponent
   b. a negative exponent.
In 82 and 83, use the following information: Meteorologists use the equation $D^3 = 216T^2$ as a model to describe the size and intensity of four types of violent storms: tornadoes, thunderstorms, hurricanes, and cyclones, where $D$ is the diameter of the storm in miles and $T$ is the number of hours the storm travels before dissipating.

82. The world’s worst recorded hurricane took place on November 13-14, 1970, in the Ganges delta islands in Bangladesh. More than 1,000,000 people died. If this storm lasted 24 hours, what was its diameter?

83. If a tornado’s diameter is 1.5 miles, how long would it be expected to last?

84. A sphere has a volume of 400 in$^3$. Another sphere has a radius half as long.
   a. What is the radius of the second sphere?
   b. What is the volume of the second sphere?

85. A spherical raindrop has radius $r$ millimeters. Through evaporation the radius decreases by .05 millimeters. If the volume of the condensed drop is 7.2 mm$^3$, what was the original radius of the drop?

Objective L. Solve real world problems by geometric sequences. (Lesson 8-3)

86. The height reached by a bouncing ball on successive bounces generates a geometric sequence. Suppose a ball reaches heights in cm of 120, 96, and 76.8 on its first three bounces. How high will the ball reach on:
   a. the next bounce?
   b. the 10th bounce?
   c. the $n$th bounce?

87. A copying machine is set to reduce linear dimensions to 95%. If an 8 in. by 10 in. original is reduced five times, what will be its dimensions?

88. A vacuum pump removes 10% of the air from a chamber with each stroke.
   a. Find a formula for $P_n$, the percent of air that remains in the chamber after the $n$th stroke.
   b. How many strokes must be taken to remove 75% of the air in the chamber?

REPRESENTATIONS deal with pictures, graphs, or objects that illustrate concepts.

There are no representation objectives in this chapter.
This weather-satellite picture was taken from an altitude of 35,800 km.
The bar graph below gives the U.S. population for each census from 1790 to 1980. The population \( P \) (in millions) is closely approximated by the equations

\[
\begin{align*}
P &= 13(1.03)^{x-1830} & \text{for } 1790-1860 \\
P &= 63(1.02)^{x-1890} & \text{for } 1870-1910 \\
P &= 151(1.013)^{x-1950} & \text{for } 1920-\text{present},
\end{align*}
\]

where \( x \) is the year of the census.

Each equation above represents an \textit{exponential function}. We need several equations to estimate the population at different times because the yearly \textit{growth factor} of the population of the United States has changed from about 1.03 to about 1.013 since 1790.

In this chapter you will learn about exponential functions and their inverses.
Consider an experiment that begins with 300 bacteria. Bacteria can quickly increase in number. Let us suppose that the population doubles every hour.

Here is the population $y$ after $x$ hours for $x = 0, 1, 2, \text{ and } 3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>300</td>
<td>600</td>
<td>1200</td>
<td>2400</td>
</tr>
</tbody>
</table>

The points for these values are graphed at the right.

The table shows that as $x$ takes on integer values increasing by 1, the values of $y$ form a geometric sequence with constant ratio 2. A formula for this sequence is $y = 300 \cdot 2^x$.

Of course, the bacteria population does not double all at once. Using fractions as exponents, you can estimate the population at intermediate times. For instance, after half an hour $y = 300 \cdot 2^{1/2} \approx 424$. Here are other approximate values at some intermediate times.

<table>
<thead>
<tr>
<th>$x$</th>
<th>.25</th>
<th>.5</th>
<th>.8</th>
<th>1.4</th>
<th>2.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>357</td>
<td>424</td>
<td>522</td>
<td>792</td>
<td>1819</td>
</tr>
</tbody>
</table>

Check these values on your calculator.

You can also find population values before the experiment started. For instance, an hour before the experiment began, $x = -1$, so $y = 300 \cdot 2^{-1} = 150$.

Here are other earlier values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>-1.5</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>150</td>
<td>106</td>
<td>75</td>
</tr>
</tbody>
</table>

At the right is the graph with values from all three tables of values included. The graph looks like it would be a continuous curve if you filled in all the gaps. Intermediate rational values of $x$ produce intermediate values of $y$. 
The same is true for irrational values of $x$. For instance, the value $x = \sqrt{5}$ yields $y = 300 \cdot 2^{\sqrt{5}}$.

To approximate $y$, first approximate $\sqrt{5}$, which as a decimal is $2.236067977...$. If you approximate $\sqrt{5}$ by 2.2, then

$$y = 300 \cdot 2^{2.2} \approx 1378.$$ 

This $y$-value is less than $300 \cdot 2^{\sqrt{5}}$. If you use 2.24 for $\sqrt{5}$,

$$y = 300 \cdot 2^{2.24} \approx 1417.$$ 

This $y$-value is more than $300 \cdot 2^{\sqrt{5}}$. If you use 2.236067 for $x$,

$$y = 300 \cdot 2^{2.236067} \approx 1413.329... .$$

If you use 2.236068 for $x$

$$y = 300 \cdot 2^{2.236068} \approx 1413.333... .$$

Thus $300 \cdot 2^{\sqrt{5}} \approx 1413.33... .$

By extending this reasoning for other irrational values of $x$, you can think of the graph as having points for all real values of $x$.

The completed graph of $y = 300 \cdot 2^x$ is shown above. The graph is called an exponential curve. The shape of an exponential curve is different from the shape of a parabola or an arc of a circle.
Example 1  Use the exponential curve above to estimate:

a. the number of bacteria after 20 minutes;
b. the time when 1500 bacteria were present.

Solution  a. 20 minutes = \( \frac{1}{3} \) hour. On the graph, when \( x = \frac{1}{3} \), \( y \) is a bit under 400. At 20 minutes there were a little less than 400 bacteria.
b. When \( y = 1500 \), \( x \approx 2\frac{1}{4} \), so there were 1500 bacteria after about 2 hours, 15 minutes.

Check  a. Substitute \( x = \frac{1}{3} \) into the equation \( y = 300 \cdot 2^x \) to get \( y \approx 378 \).
b. Substitute \( x = 2.25 \) into \( y = 300 \cdot 2^x \). You get \( y = 1427 \), so it checks.

The equation \( y = 300 \cdot 2^x \) is of the form \( y = ab^x \) and defines a function in which the independent variable \( x \) is in the exponent. Such a function is called an exponential function. In the exponential function \( f(x) = ab^x \), \( b \) is the growth factor, the amount by which \( y \) is multiplied for every unit increase in \( x \). The \( y \)-intercept \( f(0) \) is equal to \( a \). When \( b > 1 \), the situation is one of exponential growth.

The compound-interest formula \( A = P(1 + r)^t \) defines an exponential function when \( P \) and \( r \) are fixed. Then \( t \) is the independent variable, \( A = f(t) \) is the dependent variable, and \( 1 + r \) is the growth factor. The explicit formula for any geometric sequence \( g_n = g_1r^{n-1} \) also defines an exponential function with \( n \) as the independent variable, \( g_n \) as the dependent variable, and \( r \) as the growth factor.

As we noted earlier, populations can grow exponentially over short periods of time. This relationship can be used to predict future populations, an important consideration in planning or building schools, roads, airports, and other public facilities. To predict what the U.S. population will be in the year 2000, we might first examine recent censuses.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>Decade Growth Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1920</td>
<td>106,020,000</td>
<td></td>
</tr>
<tr>
<td>1930</td>
<td>123,200,000</td>
<td>1.162</td>
</tr>
<tr>
<td>1940</td>
<td>132,160,000</td>
<td>1.073</td>
</tr>
<tr>
<td>1950</td>
<td>151,330,000</td>
<td>1.145</td>
</tr>
<tr>
<td>1960</td>
<td>179,320,000</td>
<td>1.185</td>
</tr>
<tr>
<td>1970</td>
<td>203,300,000</td>
<td>1.134</td>
</tr>
<tr>
<td>1980</td>
<td>226,540,000</td>
<td>1.114</td>
</tr>
</tbody>
</table>

The decade growth factor for 1930 is \( \frac{123,200,000}{106,020,000} \) or approximately 1.162, indicating a 16.2% population increase from 1920 to 1930.
The yearly growth factor is the 10th root of the decade growth factor, \( \sqrt[10]{1.162} \) or about 1.015, indicating a mean yearly population increase of about 1.5%. Over the sixty-year period the mean yearly increase was a little under 1.3%, corresponding to a growth factor of 1.013.

Let the exponent \( x \) be the year. We will start from 1950, the middle of this period. If we take 1.013 as the growth factor and 151 (in millions) as the population in 1950, a good approximation to the U.S. population \( y \) (in millions) since 1920 is

\[
y = 151 \cdot (1.013)^{x-1950}
\]

This is the equation given on the opening pages of the chapter.

**Example 2** Use \( y = 151 \cdot (1.013)^{x-1950} \) to predict the U.S. population (a) in 1990 and (b) in 2000.

**Solution** With a calculator, the computation is straightforward.

a. For 1990: \( y = 151 \cdot (1.013)^{1990-1950} = 151 \cdot (1.013)^{40} = 253,000,000 \)

b. For 2000: \( y = 151 \cdot (1.013)^{2000-1950} = 151 \cdot (1.013)^{50} = 288,000,000 \)

Because the growth factor has been declining, these figures are likely to be high, but probably not by much.

It is natural to wonder when the population will reach 400 million or some other number. If population grows exponentially, that problem requires solving the exponential equation \( y = ab^x \) for \( x \). We examine this equation in depth beginning in Lesson 9–3 after we have looked at examples of exponential functions in which the growth factor is less than one.

**Questions**

In 1–3, refer to the bacteria experiment at the beginning of the lesson.

1. How many bacteria are there after 4 hours?

2. What does \( x = -1.5 \) mean in terms of the experiment?

3. Use the exponential curve to estimate:
   a. the number of bacteria after 1.75 hours;
   b. when 1700 bacteria were present.

4. **True or false** Because \( \sqrt{7} \) is between 2.64 and 2.65, \( 2^{2.64} < 2^{\sqrt{7}} < 2^{2.65} \).
5. Use your calculator to evaluate each power.
   a. \(2^{1.73}\)  
   b. \(2^{1.74}\)  
   c. \(2^{\sqrt{3}}\)


7. In an exponential function, for each unit increase in the \(\_\_\_\) variable, the \(\_\_\_\) variable is \(\_\_\_\) by a constant growth factor.

In 8 and 9, refer to the standard equation for an exponential model \(y = ab^x\).

8. The initial value of \(y\) is \(\_\_\_\) .

9. The constant growth factor is \(\_\_\_\) for each unit of \(\_\_\_\) .

In 10 and 11, multiple choice

10. Which equation has a graph that is an exponential curve?
   (a) \(y = 2x + 5\)  
   (b) \(y = x^2 + 5\)  
   (c) \(y = 2.5^x\)  
   (d) \(y = 2x^5\)

11. Which of the following is an equation for an exponential function?
   (a) \(y = 3x\)  
   (b) \(y = x^3\)  
   (c) \(y = 3^x\)  
   (d) \(y = 3x^{1/3}\)

12. a. Graph \(y = 2^x\).
   b. On the same axes, graph \(y = 4^x\).
   c. Where do the graphs in (a) and (b) intersect?

In 13–15, refer to the U.S. population data given in this lesson.

13. a. Suppose the yearly growth factor for all years in a decade is \(y\). What is the decade growth factor?
   b. Suppose the decade growth factor is \(d\). What is the yearly growth factor?

14. What was the yearly growth rate of the U.S. population from 1970 to 1980?

15. a. Examine the formula for the years 1790–1860. By what percent was the population growing each year?
   b. What would the 1980 population have been if the population had kept growing by the 1790–1860 rate?

16. In 1983 Kenya was the fastest growing nation on Earth. Its population was 18.5 million people and its yearly growth rate was about 4.1%.
   a. Write an equation expressing Kenya’s population \(y\) in the year \(x\).
   b. Project Kenya’s population for the year 2015 if the growth rate remains the same.
   c. Under this model, what was Kenya’s population in 1976?

17. Let \(f(x) = 2^x\), \(g(x) = 3 \cdot 2^x\), and \(h(x) = 10^x\).
   a. Graph \(f, g,\) and \(h\) on the same set of axes.
   b. For what value(s) of \(x\) does \(f(x) = h(x)\)?
   c. For what value(s) of \(x\) does \(g(x) = h(x)\)?
   d. Write two or three sentences describing properties of all exponential growth functions.
18. Suppose you invest $400 in a bank that pays 6% interest compounded annually. Your money is left in the bank for 18 months. (Lesson 8-4)
   a. Write the expression representing your final balance.
   b. Calculate the balance.

In 19 and 20, (a) state whether the sequence is arithmetic, geometric, or neither; (b) write an explicit formula for the sequence; (c) write a recursive formula for the sequence. (Lessons 3-6, 3-7, 8-3)

19. 100, 90, 81, 72.9, 65.61, ...

20. 100, 91, 82, 73, 64, ...

21. Find an equation for $H^{-1}$ when $H(x) = x^3$. (Lesson 7-6)

22. This program calculates postage for first class mail (1988 prices). What will be printed if 4.5 ounces is input for W? (Lesson 7-4)

```
10 INPUT "WEIGHT OF A FIRST CLASS LETTER"; W
20 P = .25 - .20 * INT(1 - W)
30 PRINT "THE POSTAGE FOR YOUR LETTER IS"; P
40 END
```

23. A teacher finds that her grades are low and need to be rescaled. A 100 will remain a 100, but a 65 will become an 80.
   a. If the rescaling is linear, find a relationship between the new score $y$ and the old score $x$. (Hint: write the scores as ordered pairs.)
   b. What will an old score of 51 become? (Lesson 3-5)

24. For what reasons may a population's growth rate decrease? increase? Which of these factors were present in the United States between 1920 and 1980?

25. China and India are the two most populous countries in the world. Find an estimate for the current population and growth rate in each country. Use these figures to estimate what their populations will be in the year 2000.

26. Graph the population equations on the first page of this chapter using a function grapher or graphing calculator. For what year do these equations most poorly model the population?
In Lesson 9-1, the populations studied were increasing, so the constant growth factor is greater than one. Sometimes a growth factor is less than one. When this is true, the value of the function decreases over time. These situations are sometimes called exponential decay or depreciation.

**Example 1**

A Trans Am Firebird cost $6490 new in 1978. Suppose its value decreased exponentially and that the car depreciated 44% during its first 7 years.

**a.** What equation models its value?

**b.** Find the car's value in 1988.

**Solution**

**a.** Because the car is depreciating in value by a constant factor, the situation can be modeled by an exponential function with equation \( y = ab^x \). Here \( y \) is the value of the car; \( a \) is the original value, 6490; \( b = 0.56 \) (depreciating by 44% means the growth factor is \( 1 - .44 = 0.56 \)); and \( x \) is the number of 7-year intervals after 1978. Thus an equation modeling the value is

\[
y = 6490(0.56)^x.
\]

**b.** The year 1988 is 10 years after 1978. This is \( \frac{10}{7} \) of a 7-year interval.

\[
x = \frac{10}{7},
\]

\[
y = 6490(0.56)^x
\]

\[
y = 6490(0.56)^{10/7}
\]

\[
y = 2835.
\]

In 1988 the Trans Am was worth approximately $2835.
Example 2

Radioactive carbon-14 ($\text{C}_{14}$) decays exponentially. In 5730 years, half of the original amount decays (5730 is called the half-life of $\text{C}_{14}$) and half remains. Suppose you start with a sample that has 100 g of $\text{C}_{14}$. For this situation determine:

a. an equation for the amount of $\text{C}_{14}$ remaining in the original sample, and

b. the graph for the equation.

Solution

a. Since half remains in a fixed time period, $b = \frac{1}{2}$ or 0.5. Here the initial amount is 100 g. Use the exponential equation $y = ab^x$, with $a = 100$ and $b = 0.5$. The equation is

$$y = 100(0.5)^x,$$

where $x$ is the number of 5730-year intervals.

b. Evaluate $y$ for various values of $x$, plot these points, and sketch the curve. The completed graph is shown below.

![Graph of C14 decay]

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>25</td>
<td>12.5</td>
<td>6.25</td>
</tr>
</tbody>
</table>

Notice that 5730 years before the time the amount of $\text{C}_{14}$ was measured, there would have been 200 g of $\text{C}_{14}$ in the sample. In 11,460 years after the measurement (when $x = 2$), only 25 g would be left.

These examples of growth and decay fit a general model called the exponential growth model.

**Exponential growth model:**

If a quantity $a$ grows by a factor $b$ in each unit period, then after a period of length $x$, there will be $ab^x$ of the quantity.
When the growth factor \( b \) is less than 1, the value of \( y = ab^x \) decreases as \( x \) increases and so the graph of \( y = ab^x \) goes down to the right. Decay curves have the same shape as in exponential growth; they are reflection images of growth curves over the \( y \)-axis.

By looking at the graphs of the exponential functions \( y = ab^x \) in this and the previous lesson, you should observe several properties.

1. The domain of each function is the real numbers.
2. The graph never crosses the \( x \)-axis; exponential curves have no \( x \)-intercepts.
3. Each graph has a \( y \)-intercept \( a \), where \( a \) is the value of the function at \( x = 0 \).
4. The range of each function is the positive real numbers.
5. In one of its two quadrants, the graph gets closer and closer to the \( x \)-axis. The \( x \)-axis is an asymptote of the graph.
6. When the constant growth factor \( b \) is greater than one, as in Lesson 9-1, the graph is increasing. However, if \( 0 < b < 1 \), as in Example 2 above, the graph is decreasing.

Questions

1. If \( y = ab^x \) and \( 0 < b < 1 \), then \( y \) \underline{?} as \( x \) increases.

2. Multiple choice Which equation represents exponential decay?
   - (a) \( f(x) = \frac{1}{3}x \)
   - (b) \( f(x) = \sqrt[3]{x} \)
   - (c) \( f(x) = 3^x \)
   - (d) \( f(x) = 3^x \)

3. Refer to Example 1. If you assume that the depreciation model continues to be valid, what would the Firebird be worth in 1992?

4. If an item decreases in value by 10% each year, what is the yearly growth factor?

In 5–7, refer to Example 2.

5. What does the term “half-life” mean?

6. Find how much C\(_{14}\) remains after 17,190 years.

7. Find how much C\(_{14}\) existed 1910 years before the 100 g were measured.

8. True or false
   - a. Unless specifically stated, there are no restrictions on the domain of an exponential function.
   - b. The \( y \)-axis is an asymptote for an exponential curve.
9. **Multiple choice** Which graph could represent exponential decay?

(a) ![Graph A](image)

(b) ![Graph B](image)

(c) ![Graph C](image)

(d) ![Graph D](image)

10. The half-life of iodine-123 (I\textsubscript{123}) is about 13 hours. Suppose you begin with a sample of 20 grams.

a. Write an equation to model the decay.

b. Copy and complete the table below.

<table>
<thead>
<tr>
<th>(x) = number of 13-hour periods</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y) = amount of I\textsubscript{123}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Plot these points and graph the function.

d. Use the graph to estimate the number of hours needed for 20 g to decay to 4 g.

11. **Applying the Mathematics**

a. Graph \(y = 3^x\) for values of \(x\) between -4 and 4.

b. Graph \(y = (\frac{1}{3})^x\) for values of \(x\) between -4 and 4, on the same axes.

c. The graphs in parts a and b are related to each other. How are they related, and why are they so related?

12. Suppose a car costs $10,000 and loses value every year. Let \(N\) be its value after \(t\) years.

a. Assume the depreciation is exponential, with 20% of the value lost per year. Then \(N = 10,000(0.8)^t\). Complete a table like this one.

<table>
<thead>
<tr>
<th>(t)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Assume the depreciation is linear with $1000 lost per year. Then \(N = 10,000 - 1000t\). Fill in a table.

<table>
<thead>
<tr>
<th>(t)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. If this were your car and you were trading it in after 4 years, explain why you would probably prefer that the car dealer assumed the model of part b.

d. Under what circumstances would you prefer that the dealer assume the model in part a? Justify your answer using tables or graphs.
13. Use the graph at the right. (Lesson 9-1)
   a. When \( x = \frac{2}{3} \), \( y = 100 \).
   b. Multiple choice
   An equation for the graph could be
   (i) \( y = 10x \).
   (ii) \( y = 10 + x \).
   (iii) \( y = 10^{-x} \).
   (iv) \( y = \left( \frac{1}{10} \right)^x \).
   c. The domain of the function is ___.
   d. The range of the function is ___.

14. Suppose an experiment begins with 120 bacteria which double every hour.
   a. About how many bacteria will there be after 3 hours? (Lesson 8-3)
   b. What is an equation for the number \( y \) of bacteria after \( x \) hours?
      (Lesson 9-1)

15. Use the equation on the first page of this chapter to estimate the
    U.S. population in 1995. (Lesson 9-1)

16. Give a decimal approximation to the nearest hundredth. (Lessons 8-6, 9-1)
   a. \( 4^3 \)
   b. \( 4^{7/2} \)
   c. \( 4^{\sqrt{10}} \)

17. Explain why the answer to Question 16(c) should be between the
    answers to 16(a) and 16(b). (Lesson 9-1)

18. Write as a decimal. (Lesson 8-6)
   a. \( 10^{10.8} \)
   b. \( 10^{8.4} \)
   c. \( \frac{10^{10.8}}{10^{8.4}} \)

19. The inflation rate is typically reported monthly. Suppose a monthly
    rate of 0.5% is reported for January. Assume this rate continues for
    1 year. (Lesson 8-2)
   a. What is the inflation rate for the year?
   b. The value 0.5% has been rounded. The actual value could be any
      number equal to or greater than 0.45%, and less than 0.55%.
      Write an inequality for \( r \), the annual inflation rate, based on those
      two extreme values.

20. a. Graph \( y = 10^x \) using a function plotter. Use the graph to estimate
      solutions to the following pairs of equations:
      (i) \( 10^x = 3 \) and \( 10^x = \frac{1}{3} \)
      (ii) \( 10^x = 2 \) and \( 10^x = \frac{1}{2} \).
   b. What generalization have you verified?

21. Consult a car dealer or books or magazines about automobiles. Find
    out how automobile depreciation is typically calculated. Is automobile
    depreciation described better by linear (Question 12(b)) or exponential
    (Question 12(a)) models?
An earthquake is the sudden release of energy in the form of vibrations caused by rock suddenly moving along a fault, which is an edge of the earth’s crust. Perhaps 100,000 earthquakes that can be felt occur each year, and about 1,000 cause damage. This release can be measured in joules; the intensity of a destructive earthquake may be a million times the intensity of a minor tremor. A million is too wide a range to fit on a normal number line. So, instead, a scale based on exponents is used. The scale most widely reported in the United States is the Richter scale, named after Charles F. Richter (1900–1985), a seismologist at the California Institute of Technology, its inventor. The table below describes the effect of an earthquake of a particular Richter scale value magnitude at the epicenter, the place on the earth’s surface above the location of the release of energy.

<table>
<thead>
<tr>
<th>Richter magnitude</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>cannot be felt except by instruments</td>
</tr>
<tr>
<td>2</td>
<td>cannot be felt except by instruments</td>
</tr>
<tr>
<td>3</td>
<td>cannot be felt except by instruments</td>
</tr>
<tr>
<td>4</td>
<td>like vibrations from a passing train</td>
</tr>
<tr>
<td>5</td>
<td>strong enough to wake sleepers</td>
</tr>
<tr>
<td>6</td>
<td>very strong; walls crack, people injured</td>
</tr>
<tr>
<td>7</td>
<td>ruinous; ground cracks, houses collapse</td>
</tr>
<tr>
<td>8</td>
<td>very disastrous; few buildings survive, landslides</td>
</tr>
</tbody>
</table>

An increase of 1 on the Richter scale roughly corresponds to a multiplication of the energy released by a factor of 10. Thus the energy released by an earthquake with Richter magnitude 6.4 is ten times that of an earthquake with magnitude 5.4. An increase of 2 corresponds to a multiplication of the energy by a factor of 100. This pattern is easily described algebraically; a value of x on the Richter scale corresponds to an energy release of \( k \cdot 10^x \), where the constant \( k \) depends on the units being used.
Example  
An earthquake in Alaska in 1964 had a Richter magnitude of 8.6. The famous San Francisco earthquake of 1906 is estimated to have had a Richter magnitude of 8.3. How many times more intense was the Alaskan earthquake?

Solution  
Divide the larger energy release by the smaller.

\[
\frac{\text{Alaska energy release}}{\text{San Francisco energy release}} = \frac{k \cdot 10^{8.6}}{k \cdot 10^{8.3}} = 10^{0.3} \approx 2.0
\]

The Alaskan earthquake was twice as intense. However, its epicenter was in an unpopulated area, and so damage from it, though substantial, was quite a bit less than the damage from the San Francisco earthquake.

The Richter scale is called an exponential scale or a logarithmic scale because it is calculated using exponents of numbers with the same base. The word logarithm literally means "ratio of numbers."

Another logarithmic scale describes the intensity of sound. The quietest sound that a human can hear has an intensity of about $10^{-12}$ watts per square meter ($\text{w/m}^2$). The human ear can also hear sounds with an intensity as large as $10^2 \text{ w/m}^2$. Because the range from $10^{-12}$ to $10^2$ is so large, it is convenient to use another unit, the decibel (dB), to measure sound intensity. The decibel is $\frac{1}{10}$ of a bel, a unit named after Alexander Graham Bell (1847–1922), the inventor of the telephone.

The chart below gives the decibel and the corresponding $\text{w/m}^2$ values for some common sounds.

![Decibel chart](chart.png)
As the decibel values in the right column increase by 10, the corresponding intensities in the left column multiply by 10. Thus, if the number of decibels is increased by 20, the sound intensity is multiplied by 100. If you increase the sound intensity by 40 dB, you multiply the watts per square meter by 10,000. In general, an increase of $n$ dB multiplies the intensity by $10^{n/10}$. The 120 dB intensity of loud rock music is $10^{60/10} = 10^6 = 1,000,000$ times the 60 dB intensity of normal conversation.

Logarithmic scales are different from linear scales. On a linear scale the units are spaced so that the difference between successive units is the same.

The w/m$^2$ scale is an example of a logarithmic scale. On a logarithmic scale the units are spaced so that the ratio between successive units is the same.

Logarithmic scales are often used to model data with a very wide range of values.

Some other examples of logarithmic scales include the pH scale for measuring the acidity of solutions and the scales used on radio dials.

**Questions**

In 1–3, refer to the chart of Richter scale values.

1. Why is a scale like the Richter scale used?

2. A logarithmic scale is a scale calculated using what numbers?

3. On October 1, 1986, an earthquake measuring 6.1 on the Richter scale struck southern California. Was this a strong quake?

4. An increase of one unit on the Richter scale corresponds to multiplying the energy of a quake by what number?

5. To what factor does an increase of two units on the Richter scale correspond?

6. How many times more intense is an earthquake with a Richter magnitude of 6.3 than one with magnitude 4.7?
In 7–10, refer to the chart of sound intensity levels.

7. What is the intensity of sound which is barely audible to human beings?

8. Give an example of a sound that is 100 times more intense than a noisy kitchen.

9. How many times more intense is normal conversation than a soft whisper?

10. The intensity level of a jet plane at 600 m is 20 dB more than that of a pneumatic drill at 15 m. How many times more intense is the sound of the jet?

11. What is the major difference between a linear scale and a logarithmic scale?

12. Why is a logarithmic scale better than a linear scale for illustrating the data below?
   $$5 \times 10^{-34} \text{ kg}; 1.6726 \times 10^{-27} \text{ kg}; 10^{-21} \text{ kg}; 3.15 \text{ kg}; 1.38 \times 10^5 \text{ kg}$$

In 13–15, refer to the pH scale below. The pH scale is a logarithmic scale that is used to measure how acidic or alkaline a solution is. This is done by measuring the concentration of hydronium ions, $\text{H}_3\text{O}^+$, in the solution. The concentration is expressed as a power of 10 and is then converted to a pH value as shown in the graph at the right. Pure water has a pH of 7. Acidic solutions have pH values less than 7. Alkaline or basic solutions have pH values greater than 7.

13. The gastric juice in your digestive system has a pH of 2.0 and many soft drinks have pH of 3.0.
   a. Which is more acidic, gastric juice or soft drinks?
   b. What is the concentration of $\text{H}_3\text{O}^+$ ions in the more acidic solution?

14. Seawater has pH of 8.5.
   a. Is seawater acidic or basic?
   b. What is the concentration of $\text{H}_3\text{O}^+$ ions in seawater?
   c. Rewrite your answer to part b in scientific notation.

15. An acidic solution is increased in strength from pH 5 to pH 1. How many times more concentrated is the solution?
16. **Multiple choice** A culture of 8000 bacteria triples every 40 minutes. Let \(P\) be the population and \(t\) be the number of minutes after the start. Which equation models the population size? (*Lesson 9-1*)

(a) \(P = 8000 + 40t\)  
(b) \(P = 40t^2 + 8000\)  
(c) \(P = 8000 \cdot 3^{t/40}\)  
(d) \(P = 3 \cdot 8000^{t/40}\)  
(e) \(P = 8000 + 3 \cdot 40t\)

17. Plot points and then graph \(y = -300(2)^t\). Is this an example of exponential decay? (*Lesson 9-2*)

18. Suppose that the inverse of function \(f\) is also a function. If the domain of \(f\) is the set of all real numbers and the range is the set of positive real numbers, find the domain and range of \(f^{-1}\). (*Lesson 7-6*)

In 19 and 20, write each expression as the square of a sum or difference. (*Lesson 6-1, Previous Course*)

19. \(m^2 + 6mp + 9p^2\)  
20. \(49y^2 - 14y + 1\)

In 21–23, solve for \(x\). (*Lessons 8-11, 6-6, 1-7*)

21. \(3x + 6(x - 7) = 93\)  
22. \(3x^2 + 6(x - 7)^2 = 93\)

23. \(3 + 6(x - 7)^{1.25} = 93\)

24. Acid rain is a serious environmental issue in many parts of the world. Find the pH level of acid rain and how acid rain affects the pH level of lakes. How are biologists trying to make lakes less acidic?

25. Not all logarithmic scales are based on 10. Apparent magnitudes of stars are given on a logarithmic scale. A difference of 5 magnitudes means that the star with the lower magnitude is 100 times brighter than the star with the higher magnitude.

a. How much brighter is the sun, magnitude -26.5, than Sirius (the brightest star in the sky), with magnitude -1.5?

b. Find out some other star magnitudes.
The Richter and decibel scales discussed in Lesson 9-3 are based on powers of 10. We say that 10 is the base of these logarithmic scales. Here is a number line with this kind of scale.

To graph a number on this scale, you need to write it as a power of 10. For instance $\sqrt{10}$ is $10^{1/2}$ or $10^{0.5}$, so $\sqrt{10}$ is graphed between $10^0 = 1$ and $10^1 = 10$ on a logarithmic scale.

The sentence $10^{0.5} \approx 3.162$ tells us how to write 3.162 as a power of 10. The exponent 0.5 is called the logarithm or log of 3.162 to the base 10.

**Definition:**

$n$ is the logarithm of $m$ to the base 10, written $n = \log_{10} m$, if and only if

$$10^n = m.$$ 

For instance $10^2 = 100$, so $2 = \log_{10} 100$. The logarithm of 100 to the base 10 is simply the power of 10 which gives 100. Thus, a logarithm is an exponent. Logarithms to the base 10 are called common logarithms and written without the 10. That is, $\log_{10} m = \log m$.

Scientific calculators contain a \[\log\] key to calculate logarithms to the base 10. Use your calculator’s \[\log\] key by pressing 100 \[\log\] to check that $\log_{10} 100 = 2$. Now press 3.162 \[\log\] to show that $\log 3.162 \approx 0.5$.

Some common logarithms can be found without using a calculator.

**Example 1** Evaluate without a calculator.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\log_{10}1,000,000$</td>
<td>b. $\log .1$</td>
</tr>
</tbody>
</table>

**Solution** First write each number as a power of ten. Then apply the definition of logarithm.

a. $1,000,000 = 10^6$. Since 1,000,000 is the 6th power of 10, $\log_{10}1,000,000 = 6$.

b. You need to find $n$ such that $10^n = .1$. Since $.1 = 10^{-1}$, $\log .1 = -1$.

c. $\sqrt[3]{10} = 10^{1/3}$. By definition, $\log \sqrt[3]{10} = \frac{1}{3}$. 

504
Check  Use your calculator.

- Press 1000000 \( \log \). The number 6 is displayed.
- Press .1 \( \log \). The number -1 is displayed.
- Press \( 10 \left[ \sqrt[3]{1} + 3 \right] \) \( \log \). The calculator displays 0.3333333. Notice how much simpler it is to use the definition of logarithm to evaluate \( \log \sqrt[3]{10} \) than it is to use a calculator. All three answers check.

You can estimate the common logarithm of a number by putting it into scientific notation.

**Example 2**  Estimate \( \log 4598 \).

**Solution 1**  Since 4598 is between \( 10^3 \) and \( 10^4 \), \( \log 4598 \) is between 3 and 4.

**Solution 2**  \( 4598 = 4.598 \cdot 10^3 \). This indicates that \( \log 4598 \) will be slightly larger than 3.

**Check**  A calculator shows that \( \log 4598 \approx 3.66 \ldots \).

By using the definition of common logarithms you can solve logarithmic equations. In this chapter you may use either radical form or a decimal approximation for answers unless told otherwise.

**Example 3**  Solve for \( x \): \( \log x = 1.5 \).

**Solution 1**  \( \log x = 1.5 \) if and only if \( 10^{1.5} = x \).

(radical form)  \( 10^{1.5} = 10^{3/2} = (\sqrt{10})^3 = (\sqrt{10} \cdot \sqrt{10} \cdot \sqrt{10}) = 10\sqrt{10} \).
Thus \( x = 10\sqrt{10} \).

**Solution 2**  Begin as in Solution 1, but evaluate \( 10^{1.5} \) with a calculator:

(decimal form)  \( 10 \left[ 1.5 \right] \). Your calculator displays 31.622777\ldots.
Thus, \( x = 31.62 \).

The table below shows some solutions to the equation \( y = \log x \).
The ordered pairs (.1, -1), and \( \left( \frac{3}{10} \sqrt{10}, \frac{1}{3} \right) \) are derived from Example 1. The other pairs were obtained by evaluating the expression \( \log_{10} x \) using the definition or a calculator.

<table>
<thead>
<tr>
<th>( x )</th>
<th>.1</th>
<th>1</th>
<th>( 3/10 ) ((= 2.15))</th>
<th>( \sqrt{10} ) ((= 3.16))</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>31.6*</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \log_{10} x )</td>
<td>-1</td>
<td>0</td>
<td>( 1/3 )</td>
<td>.5</td>
<td>.7*</td>
<td>.9*</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
</tr>
</tbody>
</table>

*These values have been rounded to the nearest tenth.
When these and other solutions to \( y = \log x \) are plotted, the graph shown below is the result. This graph is called a logarithmic curve, and it has the shape of an exponential curve. In fact, the functions \( y = 10^x \) and \( y = \log x \) are inverses. Here is why.

The equation \( y = \log_{10} x \) is equivalent to \( x = 10^y \). When \( x \) and \( y \) are switched in \( x = 10^y \), the resulting equation \( y = 10^x \) is an equation for the inverse of the common log function. Thus, the exponential and logarithmic functions with base 10 are inverses of each other. The graphs below verify that each graph is the reflection image of the other over the line \( y = x \).

![Graph of logarithmic functions](image)

Notice several key features of the graph of \( y = \log x \). Its \( x \)-intercept is 1 (the log of 1 is 0). It increases as you go to the right (the larger a number, the larger its logarithm). Because real powers of 10 are never negative, there are no real logarithms of negative numbers. Thus the domain of \( y = \log x \) is the set of positive real numbers. Since any number can be an exponent of 10, the range of \( y = \log x \) is the set of all real numbers.

Although the graph of \( y = \log x \) gets very close to the \( y \)-axis, it has no \( y \)-intercept. If it did, there would be a value of \( y \) satisfying \( y = \log_{10} 0 \) or \( 10^y = 0 \), which is impossible. By evaluating \( \log x \) for small positive values you can check that the \( y \)-axis is an asymptote of the logarithmic curve. The closer \( x \) gets to zero, the smaller \( y \) gets.

**Questions**

1. What is a logarithm?
2. \( \log_{10} 6 \) is read **?**.
3. If \( x = \log_{10} y \), what other relationship exists between \( x \), 10, and \( y \)?
In 4–9, evaluate without using a calculator.

4. \( \log 1,000,000 \)  
5. \( \log 10^8 \)  
6. \( \log 0.0001 \)  
7. \( \log \sqrt{10} \)  
8. \( \log 1 \)  
9. \( \log -10 \)

In 10–12 evaluate to the nearest thousandth with a calculator.

10. \( \log 2 \)  
11. \( \log 0.00046 \)  
12. \( \log 4,600,000 \)

13. **Multiple choice** The common logarithm of 1,000,529 is approximately
   (a) 3  
   (b) 6  
   (c) 7  
   (d) 10.

In 14 and 15 solve for \( x \).

14. \( \log x = 4 \)  
15. \( \log x = 2.5 \)

16. Consider the graph of \( y = \log_{10} x \).
   a. Name its \( x \)- and \( y \)-intercepts.
   b. Name three points on the graph.
   c. Name three corresponding points on the graph of \( y = 10^x \).
   d. What are the domain and range of the given logarithmic function?
   e. What are the domain and range of the exponential function in part c?
   f. The functions \( f \) and \( g \) with equations \( f(x) = \log_{10} x \) and \( g(x) = \frac{1}{x} \) are inverses of each other.

17. If a number is between 100 and 1000, its common logarithm is between \( \) and \( \).

18. The common logarithm of \( 0.0012 \) is closest to what integer?

19. **Multiple choice** If \( \log t = -0.3098 \), then \( t \) is close to
   (a) -4.9  
   (b) 0.49  
   (c) 4.9  
   (d) 49  
   (e) 490.

20. The common logarithm of a number is -2. What is the number?

21. A formula which relates the intensity of sound in decibels \( B \) to its intensity \( I \) in w/m\(^2\) is
   \[
   B = 10 \log \left( \frac{I}{10^{12}} \right). 
   \]
   Find \( B \) when \( I = 10^8 \).

22. The rate at which potassium-40 decays to argon-40 is known. This enables geologists to estimate the age of rocks by use of the formula
   \[
   t = 1.26 \cdot 10^9 \left( \frac{\log (1 + 8.33 A/P)}{\log 2} \right) 
   \]
   where \( t \) is the age in years and \( \frac{A}{P} \) is the ratio of argon atoms to potassium atoms in the sample. If the \( \frac{A}{P} \) ratio of a rock is 0.421, find the age of the rock.
23. A foundation decided to hold the following contest. It made a test with ten very hard questions. It offered 2¢ to anyone correctly answering one question. It would give ten times that for two questions correctly answered, ten times that for 3 questions, and so on. Construct a logarithmic scale showing the number of questions and the amount of money offered. (Lesson 9-3)

In 24–29, simplify without a calculator. (Lessons 8-1, 8-4, 8-5, 8-6, 8-7, 8-8)

24. \((8^2 \cdot 8^{-4})^5\)

25. \(\sqrt[12]{10}\)

26. \(27^{1/3} \cdot 64^{2/3}\)

27. \(\left(\frac{1}{4}\right)^{1/2}\)

28. \(\left(\frac{3}{4}\right)^5 \cdot \left(\frac{1}{3}\right)^{-5}\)

29. \(\sqrt{x} \cdot \sqrt[3]{x} \cdot \sqrt[5]{x}\)

30. Translucent materials allow light to pass through them but reduce the amount or intensity of the light. Suppose each 1 mm thickness of a translucent plastic reduces the intensity of light by 30% of the original amount. (Previous course, Lesson 9-2)

a. What percent of light does pass through 1 mm?
b. What percent of light passes through 2 mm?
c. Complete a table like the one below.

<table>
<thead>
<tr>
<th>(x =) thickness</th>
<th>0 mm</th>
<th>1 mm</th>
<th>2 mm</th>
<th>3 mm</th>
<th>4 mm</th>
<th>5 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = % ) of light passing through</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d. Write an equation to describe the data.
e. Why is your equation not meaningful for \(x < 0\) in

31. Graph \(f(x) = \frac{6}{x}\) and refer to the graph to answer the questions. (Lessons 2-5, 2-7, 4-6, 9-2)

a. State equations for both asymptotes.
b. State equations for all lines of symmetry.
c. Graph the image of \(f(x)\) under \(R_{45}\). What are equations for the lines of symmetry of the image?

32. An earthquake in December, 1988, in Armenia had a Richter magnitude of 6.9.
a. What kind of damage should you expect from this earthquake?
b. How many times as strong would this quake be than one of magnitude 6.2? (Lesson 9-3)

33. Refer to the U.S. census bar graph at the beginning of this chapter.
a. Determine \(\log P\) for each population \(P\).
b. Graph \(x\) on the horizontal axis, and \(\log P\) on the vertical axis.
c. What do you notice about the points you plotted?

<table>
<thead>
<tr>
<th>(x)</th>
<th>(P)</th>
<th>(\log P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1800</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1810</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1980</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Any positive number except 1 can be the base of a logarithm.

**Definition:**

Let \( b > 0 \) and \( b \neq 1 \). Then \( n \) is the **logarithm of** \( m \) **to** the base \( b \), written \( n = \log_b m \), if and only if

\[ b^n = m. \]

For example, because \( 2^6 = 64 \) we say that ‘‘6 is the logarithm of 64 to the base 2’’ or ‘‘6 is \( \log \) 64 to the base 2’’ and write \( \log_2 64 = 6 \).

Here are some other logs to the base 2.

<table>
<thead>
<tr>
<th>Exponential Form</th>
<th>Logarithmic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^5 = 32 )</td>
<td>( \log_2 32 = 5 )</td>
</tr>
<tr>
<td>( 2^4 = 16 )</td>
<td>( \log_2 16 = 4 )</td>
</tr>
<tr>
<td>( 2^3 = 8 )</td>
<td>( \log_2 8 = 3 )</td>
</tr>
<tr>
<td>( 2^2 = 4 )</td>
<td>( \log_2 4 = 2 )</td>
</tr>
<tr>
<td>( 2^1 = 2 )</td>
<td>( \log_2 2 = 1 )</td>
</tr>
<tr>
<td>( 2^0 = 1 )</td>
<td>( \log_2 1 = 0 )</td>
</tr>
<tr>
<td>( 2^{-1} = \frac{1}{2} )</td>
<td>( \log_2 \left( \frac{1}{2} \right) = -1 )</td>
</tr>
<tr>
<td>( 2^{-2} = \frac{1}{4} )</td>
<td>( \log_2 \left( \frac{1}{4} \right) = -2 )</td>
</tr>
<tr>
<td>( 2^{-3} = \frac{1}{8} )</td>
<td>( \log_2 \left( \frac{1}{8} \right) = -3 )</td>
</tr>
</tbody>
</table>

The domain of the exponential function \( y = 2^x \) is the set of all real numbers. Its range is the set of positive real numbers. That means that any positive real number can be written as a power of 2. Just as in the case of common logs, the inverse of \( y = 2^x \) is the function \( y = \log_2 x \). The graphs of these functions are shown below. You are asked to find the domain and range of \( y = \log_2 x \) in the questions. Notice that logarithms to the base 2 can be negative, but you cannot have the logarithm of a negative number. The methods of evaluating logs and solving equations of logs with bases other than 10 are very similar to the methods you used in the last lesson with common logs.
Example 1 Evaluate
a. \( \log_2 16 \);
b. \( \log_6 \sqrt{6} \).

Solution You cannot simply use the \( \log \) key on your calculator. The \( \log \) key is for common logarithms only.

a. Let \( x = \log_2 16 \). Apply the definition to rewrite the equation in exponential form.

\[
2^x = 16
\]
\[
x = 4.
\]
Therefore, \( \log_2 16 = 4 \).

b. Let \( \log_6 \sqrt{6} = x \). By definition, that means \( 6^x = \sqrt{6} \). Since \( \sqrt{6} = 6^{1/2} \), \( x = \frac{1}{2} \). Thus, \( \log_6 \sqrt{6} = \frac{1}{2} \).

Notice that when you solve a problem about logarithms in base \( b \), just as with base 10 logarithms, it often helps to write the equation in exponential form.

Example 2 Solve for \( x \) if \( \log_8 x = \frac{3}{4} \).

Solution Write an equivalent equation in exponential form.

\[
8^{3/4} = x
\]
Simplify the left side.

\[
243 = x
\]

To solve some logarithmic equations, you need to apply the techniques of the last chapter for solving equations with \( n^{th} \) powers.

Example 3 Find \( x \) if \( \log_x 8 = \frac{3}{4} \).

Solution Rewrite the equation in exponential form. \( x^{3/4} = 8 \)
Take the \( \frac{4}{3} \) power of each side.
Simplify each side of the equation. The left side is equivalent to \( x^{1} \), which is \( x \). The solution is \( x = 16 \).

Check Does \( \log_{16} 8 = \frac{3}{4} \)? It will if \( 16^{3/4} = 8 \), which is the case.

Note also that in the definition of the logarithm of \( m \) to the base \( b \), \( b \) cannot equal 1. We do not consider \( b = 1 \) because the inverse of \( y = 1^x \) is not a function.
Logarithms were invented by John Napier in the early 1600s. Henry Briggs first used common logarithms about 1620. In England even today logs to the base 10 are sometimes called Briggsian logarithms. Leonard Euler was the first person to realize that any real number could be an exponent. He was also the first to relate logarithms to exponents. Today most people study real exponents before logarithms, but that is not the order in which they developed historically.

Questions

1. a. \( \log_b 216 \) is the logarithm of _\_ to the base _\_.
   b. \( \log_b 216 = ? \) because _\_ to the _\_ power equals 216.

2. Suppose \( b > 0 \) and \( b \neq 1 \). When \( b^n = m \), \( n = ? \).

3. Write the equivalent logarithmic form for \( 8^7 = 2,097,152 \).

4. Write the equivalent exponential form for \( \log_2 0.5 = -1 \).

5. Write the equivalent exponential form for \( \log_b a = c \).

6. a. Calculate \( 3^2 \), \( 3^1 \), \( 3^0 \), \( 3^1 \), \( 3^2 \), and \( 3^3 \).
   b. Write six logarithmic equations that are suggested by the calculations.

   In 7-9, simplify.
   7. \( \log_{1000} 100 \)  
   8. \( \log_3 \left( \frac{1}{27} \right) \)  
   9. \( \log_5 \sqrt{5} \)

   In 10–15, find \( x \) if:
   10. \( \log_x 3 = \frac{1}{2} \)  
   11. \( \log_x 32 = 5 \)  
   12. \( \log_{100} x = -1.5 \)  
   13. \( \log_6 x = 3 \)  
   14. \( \log_{17} x = 0 \)  
   15. \( \log_6 \left( \frac{1}{243} \right) = -\frac{5}{6} \).

16. For the function \( y = \log_2 x \), state
   a. the domain;
   b. the range.

17. a. Graph \( y = 3^x \) and \( y = \log_3 x \) on the same set of axes.
   b. True or false \( \) The domain of \( y = 3^x \) is the range of \( y = \log_3 x \).

18. The population (in billions) of the Earth in the year \( Y \), using the year 1975 as a base, can be described by the equation
   \[ P = 2^{(Y-1975)/35} + 2. \]
   a. Write this equation in logarithmic form.
   b. When \( P = 8 \), what is \( Y \)?
   c. What does your answer in part b mean?
19. a. Calculate $\log_5 125$ and $\log_{125} 5$ without a calculator.
    b. Calculate $\log_4 16$ and $\log_{16} 4$ without a calculator.
    c. Generalize the results of parts a and b.

20. When $\log (\frac{x}{y})$ is rounded to the nearest whole number, the result is sometimes called the "number of orders of magnitude difference between $x$ and $y."$ (Each order of magnitude roughly means a substantial difference between the things being measured.) Find the number of orders of magnitude difference between the given numbers.
    a. 1 billion, the population of China, and 1 million, the population of Detroit.
    b. 4,000,000,000,000 km, the distance from Earth to the second nearest star, Alpha Centauri, and 149,000,000 km, the distance from Earth to the sun. (Lesson 9-4)

In 21 and 22, evaluate without a calculator. (Lesson 9-4)

21. $\log 10^5$  
22. $\log 0.00001$

23. Solve $\log x = 9.$ (Lesson 9-4)

24. Suppose one sound has an intensity of 80 decibels and a second sound has an intensity of 40 decibels. How many times more intense is the first sound? (The answer is not 2.) (Lesson 9-3)

In 25–27, simplify each expression. (Lessons 8-1, 8-6, 8-7)

25. $b^{1.6} \cdot b^{-3/4}$  
26. $53^a \cdot 53^b \cdot 53^{(a+b)}$  
27. $\frac{x^{3r}t}{x^t}$

28. The world population passed 5 billion in 1987 and was growing at a rate that would cause it to double in 40 years. With this assumption, what would the world population be in 2050? (Hint: Find the number of 40-year periods.) (Lesson 9-1)

29. Mona invested $800 in a 6.7% account compounded daily for five years. What was her final balance? (Lesson 8-2)

30. a. Solve $\log_3 x = \log_5 x$
    b. Generalize the idea of part a.
As you have seen, some logarithms can be found without a calculator. You know that $b^0 = 1$ for any nonzero $b$. If you rewrite this equation in logarithmic form, it becomes $\log_b 1 = 0$.

**Logarithm Theorem 1:**

For any nonzero base $b$, $\log_b 1 = 0$.

You know that the common log of $10^{7.2}$ is 7.2; that is, $\log_{10} 10^{7.2} = 7.2$. This property can be generalized. Start from the definition of logarithm: $b^n = m$ means $\log_b m = n$. Since $b^n = m$, substitute $b^n$ for $m$ in the left side of the equation $\log_b m = n$. Then you get $\log_b b^n = n$.

**Logarithm Theorem 2:**

For any nonzero base $b$, $\log_b b^n = n$.

In words, if a number can be written as a power of the base, the exponent of the number is its logarithm; so, for instance, $\log_4 4^5 = 5$. This means it is not necessary to calculate $4^5$ in order to find its logarithm in base 4.

The other basic properties of logarithms come from properties of powers. The Product of Powers Property states that to multiply two powers, add their exponents. In particular, for any base $b$ ($b > 0$, $b \neq 1$) and any real numbers $m$ and $n$,

$$b^m \cdot b^n = b^{m+n}.$$  

Now let $x = b^m$, $y = b^n$, and $z = b^{m+n}$. Then $z = xy$. However, from the definition of log,

$$\log_b x = m,$$
$$\log_b y = n,$$
and
$$\log_b z = m + n.$$  

By substitution, $$\log_b z = \log_b x + \log_b y.$$  

Since $z = xy$, we have proved the following theorem.

**Logarithm Theorem 3 (Product Property):**

For any nonzero base $b$ and positive real numbers $x$ and $y$,

$$\log_b (xy) = \log_b x + \log_b y.$$  

In words, the log of a product equals the sum of the logs of the factors.
Example 1  
Find $\log_6 2 + \log_6 108$. 

Solution  
By the Product Property of Logarithms, 
\[ \log_6 2 + \log_6 108 = \log_6 (2 \cdot 108) = \log_6 216 = 3. \]

There is also a Quotient Property of Logarithms, the proof of which is very similar to that of the Product Property. The Quotient Property follows from the related Quotient of Powers Property:
\[ b^m \div b^n = b^{m-n}. \]

The proof is derived in Question 18.

Logarithm Theorem 4 (Quotient Property):  
For any nonzero base $b$ and for any positive real numbers $x$ and $y$,  
\[ \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y. \]

Example 2  
The formula $B = 10 \log \left( \frac{l}{10^{-12}} \right)$ is used to compute the number of decibels $B$ from the intensity $l$ of sound when measured in watts/m². Use the theorem about the log of a quotient to find an equivalent formula.

Solution  
By Theorem 4 above:
\[ \log \left( \frac{l}{10^{-12}} \right) = \log l - \log 10^{-12} = \log 10^{-12} = -12. \]
\[ B = 10 \log \left( \frac{l}{10^{-12}} \right) = 10(\log l - (-12)) = 10(\log l + 12). \]

The last basic property of logarithms comes from the Power of a Power Property for exponents
\[ (b^m)^n = b^{mn}. \]

Logarithm Theorem 5 (Powering Property):  
For any nonzero base $b$ and for any positive real number $x$,  
\[ \log_b (x^n) = n\log_b x. \]
Question 19 asks you to complete the proof of this theorem.

Example 3  Rewrite \( \log x^7 \) in terms of \( \log x \).

Solution  By the Powering Property, \( \log x^7 = 7 \cdot \log x \).

Check  Let \( x = 2 \). Does \( \log 2^7 = 7 \cdot \log 2 \)? \( 2^7 = 128 \), so \( \log 2^7 = \log 128 = 2.107 \). \( \log 2 = 0.30103 \), so \( 7 \cdot \log 2 = 2.107 \). It checks.

Before calculators were invented, these properties of logarithms were applied to perform difficult multiplications, divisions, and powerings. For example, to compute

\[ N = 507 \cdot 386^{1.4}, \]

people would take the common logarithm of both sides:

\[ \log N = \log (507 \cdot 386^{1.4}). \]

Then they would use the properties of logarithms.

\[ = \log 507 + \log (386^{1.4}) = \log 507 + 1.4 \log 386 \]

They would look up these logarithms in a table and then do the arithmetic.

\[ \approx 2.7050 + 1.4(2.5866) \]

\[ \log N \approx 6.3262. \]

Then they would find \( N \) from the tables: \( N \approx 2,119,000. \)

Even though this is a long, complicated procedure, it was the only reasonable way to calculate powers and it simplified some multiplications and divisions. Calculators have eliminated the need to do problems this way. It is interesting (and a little ironic) that calculators perform many of their arithmetic operations by using ideas of logarithms.

Questions

In 1–5, simplify without a calculator.

1. \( \log_7 7^{26.8} \)  
2. \( \log_m (m^n) \)  
3. \( \log_\pi 1 \)

4. \( \log_{12} 3 + \log_{12} 4 \)  
5. \( \log_5 40 - \log_5 8 \)
In 6–8, express as the logarithm of a single number.

6. \( \log_2 25 + \log_2 7 \)  
7. \( \log 85 - \log 17 + \frac{1}{2} \log 25 \)
8. \( \log_b x + \log_b y - \log_b z \)

9. How did people compute expressions like \((9.76)^{3.2} \cdot (16.8)\) before calculators? Why was this an efficient technique?

10. How have calculators been programmed to do some of the more complicated calculations?

In 11–13, use \( \log_2 5 = 2.3219 \) and \( \log_2 6 = 2.5850 \) to evaluate.

11. \( \log_2 30 \)
12. \( \log_2 1.2 \)
13. \( \log_2 25 \)

In 14–17, true or false; if false, correct it to make it true.

14. \( \log (M + N) = \log M \cdot \log N \)
15. \( \frac{\log 4}{\log 3} = \log \left( \frac{4}{3} \right) \)
16. \( \log x^{10} = 10 \log x \)
17. \( \log_b (3x) = 3 \log_b x \)

18. Fill in the blanks in this proof of the Quotient Property of Logarithms.

Let \( x = b^m \), \( y = b^n \), and \( z = b^{m-n} \). Assume \( b > 0, b \neq 1 \).

Since \( x = b^m \), \( y = b^n \), \( \frac{x}{y} = b^{m-n} \)  
(definition of logarithm)
Quotient of Powers Property

\( \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \) Substitution

19. Justify each step in the proof of the Powering Property of Logarithms given here. Let \( \log_b x = m \).

Then \( x = b^m \)  
(definition of logarithm)
\( x^n = (b^m)^n \)  
(definition of logarithm)
\( x^n = b^{mn} \)  
(definition of logarithm)
\( x^n = b^{nm} \)  
(definition of logarithm)
\( \log_b x^n = nm \)  
(definition of logarithm)
Therefore, \( \log_b x^n = n \log_b x \).  
(definition of logarithm)

20. Solve for \( x \): \( \log x = 4 \log 2 + \log 3 \).

In 21 and 22 use the following information. The \( \text{pH} \) of a chemical solution is defined to be \( \text{pH} = -\log x \), where \( x \) is the \( \text{H}_3\text{O}^+ \) ion concentration. Determine the \( \text{pH} \) of each of the following substances with the given \( \text{H}_3\text{O}^+ \) ion concentration.

21. black coffee: \( 1 \times 10^{-5} \)  
22. hydrochloric acid (HCl): \( 6.3 \times 10^{-3} \)
23. The Henderson-Hasselbach formula \( \text{pH} = 6.1 + \log \left( \frac{B}{C} \right) \) gives the pH of a patient’s blood as a function of the bicarbonate concentration \( B \) and the carbonic acid concentration \( C \). The normal pH is about 7.4.

   a. Rewrite this equation using the Quotient Property of Logarithms.
   b. A patient has a bicarbonate concentration of 24 and a pH reading of 7.2. Find the concentration of carbonic acid. (Hint: first solve the equation in part a for \( \log C \).)

24. Simplify these logs to the base 64 without a calculator. (*Lesson 9-5*)
   a. \( \log_{64} 64 \)
   b. \( \log_{64} 8 \)
   c. \( \log_{64} 2 \)
   d. \( \log_{64} 1 \)
   e. \( \log_{64} \frac{1}{4} \)
   f. \( \log_{64} \frac{1}{8} \)

25. Solve \( \log_x 81 = 4 \). (*Lesson 9-5*)

26. Solve \( \log y = -1 \). (*Lesson 9-4*)

27. Refer to the pH scale described in Lesson 9-3. Lemons have a pH of 2.3 and milk of magnesia has a pH of 10.5. Which of these has a higher concentration of \( \text{H}_3\text{O}^+ \) ions? (*Lesson 9-3*)

28. Newton’s Law of Cooling states that the difference in the temperatures of a warm body and its cooler surroundings decreases exponentially. Suppose a bowl of soup is 100°C. In a room which is 20°C, its cooling is described by the equation \[ y = 80(0.875)^t \]
   where \( y \) is the temperature difference in °C between the soup and the room at time \( t \) in minutes.
   a. What will be the temperature of the soup after 5 minutes?
   b. According to this equation will the soup ever be 20°C? (*Lesson 9-1*)

29. The charge to park a car at a city lot is 90¢ for the first hour and 75¢ for each additional hour or fraction thereof. Let \( t \) = the number of hours parked. (*Lesson 7-4*)
   *Multiple choice.* Which equation models this situation?
   (i) \( f(t) = 90 + 75t \)
   (ii) \( f(t) = 90 + 75[t] \)
   (iii) \( f(t) = 75 + 90[t] \)
   (iv) \( f(t) = [90 + 75(t \cdot 60)] \)

30. When asked why he memorized that \( \log 2 \) is about 0.301 and \( \log 3 \) is about 0.477, Leonhard answered, “Of course I know \( \log 1 \) and \( \log 10 \). Using \( \log 2 \) and \( \log 3 \), I can get the logs of all but one of the other integers from 1 to 10.” Which logs between 4 and 9 can be found from the logs of 2 or 3, and what are they?
Recall the General Compound Interest Formula

\[ A = P \left(1 + \frac{r}{n}\right)^n, \]

where an amount \( P \) is invested in an account paying an annual interest rate \( r \) and the interest is compounded \( n \) times per year for \( t \) years. The number of compoundings can make quite a difference. Suppose you begin with \( P = \$1 \) and a bank pays 100% interest (don’t we wish!). The following table shows the value of \( A \) at the end of one year \((t = 1)\) for successively shorter compounding periods. Check the values of \( A \) with your calculator.

<table>
<thead>
<tr>
<th>( n = (\text{compoundings per year}) )</th>
<th>( P \left(1 + \frac{r}{n}\right)^n )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>annually</td>
<td>1</td>
<td>( \left(1 + \frac{1}{1}\right)^1 )</td>
</tr>
<tr>
<td>semi-annually</td>
<td>2</td>
<td>( \left(1 + \frac{1}{2}\right)^2 )</td>
</tr>
<tr>
<td>quarterly</td>
<td>4</td>
<td>( \left(1 + \frac{1}{4}\right)^4 )</td>
</tr>
<tr>
<td>monthly</td>
<td>12</td>
<td>( \left(1 + \frac{1}{12}\right)^{12} )</td>
</tr>
<tr>
<td>daily</td>
<td>365</td>
<td>( \left(1 + \frac{1}{365}\right)^{365} )</td>
</tr>
<tr>
<td>hourly</td>
<td>8760</td>
<td>( \left(1 + \frac{1}{8760}\right)^{8760} )</td>
</tr>
<tr>
<td>by the second</td>
<td>31,536,000</td>
<td>( \left(1 + \frac{1}{31,536,000}\right)^{31,536,000} )</td>
</tr>
</tbody>
</table>

The sequence of values for the total amount gets closer and closer to the number 2.71828.... Euler proved that this is indeed the case. We call this situation **continuous** or **instantaneous compounding**. In his honor this limiting number is called \( e \). Like the number \( \pi \), \( e \) looks like a variable, but it is a particular irrational number which can be expressed as an infinite, non-repeating decimal. Here are the first fifty places.

\[ 2.7182818284590452353602874713526624977572709369995... \]
In a more realistic situation, suppose a bank pays 5% interest on $1 for one year. Here are some values of $A$ for different compounding periods.

<table>
<thead>
<tr>
<th>compounding method</th>
<th>$P \left( 1 + \frac{r}{n} \right)^n$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>annually</td>
<td>$1 \left( 1 + \frac{.05}{1} \right)^1$</td>
<td>$1.05$</td>
</tr>
<tr>
<td>semi-annually</td>
<td>$1 \left( 1 + \frac{.05}{2} \right)^2$</td>
<td>$1.050625$</td>
</tr>
<tr>
<td>quarterly</td>
<td>$4 \left( 1 + \frac{.05}{4} \right)^4$</td>
<td>$1.050945$</td>
</tr>
<tr>
<td>daily</td>
<td>$1 \left( 1 + \frac{.05}{365} \right)^{365}$</td>
<td>$1.051267$</td>
</tr>
<tr>
<td>hourly</td>
<td>$1 \left( 1 + \frac{.05}{8760} \right)^{8760}$</td>
<td>$1.051271$</td>
</tr>
</tbody>
</table>

Notice that the total amount seems to be getting closer to $1.051271\ldots$ This number is very close to the value of $e^{0.05} = 1.051271\ldots$ In fact, $1$ compounded continuously at 5% annual interest for one year will be worth exactly $e^{0.05}$. To evaluate $e^{0.05}$ on your calculator press .05 $e^x$. (On some calculators you may need to press [INV], [2nd], or [f] before the $\ln x$ key.) If your calculator does not have an $e^x$ key you can approximate $e^{0.05}$ by calculating $(2.71828)^{0.05}$ using the powering key $y^x$.

Thus for situations where interest is compounded continuously, the general compound interest formula can be greatly simplified.

**Continuously Compounded Interest Formula:**

If an amount $P$ is invested in an account paying an annual interest rate $r$ compounded continuously, the amount $A$ in the account after $t$ years will be

$$A = Pe^{rt}. $$
Example 1
If $850 is invested at an annual interest rate of 6% compounded continuously, what is the amount in the account after 10 years?

Solution 1
Use the formula $A = Pe^{rt}$, where $P = 850$, $r = 0.06$, and $t = 10$.

$$A = 850e^{0.06(10)}$$
$$A = 850e^{0.6}$$

One calculator key sequence is: 850 [×] .6 [e^x] =

$$A \approx 1548.80$$

Solution 2
If your calculator does not have an $e^x$ key, use the approximation 2.71828 for $e^x$ and substitute.

$$A = 850e^{0.6} = 850(2.71828)^{0.6} \approx 1548.80$$

The balance will be about $1548.80 after 10 years.

The exponential function $y = e^x$ has special properties that make it particularly suitable for applications. Some of these properties are studied in calculus. For now, you only need to know that many formulas for growth and decay are written using $e$ as the base. This is why most scientific calculators have a key to find values of $e^x$.

Example 2
The amount $L$ of a certain radioactive substance remaining after $t$ years decreases according to the formula $L = Be^{-0.0001t}$. When $t = 0$, $L = B$, so $B$ is the original amount of the substance. If 2000 $\mu$ (micrograms) are left after 6000 years, how many micrograms were present initially?

Solution
When $t = 6000$, $L = 2000$. Substitute these values and solve for $B$.

$$L = Be^{-0.0001t}$$
$$2000 = Be^{-0.0001(6000)}$$
$$2000 = B(0.54881)$$
$$3644 \approx B$$

About 3600 $\mu$ were present initially.
Formulas such as those in Examples 1 and 2,

\[ A = 850e^{0.06t} \]

\[ L = Be^{-0.00011t}, \]

are instances of a general model for situations involving continuous change. The continuous-change model is often described using function notation. Let \( N_0 \) (read "\( N \) naught") be the initial amount, and let \( r \) be the rate of continuous growth or decay per unit of time \( t \). Then \( N(t) \), the amount at time \( t \), is given by the equation

\[ N(t) = N_0e^{rt}. \]

This equation is an exponential equation of the form

\[ y = ab^x \]

where \( a = N_0 \), \( x = t \), and the growth factor \( b = e^r \). If \( r \) is positive, then \( e^r > 1 \), so there is growth. If \( r \) is negative, then \( 0 < e^r < 1 \) and there is decay.

Technicians working with radioactive substance

Questions

In 1 and 2, \( P = \$1 \), \( r = 100\% \), and \( t = 1 \) year. Use the table in the lesson.

1. As \( n \) increases, the value of \( A \) becomes closer and closer to what number?

2. Write the key sequence for your calculator to verify that \( A \approx \$2.71457 \) when interest is compounded daily.

3. In whose honor did the number \( e \) get its name?

4. Approximate \( e \) to the nearest hundred-thousandth.
5. If $1 is invested at 10% interest compounded continuously, find its value at the end of one year.

6. The amount of $700 is invested at an annual interest rate of 5% compounded continuously. How much is in the account after 10 years?

7. Consider the function $N(t) = N_0e^{rt}$.
   a. What does $N_0$ represent?
   b. What does $r$ represent?
   c. What is true about $r$ when this function models exponential decay?

8. Refer to Example 2. If at the end of 2000 years there are 1000 micrograms of the substance remaining, how many micrograms were present initially?

9. Suppose $3000 is invested at 7.5% interest compounded continuously for 8 years.
   a. How much is the investment worth at the end of that period?
   b. How much would the investment be worth if the 7.5% interest were compounded annually?

10. In 1985 the population of Nairobi, Kenya was 1 million. At that time the formula $N(t) = N_0e^{0.08t}$ was being used to project the population $t$ years later.
    a. What annual rate of population increase was assumed?
    b. Find $N(15)$, the projected population for the year 2000.

11. Graph $y = e^x$ for values of $x$ between -3 and 3, inclusive.

12. Complete with $>$, $<$, or $=$: $\pi e \text{ __ } 2e^\pi$.

13. A machine used in an industry depreciates so that its value after $t$ years is given by $N(t) = N_0e^{-0.25t}$.
    a. What is the annual rate of depreciation of the machine?
    b. After 3 years the machine is worth $12,000. What was its original value?
14. Rumor spreads like an epidemic at a rate directly proportional to the number of people who have heard the rumor (and thus perpetuate it). Rumor spreading can be modeled by the equation

\[ H = \frac{C}{1 + (C - S)e^{-0.4t}}, \]

where \( C \) is the total number of people in the community, \( S \) is the number of people who initially spread the rumor, and \( H \) is the number of people who have heard the rumor after \( t \) minutes. In a school of 1800 students, one student going to lunch on Friday overhears the principal saying the following Monday there will be a surprise school holiday. About how many students will have heard the rumor after 45 minutes?

15. Solve for \( z \): \( \log z = \frac{2}{3} \log 8 + \log 3 \). (Lesson 9-6)

16. Write \( \log (pq^2) \) in terms of \( \log p \) and \( \log q \). (Lesson 9-6)

In 17 and 18, suppose \( f(x) = \log x \). Then find: (Lesson 9-4)

17. \( f(.1) \)

18. \( f^{-1}(3) \)

19. Find \( \log_4 \left( \frac{1}{2} \right) \) without a calculator. (Lesson 9-5)

In 20 and 21, solve for \( x \). (Lesson 1-7)

20. \( 25 = \frac{3}{x} \)

21. \( \frac{x}{3} = 25 \)

22. a. Graph: \( g: x \to 3x + 2 \).

b. Give a formula for \( g^{-1}(x) \) and graph on the same axes. (Lesson 7-7)

23. For what value of \( m \) does the equation \( mx^2 + 12x + 9 = 0 \) have exactly one solution for \( x \)? (Lesson 6-7)

24. Another way to get an approximate value of \( e \) is to evaluate the infinite sum

\[ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \]

(recall that \( n! \) is the product of all integers from 1 to \( n \) inclusive). Use your calculator to calculate

\[
\begin{align*}
1 + & \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \\
1 + & \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \\
1 + & \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}
\end{align*}
\]

and so on

until you have approximated \( e = 2.71828\ldots \) to the nearest thousandth. What is the last term you need to add to do this?
Logarithms to the base \( e \) are called **natural logarithms**. Sometimes they are called **Napierian logarithms** after John Napier (1550–1617), the first person to use logarithms of any kind. Just as \( \log x \) (without any base named) is a shorthand for \( \log_{10} x \), so there is an abbreviation for \( \log_e x \), namely \( \ln x \).

**Definition:** \( \ln m = n \) if and only if \( m = e^n \).

The symbol \( \ln x \) is read “the natural log of \( x \).” Natural logarithms of powers of \( e \) can be determined mentally.

\[
\ln 1 = 0 \text{ because } 1 = e^0.
\]
\[
\ln e = 1 \text{ because } e = e^1.
\]

In general, \( \ln(e^x) = x \) is a special case of the theorem \( \log_b(b^x) = x \).

To determine natural logarithms of numbers not in \( e^x \) form, you need a calculator, computer, or table of values. On a scientific calculator use the key sequence \( x \ln x \). For instance, to find \( \ln 10 \) press \( 10 \ln x \); you should get approximately 2.3026. This means that \( e^{2.3026} \approx 10 \). In some computer languages the natural logarithm function is denoted \( \text{LOG} \). This can be confusing because \( \log \) usually refers to base 10.

Natural logarithms are frequently used in formulas.

**Example 1** Ignoring the force of gravity, the maximum velocity \( v \) of a rocket is given by the formula \( v = c \ln R \), where \( c \) is the velocity of the exhaust and \( R \) is the ratio of the mass of the rocket with fuel to its mass without fuel. To achieve a stable orbit a spacecraft must attain a velocity of about 7.8 km/s.

**a.** With a small payload a solid-propellant rocket could have a mass ratio of about 19. A typical exhaust velocity for such a rocket might be about 2.4 kilometers per second. Could a spacecraft propelled by this rocket achieve a stable orbit?

**b.** Find \( R \) for a V-2 rocket if \( c = 6440 \) ft/sec and \( v = 6630 \) ft/sec.
Solution  a. For the solid propellant rocket, $R \approx 19$ and $c \approx 2.4 \text{ km/s}$. Find $v$ using the formula above.

$$
\begin{align*}
&= 2.4 \ln 19 \\
v &= 2.4 (2.944) \\
&\approx 7.1
\end{align*}
$$

Notice that the maximum velocity of this rocket is less than the velocity needed for orbit. This rocket could not propel a spacecraft into orbit.

b. Substitute the given values for $c$ and $v$, and solve for $R$.

$$
\begin{align*}
v &= c \ln R \\
6630 &= 6440 \ln R
\end{align*}
$$

To solve for $R$, you must first solve for $\ln R$.

$$
1.0295 = \ln R
$$

Now, use the definition of natural logarithm to solve for $R$.

$$
R = e^{1.0295}
$$

$$
\approx 2.7997
$$

The mass of the rocket with fuel is about 2.8 times the mass of the rocket without fuel. (Thus the mass of the fuel is 1.8 times the rocket's mass.)

In previous lessons, you have seen that $y = \log x$ is the inverse function of $y = 10^x$, and that $y = \log_2 x$ is the inverse of $y = 2^x$. Similarly, $y = \ln x$ is the inverse of $y = e^x$, as can be seen in the graph below. Each is the reflection image of the other over the line with equation $y = x$.

<table>
<thead>
<tr>
<th>$e^x = y$</th>
<th>$\ln x = y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>-1</td>
<td>0.37</td>
</tr>
<tr>
<td>0</td>
<td>1.00</td>
</tr>
<tr>
<td>1</td>
<td>2.72</td>
</tr>
<tr>
<td>1.6</td>
<td>4.95</td>
</tr>
<tr>
<td>2</td>
<td>7.39</td>
</tr>
</tbody>
</table>

Because the functions above are inverses, formulas involving $e^x$ can lead to finding natural logarithms.

LESSON 9-8  Natural Logarithms  525
Example 2  Jamie has $600 to invest in an account which compounds interest continuously at a rate of 8%. How long should the money be in the account in order for it to double in value?

Solution  Use the formula for continuously compounded interest.

\[ A = Pe^{rt} \]

Substitute \( P = 600, r = .08, \) and \( A = 1200. \)

\[ 1200 = 600e^{0.08t} \]

\[ 2 = e^{0.08t} \]

To solve for \( t, \) use the definition of natural log.

\[ \ln 2 = 0.08t \]

Divide by 0.08.

\[ t = \frac{\ln 2}{0.08} \approx 8.7 \]

Jamie should leave the money in the account for about 8.7 years.

All the properties of logarithms derived in Lesson 9-6 apply to natural logarithms. For all \( x > 0 \) and \( y > 0, \)

\[ \ln (xy) = \ln x + \ln y, \]

\[ \ln \left(\frac{x}{y}\right) = \ln x - \ln y, \]

and \( \ln (x^n) = n \ln x. \)

For instance, \( \ln 64 = \ln 2^6 \)

\[ = 6 \ln 2. \]

Because \( \ln 2 \approx 0.6931, \)

\[ \ln 64 \approx 6(0.6931) \]

\[ \approx 4.159. \]

Questions

1. The base of natural logarithms is ?

2. Multiple choice \( y = \ln x \) is the same as:
   (a) \( x = \log_e y \)
   (b) \( x = \log_y e \)
   (c) \( y = \log_e x \)
   (d) \( y = \log_x e \)

In 3 and 4, write the equivalent exponential form.

3. \( \ln 1 = 0 \)

4. \( \ln 300 \approx 5.70 \)

In 5 and 6, write in logarithmic form.

5. \( e^2 \approx 7.39 \)

6. \( e^{0.06} \approx 1.06 \)
7. The graph of what function is the reflection image of the graph of \( y = \ln x \) over the line \( x = y \)?

8. Name a point on the graph of \( y = \ln x \) whose coordinates are not given on the previous page.

In 9 and 10, approximate each value to the nearest thousandth.

9. \( \ln 400 \)  
10. \( \ln 1.6161 \)

In 11 and 12, refer to Example 1.

11. The space shuttle has an \( R \) value of about 3.5. Its main engines can produce an exhaust velocity of about 4.6 km/s. Can the space shuttle achieve a stable orbit with its main engines?

12. The Viking rocket has an exhaust velocity of about 190 km/s and travels without fuel at a rate of 310 km/s. Find its mass ratio \( R \).

13. In Example 2, how long should Jamie leave the money in the account in order for it to triple in value?

14. a. What happens when you try to find \( \ln (-2) \) on your calculator?  
b. Justify your answer to part a.

15. At what point does the line with equation \( x = \frac{1}{2} \) intersect the graph of \( y = \ln x \)?

16. Refer to Example 1. If the maximum velocity of a rocket is 7200 ft/sec and the mass ratio is 2.5, what is the maximum velocity of the exhaust?

In 17 and 18, suppose \( \ln x = 8 \) and \( \ln y = 4 \). Evaluate:

17. \( \ln (3xy) \)  
18. \( \ln \left( \frac{4x}{y} \right) \)

19. The percent risk \( R \) of an auto accident is exponentially related to the percent \( b \) of the alcohol blood level of the driver and is given by this formula

\[
R = e^{2.4b}
\]

a. What is the relative risk of an auto accident if the blood alcohol level is 0.10%?  
b. At what percent alcohol blood level of the driver is the car considered certain to crash?

20. In 1982, it was projected that \( t \) years later the population of the Philippines would be given by \( P(t) = 50e^{0.02t} \) million. According to this projection, what will the population be in the year 2012? (Lesson 9-7)

21. Solve for \( x \): \( \log_7 7 = 2 \). (Lesson 9-5)

22. Solve for \( A \): \( \log A = \log x + 2 \log y \). (Lesson 9-6)
23. True or false \(\log(1.7 \times 10^3) = (\log 1.7)(\log 10^3)\). (Lesson 9-6)

24. A lab assistant accidentally poisoned a bacteria culture. The bacteria died off approximately exponentially as shown in the graph below. (Lesson 9-2)
   a. About how many bacteria were there at the time of the poisoning?
   b. About how many were there 6 minutes later?
   c. Approximately what is the half-life of the poisoned culture?

In 25 and 26, solve for \(w\).

25. \(-3w < 3(1 - w)\) (Lesson 1-9)

26. \(-3w^2 = 3(w - 1)\) (Lesson 6-6)

27. Solids \(A'\) and \(A\) are similar. The ratio of similitude is 5, with \(A'\) larger. If the volume of \(A\) is 400 cubic millimeters, what is the volume of \(A'\)? (Previous Course)

28. Natural logarithms can be calculated using the series
   \[
   \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots
   \]
   Substitute 0.5 for \(x\) to estimate \(\ln 1.5\) to the nearest hundredth.

29. When a single rocket engine cannot produce enough velocity to launch a spacecraft, NASA often uses “staging.” Find out what staging means, and how many stages recent spacecraft have used.
Solving $b^x = a$

Bikini Atoll nuclear test, 1946; see question 15, page 532.

In the previous lessons, you have learned about exponential and logarithmic functions: how to work with them, what they model, and how to use them in some applications. You now have the tools to solve equations of the form $b^x = a$.

The procedure is to:
1. take the logarithm of both sides, then
2. use the Powering Property of logarithms.

**Example 1** Solve $5^x = 20$ for $x$, by
a. taking common logarithms of each side.
b. taking natural logarithms of each side.

**Solution**

**a.**

\[
\begin{align*}
5^x &= 20 \\
\log 5^x &= \log 20 \\
x \log 5 &= \log 20 \\
x &= \frac{\log 20}{\log 5} \\
x &= \frac{1.3010}{.6990} \\
x &\approx 1.86
\end{align*}
\]

**b.**

\[
\begin{align*}
5^x &= 20 \\
\ln 5^x &= \ln 20 \\
x \ln 5 &= \ln 20 \\
x &= \frac{\ln 20}{\ln 5} \\
x &\approx 2.9957
\end{align*}
\]

**Check** Note that $5^1 = 5$ and $5^2 = 25$, so it makes sense that $5^{1.86} = 20$.

In these solutions, (a) and (b), $x = \frac{\log 20}{\log 5} = \frac{\ln 20}{\ln 5}$. In fact, any base $t$ for the logs of 20 and 5 could be used. This result can be presented as a theorem.
Theorem:

When \( a > 0 \) and \( b > 0 \), and \( b \neq 1 \), the unique real number \( x \) satisfying \( b^x = a \) is \( \frac{\log_t a}{\log_t b} \) for any base \( t \), where \( t > 0, t \neq 1 \).

By the definition of logarithm \( 5^x = 20 \) is equivalent to \( x = \log_5 20 \). So, in general, there are two distinct ways of solving \( b^x = a \). Either use the definition of log to get \( x = \log_b a \) or divide \( \log_r a \) by \( \log_r b \) in any base \( t \).

Observe that in Example 1 you get the same answer using common logs as you do using natural logs. Because the same results are always obtained by using either common or natural logarithms, you may choose either one for a given situation. In some cases, one is more efficient or easier to use than the other. When the base of the exponential equation is 10, it is usually easier to use common logarithms; when the base is \( e \), it is usually easier to use natural logs.

Example 2  At what rate of interest, compounded continuously, would you have to invest your money so that it would triple in 10 years?

Solution  Use \( A = Pe^rt \).

Since \( A \), the total amount desired, is triple the starting amount \( P \),
\( A = 3P \). Here \( t = 10 \). Substituting,
\[
3P = Pe^{10r}.
\]
Dividing by \( P \),
\[
3 = e^{10r}.
\]
Take the logarithm of each side to the base \( e \). (This gives the same result as applying the definition of the natural logarithm.)
\[
\ln 3 = 10r
\]
\[
r = \frac{\ln 3}{10} = 0.10986
\]

It takes an interest rate of about 11%, compounded continuously, to triple your money in 10 years.

Decay or depreciation problems are often modeled by exponential equations with negative powers. Such equations can also be solved by the above techniques.
Example 3

The intensity $I_t$ of light through ordinary glass of thickness $t$ (in centimeters) is modeled by the exponential equation

$$I_t = I_0 10^{-0.0434t}$$

where $I_0$ is the intensity before entering the glass. How thick must the glass be to block out 10% of the light?

Solution

Blocking out 10% of the light means that $I_t = 0.90 I_0$. Substituting into

$$I_t = I_0 10^{-0.0434t}$$

we get

$$0.90 I_0 = I_0 10^{-0.0434t}$$

Dividing by $I_0$,

$$0.90 = 10^{-0.0434t}$$

Taking the log of both sides,

$$\log 0.90 = -0.0434t$$

Dividing by -0.0434,

$$t = \frac{\log 0.90}{-0.0434} = 1.05432$$

To block out 10% of the light, ordinary glass should be a bit more than 1 cm thick.

Questions

1. Refer to Example 1. True or false

   $\frac{\log 20}{\log 5} = \frac{\ln 20}{\ln 5}$

2. Solve for $7^x = 15$ by
   a. taking common logarithms.
   b. taking natural logarithms.
3. **True or false** If you can take the natural logarithm of both sides of an equation, then you do not change the solutions to that equation.

In 4–6, solve to the nearest hundredth.

4. \(2^x = 3\)  
   5. \(3^y = 12\)  
   6. \(25.6^z = 2.89\)

In 7 and 8, refer to Example 2.

7. Why is this problem more efficiently solved with natural logarithms than with common logarithms?

8. What interest rate would it take to double your money in 7 years?

9. In Example 3, find the thickness of glass needed to block out 40% of the light.

10. Suppose you invested $200 in a savings account paying 6.25% interest compounded continuously. How long would it take for your account to be worth $300 if you assume that no other deposits or withdrawals are made?

In 11–13, solve.

11. \(5^{2y} = 1986\)  
12. \(8^{-4x} = 256\)  
13. \(10^{-r+1} = 2\)

14. a. Graph \(y = 5^x\) for \(-1 \leq x \leq 2\).  
   b. How does Example 1 relate to this graph?

15. The amount \(A\) of radioactivity from a nuclear explosion is estimated to decrease exponentially by \(A = A_0e^{-2t}\), where \(t\) is measured in days. How long will it take for the radioactivity to reach \(\frac{1}{1000}\) of its original intensity?

16. A colony of bacteria grows according to \(N_t = N_0e^{2t}\), where \(N_0\) is the initial number of bacteria, \(t\) is the time in hours, and \(N_t\) is the number of bacteria after \(t\) hours. How long does it take the colony to quadruple in size?

17. Suppose \(\ln a = 5\) and \(\ln b = 10\). Find \(\ln (ab)^2\). *(Lessons 9-8, 9-6)*

18. For a small 3-stage rocket, the formula  
   
   \[V = c_1 \ln R_1 + c_2 \ln R_2 + c_3 \ln R_3\]

   is used to find the velocity of the rocket at the final burnout. If \(R_1 = 1.46, R_2 = 1.28, R_3 = 1.41, c_1 = 7400\) ft/sec, and \(c_2 = c_3 = 8100\) ft/sec, find \(V\). *(Lesson 9-8)*

19. Which equations model decay situations? *(Lessons 9-7, 9-2)*  
   a. \(y = ae^{-r}, r > 0\)  
   b. \(y = ae^{-\tau}\)  
   c. \(y = 700(0.69)^x\)  
   d. \(y = 1.66(1.08)^x\)  
   e. \(y = 2\left(\frac{1}{e}\right)^x\)
20. Where is the error in the following "proof" of the inequality \(5 < 2\)? (Lessons 9-6, 9-4)

Proof:

\[
\frac{1}{32} < \frac{1}{4}
\]

\[
\log \left( \frac{1}{32} \right) < \log \left( \frac{1}{4} \right)
\]

\[
\log \left( \left( \frac{1}{2} \right)^5 \right) < \log \left( \left( \frac{1}{2} \right)^2 \right)
\]

\[
5 \log \left( \frac{1}{2} \right) < 2 \log \left( \frac{1}{2} \right)
\]

\[
5 < 2
\]

21. In 1950 an earthquake with Richter value 8.7 hit Assam, India. How many times more intense was this earthquake than the one of intensity 8.3 that hit San Francisco in 1906? (Lesson 9-3)

22. The graph at the right shows a feasible region for a certain situation. Which vertex would minimize cost if the expression

\[
15x + 14y
\]

denotes the cost? (Lesson 5-8)

In 23 and 24, solve.

23. \(3^{5/3} = 96\) (Lesson 8-6)  

24. \(\log 64 = x \log 2\) (Lesson 9-6)

25. Let \(N = \begin{bmatrix} 4 & 7 \\ 1 & 3 \end{bmatrix}\) and \(R = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 7 & -6 \end{bmatrix}\).

Find \(N \cdot R\). (Lesson 4-2)

26. Find all integers \(x\) such that \(\frac{2x - 9}{x - 5} = 5\).

(Hint: You will need to solve a quadratic equation.) (Lessons 1-5, 6-6)

Exploration

27. Suppose \(a^x = b\) and \(b^y = a\). How are \(x\) and \(y\) related? (Hint: If you cannot figure this out in general, start by letting \(a\) and \(b\) have certain values and solving for \(x\) and \(y\).)
When \( b > 0 \) and \( b \neq 1 \), the function \( f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is the exponential function with base \( b \). We write \( b^x = a \) if and only if \( x = \log_b a \), so we can write \( f \) as \( f: \log_b a \rightarrow a \). Its inverse \( f^{-1}: b^x \rightarrow x \) is the logarithm function with base \( b \). Because exponential and logarithmic functions are inverses, their graphs are reflection images of each other.

**Exponential Curves**

**Growth:** \( y = ab^x, \ b > 1 \)

- \( e.g., \ y = 2^x \)

**Decay:** \( y = ab^x, \ 0 < b < 1 \)

- \( e.g., \ y = \left(\frac{1}{2}\right)^x \)

**Logarithmic Curve**

\( y = \log_b x, \ b > 1 \)

- \( e.g., \ y = \log_2 x \)

Exponential functions find their major uses in situations of growth (e.g., population growth or compound interest) or decay (e.g., depreciation or half-life). Logarithmic functions are used to scale data having a wide range (e.g., earthquake or sound intensities) and to solve equations of the form \( b^x = a \), where \( b \) and \( a \) are positive. The solution to \( b^x = a \) is \( x = \frac{\log a}{\log b} \). Both kinds of functions appear in many formulas. The base of a log function can be 10, in which case its values are called common logarithms, or the base can be \( e \approx 2.71828 \), in which case the values are called natural logarithms.

All the basic properties of logarithms correspond to properties of powers. Let \( b \) be any base \( b > 0 \), \( b \neq 1 \). Let \( \log_b x = m \) and \( \log_b y = n \). Then \( b^m = x \) and \( b^n = y \).

<table>
<thead>
<tr>
<th>Zero exponent:</th>
<th>Power property</th>
<th>Logarithm property</th>
</tr>
</thead>
<tbody>
<tr>
<td>To multiply powers, add exponents.</td>
<td>( b^0 = 1 )</td>
<td>( \log_b 1 = 0 )</td>
</tr>
<tr>
<td>To divide powers, subtract exponents.</td>
<td>( b^m \cdot b^n = b^{m+n} )</td>
<td>( \log_b (xy) = \log_b x + \log_b y )</td>
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<tr>
<td>To take the power of a power, multiply the exponents.</td>
<td>( \frac{b^m}{b^n} = b^{m-n} )</td>
<td>( \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y )</td>
</tr>
<tr>
<td></td>
<td>( (b^m)^a = b^{ma} )</td>
<td>( \log_b (x^a) = a \log_b x )</td>
</tr>
</tbody>
</table>
Vocabulary

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description and a specific example.

**Lesson 9-1**
- exponential function
- exponential curve
- exponential growth, growth factor

**Lesson 9-2**
- exponential decay, depreciation
- half-life
- asymptote

**Lesson 9-3**
- logarithmic scale
- Richter scale
- decibel

**Lesson 9-4**
*common logarithm, logarithm of m to the base 10
- logarithmic curve

**Lesson 9-5**
*logarithm of m to the base b

**Lesson 9-6**
- Product Property of Logarithms
- Quotient Property of Logarithms
- Powering Property of Logarithms

**Lesson 9-7**
*The number e
- continuous compounding
- Continuously Compounded Interest Formula

**Lesson 9-8**
*natural logarithm, ln x
CHAPTER 9

Progress Self-Test

Take this test as you would take a test in class. Use graph paper and a calculator. Then check your work with the solutions in the Selected Answer section in the back of the book.

In 1–4, find each logarithm exactly.
1. \( \log (1,000,000) \)
2. \( \log_4 \left( \frac{1}{16} \right) \)
3. \( \ln e^{-6} \)
4. \( \log_2 1 \)

In 5 and 6, find each logarithm to the nearest hundredth.
5. \( \ln (42.7) \)
6. \( \log 25 \)

In 7–10, solve. If necessary, round to the nearest hundredth.
7. \( e^x = 412 \)
8. \( \log_x 8 = \frac{3}{4} \)
9. \( \log_{m+1} 30 = \log_{12} 30 \)
10. \( 6^x = 32 \)
11. Write in exponential form: \( \log 45 = 1.65 \).

In 12–14, true or false.
12. \( \ln (23)^{-2} = -2 \ln 23 \)
13. \( \log \left( \frac{M}{N^2} \right) = \log M - 2 \log N \)
14. \( \log_3 7 \cdot \log_3 13 = \log_3 91 \)

In 15–17, assume that bacteria decay according to the exponential model \( y = A(.92)^x \), where \( A \) cells of bacteria would become \( y \) cells \( x \) hours later.

15. What is the rate at which the bacteria decay?
16. If you start with 12,000 bacteria, how many will remain after 8 hours?
17. If at some time there are 1000 bacteria, how many were there two hours earlier?
18. Lana invested some money in an account in which interest is compounded continuously. If the rate is 7%, how long will it take her to double her money?
19. Suppose one sound measures 105 decibels while a second measures 125 decibels. How many times more intense is the second than the first?

In 20–24, consider the function \( y = \log_3 x \).
20. Name five points on the graph.
21. State the domain and range of the function.
22. Graph the function.
23. State an equation for its inverse.
24. Graph the equation from Question 23 on the same axes you used in Question 22.
SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

**SKILLS** deal with the procedures used to get answers

- **Objective A:** Determine values of logarithms. (Lessons 9-4, 9-5, 9-6, 9-8)
  In 1–8, write each number as a decimal. Do not use a calculator.
  1. \( \log 1000 \)
  2. \( \log (.000001) \)
  3. \( \ln e^9 \)
  4. \( \log_5 243 \)
  5. \( \log_{11} 11^{15} \)
  6. \( \ln 1 \)
  7. \( \log_{1/2} 8 \)
  8. \( \log_5 \sqrt[3]{5} \)

  In 9–14, find each logarithm to the nearest hundredth.
  9. \( \log 97,234 \)
  10. \( \ln (100.95) \)
  11. \( \ln 87 \)
  12. \( \log (.0003) \)
  13. \( \ln (-4.1) \)
  14. \( \ln 10 \)

**PROPERTIES** deal with the principles behind the mathematics.

- **Objective B:** Solve exponential equations. (Lesson 9-9)
  In 15–22, solve. If necessary, round to the nearest hundredth.
  15. \( 7^x = 343 \)
  16. \( 9^y = 27 \)
  17. \( 1000(1.05)^n = 2000 \)
  18. \( 3 \cdot 2^z = 1 \)
  19. \( e^z = 22 \)
  20. \( (0.4)^w = e \)
  21. \( 12^{a+1} = 1000 \)
  22. \( 3^{-2b} = 51 \)

- **Objective C:** Solve logarithmic equations. (Lessons 9-4, 9-5, 9-6, 9-8)
  In 23–30, solve. If necessary, round to the nearest hundredth.
  23. \( \log_x 37 = \log_{11} 37 \)
  24. \( \ln (4y) = \ln 9 + \ln 12 \)
  25. \( \log z = 4 \)
  26. \( \log x = 2.91 \)
  27. \( 2 \ln 15 = \ln x \)
  28. \( \log_8 x = \frac{3}{4} \)
  29. \( \log_x 64 = 3 \)
  30. \( \log_x 5 = 10 \)

- **Objective D:** Apply the definition of logarithm. (Lessons 9-4, 9-5, 9-8)
  In 31–34, write in exponential form.
  31. \( \log_8 \left( \frac{1}{27} \right) = -3 \)
  32. \( \ln (6.28) \approx 1.8 \)
  33. \( \log a = b \)
  34. \( \log_b m = n \)

  In 35–38, write in logarithmic form.
  35. \( 10^{1.2} \approx 0.0631 \)
  36. \( e^4 \approx 54.5982 \)
  37. \( x^x = z, \ x > 0, \ x \neq 1 \)
  38. \( 3^n = 12 \)
**Objective E:** Identify properties of logarithms.  
(Lessons 9-6, 9-8)

In 39–44, state the general property used in simplifying the expression.

39. \( \ln 3 + \ln 4 = \ln 12 \)
40. \( \log 40 - \log 4 = \log 10 \)
41. \( \log_{16} (13^2) = -2\log_{16} 13 \)
42. \( \ln e = 1 \)
43. \( \log_{92} 92^{31} = 31 \)
44. \( \log_{2.1} 1 = 0 \)

**Objective F:** Recognize properties of exponential and logarithmic functions.  
(Lessons 9-1, 9-2, 9-4, 9-5, 9-8)

45. What is the range of \( f(x) = e^x \)?
46. State the domain of the exponential function \( y = 2^x \).

In 47–49, true or false.

47. The domain of the log function to the base 5 is the range of the exponential function to the base 5.
48. The logarithm of a negative number is not defined.
49. The common log function is increasing.

**USES** deal with the applications of mathematics in real situations.

**Objective G:** Apply exponential models and formulas.  
(Lessons 9-1, 9-2, 9-7, 9-9)

50. A certain strain of bacteria grows according to \( N = N_0 e^{0.827t} \) where \( t \) is the time in hours. How long will it take for 30 bacteria to increase to 500 bacteria?
51. Dennis invests $2500 at 12% interest for one year. How much more money would he have if the interest is compounded continuously than if it is compounded monthly?
52. If $1000 is compounded continuously at 7.25% a year, in how many years will it triple?
53. The power output \( P \) (in watts) of a satellite is given by the equation \( P = 50e^{-0.25t} \) where \( t \) is the time in days. If the equipment aboard a satellite requires 15 watts of power, how long will the satellite be operating?
54. Strontium 90 (Sr\(^{90}\)) has a half-life of 25 years. How much will be left of 5 grams of Sr\(^{90}\) after 100 years?
**Objective H:** Apply logarithmic scales (Richter, pH, decibel), models, and formulas. (Lessons 9-3, 9-4, 9-8)

55. Sea water has a pH of 8.5 while pure water has a pH of 7. How many times more acidic is pure water than sea water?

56. The formula $B = 10 \log \left( \frac{I}{10^{12}} \right)$ converts sound intensity $I$ in w/m² into decibels $B$. Find $B$ when $I = 2.48 \times 10^9$.

57. How many times more intense is an earthquake with a Richter magnitude of 7.2 than one with a magnitude of 5.2?

58. Under certain conditions, the height $h$ in feet above sea level can be approximated by knowing the atmospheric pressure $P$ in pounds per square inch (psi) using the model

$$\frac{\ln P - \ln 14.7}{-0.000039} = h.$$  

If human blood at body temperature will boil at 0.9 psi, at what height would your blood boil in an unpressurized cabin?

**Representations** deal with pictures, graphs, or objects that illustrate concepts.

**Objective I:** Graph exponential functions. (Lessons 9-1, 9-2, 9-7)

59. Graph $y = 5^x$ using at least 5 points.

60. Graph $y = (\frac{1}{3})^x$ using at least 5 points.

61. Which graph, that of Question 59 or 60, represents decay? Why?

62. The equation of the asymptote to the graph of $y = e^x$ is ___.

**Objective J:** Graph logarithmic functions. (Lessons 9-4, 9-5, 9-8)

63. a. Graph 5 points on $y = \ln x$.

   b. Name its inverse function.

64. The graph below has the equation $y = \log_Q x$. Find $Q$.
The word *trigonometry* is derived from Greek words meaning “triangle measure,” and its study usually begins by examining relationships between sides and angles in right triangles. These ideas originated thousands of years ago. As early as 1500 B.C., the Egyptians had sun clocks. Using their ideas, the ancient Greeks created sundials by erecting a gnomon, or staff, in the ground. The shadows and the height of the gnomon created triangles that could be used to measure the angle of the sun. With these measurements, the Greeks could measure the duration of a year.
By measuring shadows and the angle of the sun, ancient people were also able to measure heights of natural or man-made objects. "Shadow reckoning" was used by the Greeks to measure heights of the Egyptian pyramids. In the 15th to 18th centuries, instruments were developed to aid in measuring the height of the sun and of various other objects. Today the shadows cast by the sun are employed to find the depths of craters on the moon or the heights of dust tornadoes on Mars.

Trigonometry is also used to describe the motion of radio and other waves, called sinusoidal motion. For instance, for a spacecraft launched from Cape Canaveral, trigonometry helps to describe its motion. The spacecraft's distance from the equator with respect to time is sinusoidal, and can be described with trigonometry.

This chapter proceeds as the history of trigonometry did; it starts with right triangle relationships, moves to the study of all triangles, and then considers the ideas needed to describe sinusoidal motion.
The Trigonometric Ratios

Suppose a flagpole casts a 12-ft shadow when the sun is at an angle of 49° with the ground. What is the height of the pole?

Problems such as this one can be solved by using trigonometry. Consider the two right triangles $ABC$ and $A'B'C'$, with $\angle A \cong \angle A'$.

By the AA Similarity Theorem, these triangles are similar, so the ratios of the lengths of corresponding sides are equal. In particular,

$$\frac{B'C'}{BC} = \frac{A'B'}{AB}.$$

Exchanging the means gives the equivalent proportion,

$$\frac{B'C'}{A'B'} = \frac{BC}{AB'}.$$

Look more closely at these ratios:

$$\frac{B'C'}{A'B'} = \frac{\text{length of leg opposite } \angle A'}{\text{length of hypotenuse of } \triangle A'B'C'}$$

and

$$\frac{BC}{AB} = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse of } \triangle ABC}.$$
In all right triangles with an angle congruent to \( \angle A \), the ratio of the length of the leg opposite that angle to the length of the hypotenuse of the triangle is the same. Likewise, in these triangles, any other ratio of sides is constant. These ratios are called \textbf{trigonometric ratios}. There are six possible ratios. All six have special names, but three of them are more important and are defined here. The Greek letter \( \theta \) (theta) is customarily used to refer to either the angle or its measure.

\textbf{Definitions:}

In a right triangle with acute angle \( \theta \),

\[
\begin{align*}
\text{sine of } \theta &= \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}}, \\
\text{cosine of } \theta &= \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}}, \\
\text{tangent of } \theta &= \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta}.
\end{align*}
\]

To follow a practice begun by Euler, we use the abbreviations \( \sin \theta \), \( \cos \theta \), and \( \tan \theta \) to stand for the above ratios. Also, the definitions can be abbreviated as follows:

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{opp}}{\text{hyp}}, \\
\cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adj}}{\text{hyp}}, \\
\tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{opp}}{\text{adj}}.
\end{align*}
\]

Most scientific calculators have a \( \text{DRG} \) key which allows you to enter angle measures in three different units: degrees, radians, or gradians. Small type on the display screen indicates what type of unit your calculator will display (DEG, RAD, GRAD). You should press the \( \text{DRG} \) key until the screen displays DEG. Then, to evaluate the sine of \( n^\circ \), enter \( n \) then press \( \text{sin} \). Your calculator will display a 7- or 8-place decimal. In this book, we will round this value to 3 places. The other ratios are evaluated in the same way.
Example 1

Find \( \tan 49^\circ \).

Solution

The key sequence 49 [tan] shows that \( \tan 49^\circ \approx 1.150 \ldots \)

Check

Use the definition and draw a 49° angle in a right triangle. In right triangle \( ABC \),
\[
\tan A = \tan 49^\circ = \frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A} = \frac{BC}{AC}.
\]
We measure the sides and find \( BC \approx 20.5 \text{ mm} \) and \( AC \approx 18 \text{ mm} \). Thus \( \tan 49^\circ \approx \frac{20.5}{18} \approx 1.139 \).

Example 2

Find the height of the flagpole mentioned in the first paragraph of this lesson.

Solution

With respect to the 49° angle, the adjacent leg is known and the opposite leg is needed. Consequently, use the tangent ratio to set up an equation.
\[
\tan 49^\circ = \frac{\text{opposite}}{\text{adjacent}}
\]
\[
\tan 49^\circ = \frac{x}{12}
\]
Solve for \( x \).
\[
12 \cdot \tan 49^\circ = x
\]
From Example 1, we know \( \tan 49^\circ \approx 1.150 \),
so \( x \approx 12(1.150) \approx 13.8 \).

The flagpole is about 13.8 ft high.

Check

Recall from geometry that within a triangle, longer sides are opposite larger angles. We have found that the side opposite the 49° angle is about 13.8 feet long. The angle opposite the 12 foot side is 41°, which is smaller than 49°. So the answer makes sense.

Trigonometry is often used in navigation. By using the path of a ship or plane, and a map with north-south and east-west lines, you can calculate the distance traveled even without a ruler.

Example 3

A rocket with a range of 200 km is launched at sea with a bearing of 30°. (A bearing is the angle measured clockwise from due north.)

a. How far north of its original position will the rocket land?
b. How far east of its original position will the rocket land?
Solution  Call the original position $Q$ and the landing position $L$. Construct right triangle $QPL$.

![Diagram of right triangle QPL with hypotenuse, leg adjacent to angle Q, and leg opposite angle Q labeled.]

a. The leg adjacent to $\angle Q$, $QP$, is needed and the hypotenuse $QL$ is known. Use the cosine ratio.

$$\cos Q = \frac{\text{adj}}{\text{hyp}} = \frac{QP}{QL}$$

$$\cos 30° = \frac{QP}{200}$$

$$QP = 200 \cdot \cos 30°$$

Use the calculator sequence $200 \times 30 \cos =$. $QP \approx 173.21$.

The rocket should land 173 km north of its original position.

b. The leg opposite $\angle Q$, $PL$, is needed. The hypotenuse $QL$ is known. Use the sine of $\angle Q$.

$$\sin Q = \frac{\text{opp}}{\text{hyp}} = \frac{PL}{QL}$$

$$\sin 30° = \frac{PL}{200}$$

$$PL = 200 \sin 30°$$

Use the calculator sequence $200 \times 30 \sin =$. $PL = 100$.

The rocket should land 100 km east of its original position.

Check 1  $\triangle PQL$ is a 30-60-90 right triangle. The leg opposite the 30° angle should be half the hypotenuse, which it is.

Check 2  The sides should agree with the Pythagorean Theorem. Does $(173.21)^2 + 100^2 = 200^2$? Does $30001.704 + 10000 = 40000$? Yes. Slight differences are due to rounding.

LESSON 10-1  The Trigonometric Ratios  545
The first person to calculate values akin to today’s sines was the Greek mathematician Ptolemy in the 2nd century A.D. The first elaborate tables of values for the sine ratio were due to Johannes Müller (1436–1476), who called himself Regiomontanus. Both Ptolemy and Regiomontanus dealt with lengths of chords in circles, not directly with right triangles. The idea of using ratios in right triangles to solve these problems is due to Georg Joachim Rheticus (1514–1576). Today’s calculators have made tables almost entirely unnecessary.

Questions

1. What is the origin of the word “trigonometry”?  
2. Name one current application where “shadow reckoning” is used.  
3. The motion of radio and other waves is called ____.  
4. In similar triangles the ratios of the lengths of corresponding sides are ____.  
5. Multiple choice Consider the ratios \( \frac{QV}{PV} \), \( \frac{RU}{PU} \), and \( \frac{ST}{PT} \) in triangles \( PQV \), \( PUR \), and \( PTS \) at the right. Which ratio is largest?  
   (a) \( \frac{QV}{PV} \)  
   (b) \( \frac{RU}{PU} \)  
   (c) \( \frac{ST}{PT} \)  
   (d) They are all equal.  
6. Refer to the figure below. Copy and complete with the correct ratio.  
   a. \( \sin \theta = ? \).  
   b. \( \cos \theta = ? \).  
   c. \( \tan \theta = ? \).  
7. Refer to right \( \triangle ABC \) at the right.  
   a. \( BC \) is the ____.  
   b. The leg opposite \( \angle B \) is ____.  
   c. The leg adjacent to \( \angle B \) is ____.  
   d. \( AB \) is the leg opposite ____.  
   e. \( \frac{AC}{AB} = ? \angle B \).  
   f. \( \frac{AC}{BC} = ? \angle B \).  
   g. \( \frac{AB}{BC} = ? \angle B \).
8. a. Measure the lengths of each side of \( \triangle DEF \) at the right to the nearest millimeter, and then estimate \( \sin D \), \( \cos D \), and \( \tan D \) using ratios.

b. \( m\angle D \approx 27^\circ \). Check your answers from part a by finding \( \sin D \), \( \cos D \), and \( \tan D \) on a calculator.

In 9–11, use your calculator to evaluate each of the following to three decimal places.

9. \( \sin 22.5^\circ \)  
10. \( \tan 45^\circ \)  
11. \( \cos 87^\circ \)

12. Refer to Example 2. If the shadow were 20 feet long, what would be the height of the flagpole? (Assume the angle of the sun is still 49\(^\circ\).

13. A ship sails 340 kilometers on a bearing of 75\(^\circ\).
   a. How far north of its original position is the ship?
   b. How far east of its original position is the ship?

14. Refer to the figure at the right. Find the lengths of \( x \) and \( z \).

15. A 20-ft ladder is placed against a wall at an angle of 50\(^\circ\) with the ground. How far from the base of the wall is the bottom of the ladder?
16. In right triangle $ABC$, $\angle C$ is the right angle and $CA = 2CB$.
   a. Draw one such triangle.
   b. Measure each side to the nearest mm and calculate $\cos A$.
   c. Why should your answer for part b be about the same as your classmates’?

17. Give the measure of each angle in a regular polygon of $n$ sides.
   (Previous course)

18. Give the coordinates of the image of $(1, 0)$ under each transformation.
   (Lessons 4-4, 4-5)
   a. $R_{90}$
   b. $R_{-90}$
   c. $R_{180}$
   d. $r_x$

19. What is the image of $(x, y)$ under $r_x$? (Lesson 4-5)

In 20 and 21, true or false.
Refer to the figure at the right where $j \parallel k$. (Previous course)
20. $\angle 3 \equiv \angle 6$
21. $\angle 2 \equiv \angle 5$

22. The number of prime numbers less than a positive integer $p$ is given
   approximately by $\frac{p}{\ln p}$. About how many primes are there less than
   1 billion? (Lesson 9-8)

23. The half-life of carbon-14 is about 5730 years. How many years does
   it take 1 kg of C$^{14}$ to decay to 250 g? (Lessons 9-2, 9-9)

   a. To how many decimal places are the values given?
   b. To how many decimal places does your calculator give values?
In the last lesson you learned how to find lengths of sides in right triangles using the trigonometric ratios. It is also possible to use the trigonometric ratios to find angle measures in right triangles.

The **angle of elevation** of the sun is the angle between the line of sight to the sun and the horizontal. From this angle the ancient Greeks, like present-day astronomers, could determine the time of day. Using other stars, they could also tell time at night.

**Example 1**

A 24-foot high tower casts a 15-foot shadow. What is the angle of elevation of the sun?

**Solution** Let $\theta$ be the angle of elevation. You are given the lengths of the sides opposite and adjacent to $\theta$ so use the tangent ratio.

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{24}{15} = 1.6$$

Now find an angle $\theta$ whose tangent equals 1.6. To find that angle, use the $\text{INV}$ or $\text{2nd}$ key on your calculator. A possible key sequence is $1.6 \text{ INV } \text{tan}$; the display should show 57.994617, which is the measure of $\theta$ in degrees. The angle of elevation is about $58^\circ$. 
Related to the angle of elevation is another angle. In the figure below, if $A$ looks up at $B$, then $\theta$ is the angle of elevation. $B$ has to look down at $A$. The angle between $B$'s line of sight and the horizontal is called the **angle of depression**. In the figure at the right, the angle of depression is labeled $\alpha$ (the Greek letter alpha). The line of sight between $A$ and $B$ is a transversal for the parallel horizontal lines. Thus $\theta$ and $\alpha$ are alternate interior angles and must be congruent. So the angle of elevation is equal to the angle of depression.

![Diagram of angle of elevation and depression](image)

**Example 2** A surveyor on top of a building finds that there is a $28^\circ$ angle of depression to the head of the 6-ft tall assistant. If the assistant is 40 ft from the building, how tall is the building?

**Solution** Draw a picture. The angle of depression, which is not inside the drawn triangle, is congruent to the angle of elevation, which is in the drawn triangle. The height of the building can be found by adding $x$ to the 6-ft height of the assistant. To find $x$, use the tangent ratio because the adjacent side is known and the opposite side is needed.

\[
\tan 28^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{40}
\]

\[
(40)(\tan 28^\circ) = x
\]

\[
(40)(0.532) \approx x
\]

\[
x \approx 21
\]

The height of the building is about $21 + 6 = 27$ ft.
In some situations you may need to draw auxiliary lines to create right triangles.

Example 3  Each edge in the regular pentagon $VIOLA$ is 7.8 cm. Find the length of diagonal $VO$.

Solution  Create a right triangle by drawing the perpendicular to $VO$ from $I$. Call the intersection point $P$, as in the drawing at the right. Recall that each angle in a regular pentagon has measure

$$\frac{180(5 - 2)}{5} = 108^\circ,$$

so $m\angle VIO = 108^\circ$. $IP$ bisects $\angle VIO$. So $m\angle VIP = 54^\circ$. Since the hypotenuse of $\triangle VIP$ is known and the opposite leg, $VP$, is needed, use the sine ratio.

$$\sin 54^\circ = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 54^\circ = \frac{VP}{7.8}$$

$$7.8 \cdot \sin 54^\circ \approx VP$$

$$6.3 \approx VP$$

The perpendicular $IP$ bisects $VO$, so the diagonal is about $2(6.3)$ or 12.6 cm long.

Questions

1. Write a key sequence to find $\theta$ if $\cos \theta = .866$.

In 2–4, find the value of $\theta$ to the nearest degree.

2. $\tan \theta = .25$  
3. $\sin \theta = .61$  
4. $\cos \theta = .80$

5. The angle of elevation is the angle made between the line of sight of the object and the ___.

6. Refer to Example 1. If a 37-ft tower casts a 6.2-ft shadow, what is the angle of elevation of the sun?
7. In the picture below, a person is standing on a cliff looking down at a boat. \( \theta \) is the angle of depression.

8. *True or false* The angle of elevation from a point A to a point B equals the angle of depression from B to A.

9. Refer to Example 2. Suppose the same assistant stands 50 ft from another building, and the angle of depression is 65°. How tall is this new building?

10. In Example 3, find IP to the nearest tenth of a centimeter.

11. Refer to \( \triangle RFK \) below. Find \( \theta \) to the nearest degree.

12. To avoid a steep descent, a plane flying at 35,000 ft starts its descent 150 miles from the airport. For the angle of descent \( \theta \) to be constant, at what angle should the plane descend?

13. A certain ski slope is 580 meters long with a vertical drop of 150 m. At what angle does the skier descend?
14. Suppose each side in regular octagon $ABCDEFGH$ below has a length of 4 cm. Find the length of $AC$.

15. To estimate the distance across a river Sir Vare marks point $A$ near one bank, sights a tree $T$ growing on the opposite bank, and measures off a distance $AB$ of 100 ft. At $B$ he sights $T$ again. If $m\angle A = 90^\circ$ and $m\angle B = 76^\circ$, how wide is the river? (Lesson 10-1)

16. There is an interesting relationship between the sine and cosine that is illustrated below. (Lesson 10-1)

   a. Copy and complete the chart with the aid of a calculator.

   
   \[
   \begin{array}{cccccccc}
   \theta & 10^\circ & 20^\circ & 30^\circ & 40^\circ & 50^\circ & 60^\circ & 70^\circ & 80^\circ \\
   \sin \theta & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
   \cos \theta & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
   \end{array}
   \]

   b. Make a conjecture. For all $\theta$ between $0^\circ$ and $90^\circ$, $\sin \theta = \_$. 

   c. Prove your conjecture. (You may wish to use the triangle below.)
In 17–19, use triangle $SKY$ given below.

17. Find the coordinates of the vertices of the image triangle under the transformation $R_{270}$. *(Lesson 4-7)*

18. Find $SK$. *(Previous course)*

19. Find $m\angle SKY$. *(Lesson 10-1)*

20. State the quadrant (I, II, III, or IV) the point $(x, y)$ is found in given the following conditions. Assume $x \neq 0$.
   a. $x$ is negative and $y$ is positive.
   b. $x$ is positive and $y$ is negative.
   c. $x = y$. *(Previous course)*

21. Recall from your geometry course the SAS, SSS, and ASA triangle congruence theorems. Explain what each of these means. *(Previous course)*

22. Solve for $x$: $mx^2 + px + t = 0$. *(Lesson 6-6)*

23. If $\log 2 + \log 3 - \log 4 = \log x$, what is $x$? *(Lesson 9-6)*

24. A British nautical mile is defined as the length of a minute of arc of a meridian (a minute is $\frac{1}{60}$ of a degree). In feet, it is approximated by

$$6,077 - 31 \cos (2\theta)$$

where $\theta$ is the latitude in degrees.
   a. Find the length of a British nautical mile where you live. (You need to find your latitude.)
   b. The U.S. nautical mile is defined to be 6080.2 feet. At what north latitude do the two definitions agree?
In this lesson we prove three important theorems relating sines and cosines. Consider triangle $ABC$ with right angle $C$. Then $m\angle A + m\angle B = 90^\circ$. So if $m\angle A = \theta$, then $m\angle B = 90^\circ - \theta$.

First, notice that $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$
and also $\cos (90^\circ - \theta) = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$.

Similarly, both $\cos \theta$ and $\sin (90^\circ - \theta)$ equal $\frac{b}{c}$.

Thus we have proved the following theorem.

**Complements Theorem:**

For all $\theta$ between $0^\circ$ and $90^\circ$,

- $\sin \theta = \cos (90^\circ - \theta)$
- $\cos \theta = \sin (90^\circ - \theta)$.

In words, if two angles are complementary, the sine of one angle equals the cosine of the other. Cosine is short for complement's sine. For instance, $\cos 23^\circ = \sin (90^\circ - 23^\circ) = \sin 67^\circ$. You should check with your calculator that both $\cos 23^\circ$ and $\sin 67^\circ$ are approximately $0.921$.

Second, notice that because $\sin \theta = \frac{a}{c}$ and $\cos \theta = \frac{b}{c}$,

$$(\sin \theta)^2 + (\cos \theta)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

$$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$= \frac{a^2 + b^2}{c^2}.$$

Triangle $ABC$ is a right triangle, so by the Pythagorean Theorem, $a^2 + b^2 = c^2$. Thus,

$$(\sin \theta)^2 + (\cos \theta)^2 = \frac{c^2}{c^2} = 1.$$
This argument proves another theorem, called the Pythagorean Identity.

**Theorem (Pythagorean Identity):**

For all θ between 0° and 90°,

\[(\cos \theta)^2 + (\sin \theta)^2 = 1.\]

The Pythagorean Identity can be used to find the value of sin θ if only cos θ is known, or vice versa.

**Example**

Suppose θ is an acute angle in a right triangle, and sin θ = 0.6. Find cos θ.

**Solution**

From the Pythagorean Identity, you know that

\[(\cos \theta)^2 + (\sin \theta)^2 = 1.\]

Substitute 0.6 for sin θ and solve for cos θ.

\[
\begin{align*}
(\cos \theta)^2 + 0.6^2 &= 1 \\
(\cos \theta)^2 + 0.36 &= 1 \\
(\cos \theta)^2 &= 0.64 \\
\cos \theta &= \pm 0.8
\end{align*}
\]

For acute angles, cos θ is always positive, so cos θ = 0.8.

**Check**

Use your calculator to find θ, and then cos θ. Key in .6 [INV] sin [cos]. The calculator displays 0.8.

Generally, calculators give decimal approximations for values of sines and cosines. However, there are a few angles for which the sine and cosine have simple exact values. The angles and the values of their sines and cosines are given below.

**Exact Value Theorem:**

- a. \( \sin 30^\circ = \cos 60^\circ = \frac{1}{2} \)
- b. \( \sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2} \)
- c. \( \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2} \)
Proof:

a. In geometry you learned that the side opposite a 30° angle is half the hypotenuse $h$. Use the figure at the right:

$$\sin 30° = \cos 60° = \frac{BC}{AB} = \frac{\frac{1}{2}h}{h} = \frac{1}{2}$$

b. From geometry you also learned that the hypotenuse of a 45°-45°-90° triangle is $\sqrt{2}$ times the length of either leg. Therefore:

$$\sin 45° = \cos 45° = \frac{EF}{DE} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

c. Use the triangle in part a. Using the Pythagorean Theorem, you can express $AC$ in terms of $h$.

$$(AC)^2 + \left(\frac{1}{2}h\right)^2 = h^2$$

$$(AC)^2 + \frac{1}{4}h^2 = h^2$$

$$(AC)^2 = \frac{3}{4}h^2$$

$$AC = \frac{\sqrt{3}}{2}h$$

$$\sin 60° = \cos 30° = \frac{AC}{AB} = \frac{\frac{\sqrt{3}}{2}h}{h} = \frac{\sqrt{3}}{2}$$

Since $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, the Exact Value Theorem yields the following decimal approximations to the sine and cosine.

(a) $\sin 30° = \cos 60° = 0.5$ (exact value)
(b) $\sin 45° = \cos 45° \approx 0.707$
(c) $\sin 60° = \cos 30° \approx 0.866$.

For sine and cosine of these angles, the exact values occur often, as do the decimal approximations. You should memorize both.
Questions

1. **Multiple choice.** Which is the measure of the complement of an angle with measure θ?
   (a) θ - 90°   (b) 180° - θ   (c) 90° - θ   (d) 90° + θ

In 2 and 3, copy and complete with the measure of an acute angle.

2. \( \cos 40° = \sin \ ? \)  
3. \( \sin 72° = \cos \ ? \)

4. State the Pythagorean Identity.

5. \( (\cos 15°)^2 + (\sin 15°)^2 = \ ? \).

In 6 and 7, assume that θ is an acute angle in a right triangle. Use the Pythagorean Identity to find \( \cos \theta \) if:

6. \( \sin \theta = .28 \)  
7. \( \sin \theta = \frac{\sqrt{3}}{2} \).

8. Name three acute angles whose sine and cosine can be found by using the Exact Value Theorem.

In 9–11, copy and complete with an exact value.

9. \( \sin 45° = \ ? \)   
10. \( \cos 60° = \ ? \)  
11. \( \cos 30° = \ ? \)

12. In \( \triangle TIP \) at the right. \( IP = 10 \).
    Find exact values of:
    a. \( IT \)
    b. \( PT \)

13. \( \triangle QLE \) at the right is an equilateral triangle with sides 6" long.
    a. Find the exact height \( h \).
    b. Find the exact area of \( \triangle QLE \).

In 14–16, use the figures below and the definition of \( \tan \theta \) to find exact values.

14. \( \tan 30° \)

15. \( \tan 60° \)

16. \( \tan 45° \)
17. The angle of elevation of a pipeline up the side of a mountain is 45°.
   a. What is the exact vertical rise of a length of pipe 20 meters long?
   b. Approximate your answer to part a to the nearest tenth of a meter.

18. Refer to the triangle at the right.
   a. Express in terms of \( x, y, \) or \( z, \) and simplify: \( \frac{\sin \theta}{\cos \theta}. \)
   b. True or false. For all \( \theta \) between 0° and 90°, \( \frac{\sin \theta}{\cos \theta} = \tan \theta. \)

19. A private plane begins its descent to an airport when it is a ground distance of 6 miles away and 1 mile high. At what constant angle of depression would it need to descend? (Lesson 10-2)

20. Find the value of \( \theta \) to the nearest degree if \( \sin \theta = .67. \) (Lesson 10-2)

21. Given that \( c^2 = a^2 + b^2 - 1.88ab, \) solve for \( a \) if \( b = 5 \) and \( c = -1.78. \) (Lesson 6-5)

22. Find the distance from \((-3, 5)\) to \((1, 9).\) (Previous course)

23. Find the image of \((1, 0)\) under the given transformation.
   a. \( R_{360} \)  
   b. \( R_{270} \)  
   c. \( R_{90} \)

24. a. Verify that:
   (i) \( \sin 60^\circ = 2 \cdot \sin 30^\circ \cdot \cos 30^\circ \)
   (ii) \( \sin 84^\circ = 2 \cdot \sin 42^\circ \cdot \cos 42^\circ. \)
   b. Generalize the result of part a and verify your generalization with some other values.
In right triangles the acute angles measure between $0^\circ$ and $90^\circ$. So the definitions of cosine and sine given in Lesson 10-1 cannot apply to measures greater than $90^\circ$. However, rotations may have any magnitude, positive or negative. So rotations can be used for defining cosines and sines in general.

The unit circle is the circle with center at the origin and radius 1. If the point $(1, 0)$ on the circle is rotated around the origin with a magnitude $\theta$, then the image point $(x, y)$ is also on the circle. The coordinates of the image point can be found using sines and cosines.

**Example 1** What are the coordinates of the image of $(1, 0)$ under $R_{70}$?

**Solution** Let $A = (x, y)$ be the image of $(1, 0)$ under $R_{70}$. Using the figure at the right, $OA = 1$, since the radius of the unit circle is 1. Draw a vertical line from $A$ to form a right triangle with $OA$ and the $x$-axis. Then $\triangle ABO$ is a right triangle with legs of length $x$ and $y$ and hypotenuse of length 1. Now use the definitions of sine and cosine.

\[
\cos 70^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x
\]
\[
\sin 70^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y
\]

The first coordinate is $\cos 70^\circ$ and the second coordinate is $\sin 70^\circ$. Thus $(x, y) = (\cos 70^\circ, \sin 70^\circ) \approx (.342, .940)$; that is, the image of $(1, 0)$ under $R_{70}$ is $(.342, .940)$.

**Check** Use the Pythagorean Identity.

Does $(.342)^2 + (.940)^2 = 1^2$?

$.117 + .884 = 1.001 = 1$, so it checks.
In Example 1, the image of \((1,0)\) under \(R_{70}\) is \((\cos 70^\circ, \sin 70^\circ)\). We generalize this idea to define the sine and cosine for a rotation of any magnitude \(\theta\).

**Definition:**

Let \(\theta\) be the magnitude of a rotation. Then for any \(\theta\), the point \((\cos \theta, \sin \theta)\) is the image of \((1,0)\) under \(R_\theta\).

Stated another way, \(\cos \theta\) is the \(x\)-coordinate of the image of \((1,0)\) under a rotation of \(\theta\); \(\sin \theta\) is the \(y\)-coordinate of the image.

The sines and cosines of angles which are multiples of \(90^\circ\) can be found without using a calculator.

**Example 2**  
Find:  
\(a.\) \(\sin 90^\circ\)  
\(b.\) \(\cos (-180^\circ)\).

**Solution**

\(a.\) The image of \((1,0)\) under \(R_{90}\) is \((0,1)\). Because \(\sin 90^\circ\) is the \(y\)-coordinate of this image point, \(\sin 90^\circ = 1\).

\(b.\) The image of \((1,0)\) under \(R_{-180}\) is \((-1,0)\). Since \(\cos (-180^\circ)\) is the \(x\)-coordinate of this point, \(\cos (-180^\circ) = -1\).

**Check**  
You can check each of these on your calculator. For instance, for part \(b\) use the key sequence \(180 [+/-] \cos\). You should get \(-1\) on the display.
Recall that rotations of magnitude greater than 360° refer to more than one complete revolution.

**Example 3**

Find:

a. \( \sin 630° \)

b. \( \cos 385° \).

**Solution**

a. \( R_{630} \) equals one complete revolution around the circle, followed by \( R_{270} \). Because the image of \((1, 0)\) under \( R_{630} \) is \((0, -1)\), \( \sin 630° = -1 \).

b. \( R_{385} \) equals one complete revolution followed by a rotation of 25°, thus, \( \cos 385° = \cos 25° = .906 \).

Generally, the unit circle is a good tool to use along with your calculator to help solve problems involving angles. The definitions of sine and cosine in terms of rotations on the unit circle determine the trigonometric values of any angle. They can also be used to prove that both the Complements Theorem and the Pythagorean Identity proved in the previous lesson apply to all real numbers \( \theta \).

**Questions**

1. If \((1, 0)\) is rotated \( \theta \) around the origin:
   a. \( \cos \theta \) is the \( \_ \_ \_ \)-coordinate of its image.
   b. \( \sin \theta \) is the \( \_ \_ \_ \)-coordinate of its image.

2. True or false. The image of \((1, 0)\) under \( R_{23} \) is \((\sin 23°, \cos 23°)\).

3. \( R_0(1, 0) = \_ \_ \_ \), so \( \cos 0° = \_ \_ \_ \) and \( \sin 0° = \_ \_ \_ \).
In 4–7, use the unit circle without your calculator to find:

4. \( \cos 90° \)  
5. \( \sin 180° \)  
6. \( \cos 270° \)  
7. \( \sin (-90°) \).

8. a. A rotation of 540° equals a rotation of 360° followed by \( ? \).
   
b. The image of \((1, 0)\) under \( R_{540} \) is \( ? \).
   
c. Evaluate \( \sin 540° \).

In 9–12, Evaluate without using a calculator.

9. \( \cos 450° \)  
10. \( \sin 450° \)  
11. \( \cos (-630°) \)  
12. \( \sin (-720°) \)

In 13 and 14, use a calculator to approximate to the nearest thousandth.

13. \( \cos 392° \)  
14. \( \sin 440° \)

In 15–17, suppose \( A = (1, 0) \), \( B = (0, 1) \), \( C = (-1, 0) \), and \( D = (0, -1) \). Which point is the image of \((1, 0)\) under each rotation?

15. \( R_{450} \)  
16. \( R_{540} \)  
17. \( R_{(-720)} \)

In 18–23, which letter in the figure at the right could stand for each of the following?

18. \( \cos 80° \)  
19. \( \sin 80° \)  
20. \( \cos (-280°) \)  
21. \( \sin 800° \)  
22. \( \cos 380° \)  
23. \( \sin (-340°) \)

24. If \( 0° \leq \theta \leq 360° \),
   
a. What is the largest possible value of \( \cos \theta \)?
   
b. What is the smallest possible value of \( \sin \theta \)?

In 25 and 26, verify by substitution that the statement holds for the given value of \( \theta \).

25. \( (\cos \theta)^2 + (\sin \theta)^2 = 1; \theta = 630° \)

26. \( \sin \theta = \cos (90° - \theta); \theta = -90° \)

In 27 and 28, find the exact value without using a calculator.

27. \( \cos 420° \)  
28. \( (\sin 405°)^2 + (\cos 405°)^2 \)
29. At 65 feet up in a lookout tower, a ranger sights a fire. The angle of depression to the fire measures 4°. How far from the base of the tower is the fire? (Lesson 10-2)

30. Refer to the diagram below. A roof has a pitch of \( \frac{1}{12} \). What angle \( \theta \) does it make with the horizontal? (Lesson 10-2)

31. Use the equation \( y = 5x^2 - 7x + 4 \). Determine the number of \( x \)-intercepts of the graph of this parabola. (Lesson 6-7)

32. a. Draw a set of coordinate axes on graph paper.
   b. With the origin as the center, use a compass to draw a unit circle with radius 1 inch.
   c. Label \( A = (1, 0) \).
   d. With a protractor, locate the image of \( A \) under \( R_{150} \).
   e. Use your drawing to estimate \( \cos 150^\circ \) and \( \sin 150^\circ \).
   f. Compare your estimated value from part e to those displayed by your calculator.
Every value of \( \cos \theta \) or \( \sin \theta \) is a coordinate of a point on the unit circle. With a calculator, it is easy to determine these values. If \( 0^\circ < \theta < 90^\circ \), you can check calculator values by drawing a right triangle. In this lesson, we show how to find and check other values.

Think of the image of \((1, 0)\) under \( R_\theta \). Unless \( \theta \) is a multiple of \( 90^\circ \), the image is in one of the four quadrants. As the diagram below at the left shows, each quadrant is associated with a range of values of \( \theta \).

The quadrants enable you to determine quickly whether \( \cos \theta \) and \( \sin \theta \) are positive or negative. Refer to the graph just above. Because \( \cos \theta \) is the first or \( x \)-coordinate of the image, it is positive in Quadrants I and IV and negative in Quadrants II and III. The sine, which is the \( y \)-coordinate of the image, is positive in Quadrants I and II and negative in Quadrants III and IV.

You do not need to memorize these ideas. You can always visualize them on the unit circle.
Example 1  Is sin 150° positive or negative?

Solution  150° is between 90° and 180°, so it refers to a point in Quadrant II. The sine is the second coordinate. In this quadrant the second coordinate is positive, so sin 150° is positive.

Once you know the sign of the cosine or sine, find its value by referring to points in the first quadrant. For instance, you know sin 150° is the second coordinate of a point in the second quadrant. By reflecting this point over the y-axis, you get a reference or image point in the first quadrant, namely (cos 30°, sin 30°). Notice that the angles formed with the x-axis are congruent.

The first coordinates of these points are opposites, so

\[ \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}. \]

The second coordinates are equal, so sin 150° = sin 30° = 1/2.

When the image is in Quadrant III, rotating the point 180° gives a corresponding point in the first quadrant.

Example 2  Show that sin (-125°) = -sin 55°.

Solution  Make a sketch. The point \( P = (\cos (-125°), \sin (-125°)) \) is in the third quadrant, so the sine is negative. Rotate \( P \) 180° to get a first quadrant point \( P' \). This image point has coordinates \((\cos 55°, \sin 55°)\) because \(180° - 125° = 55°\). Because the image of \((x, y)\) under a rotation of 180° is \((-x, -y)\), \(P'\) also has coordinates \((-\cos (-125°), -\sin (-125°))\). Thus \(\sin (-125°) = -\sin 55°\).

Check  Using a calculator, \(\sin (-125°) \approx -0.819\) and \(\sin 55° \approx 0.819\), so it checks.
Points in Quadrant IV are reflection images, over the $x$-axis, of points in the first quadrant.

**Example 3** Find an exact value for $\cos 315^\circ$.

**Solution** $\cos 315^\circ$ is the first coordinate of a point in Quadrant IV, so the cosine is positive. Reflect $(\cos 315^\circ, \sin 315^\circ)$ over the $x$-axis. The image forms an angle of $360^\circ - 315^\circ = 45^\circ$, so the image point is $(\cos 45^\circ, \sin 45^\circ)$. Since the first coordinates of these points are equal,

$$\cos 315^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$ 

![Diagram](image)

**Check** A calculator shows $\cos 315^\circ \approx 0.707$. Recall that 0.707 is an approximation to $\frac{\sqrt{2}}{2}$, so it checks.

If you add or subtract multiples of $360^\circ$ to $315^\circ$, you will get the same value for the cosine. This is because $R_{315^\circ} = R_{675^\circ} = R_{1035^\circ}$, and so on. Thus $\cos 315^\circ = \cos 675^\circ = \cos 1035^\circ$. Similarly, $\sin 315^\circ = \sin 675^\circ = \sin 1035^\circ$. The fact that values of sines and cosines repeat is a very important property. You will learn more about this later in this chapter.

**Questions**

In 1 and 2, *multiple choice*. Select from the following choices.

(a) is always positive
(b) is always negative
(c) may be positive or negative

1. If $R_\theta(1, 0)$ is in quadrants II or III then $\cos \theta$ . 
2. If $180^\circ < \theta < 360^\circ$, $\sin \theta$ . 

In 3 and 4, (a) draw the corresponding point on the unit circle; (b) without using a calculator, state whether the value is positive or negative.

3. $\sin 343^\circ$ 
4. $\cos 217^\circ$
5. Evaluate (a) \( \cos 118^\circ \) and (b) \( \sin 118^\circ \) with a calculator.

In 6 and 7, find \( \theta \) if \( 0^\circ < \theta < 90^\circ \).
6. \( \sin 182^\circ = -\sin \theta \)
7. \( \cos 295^\circ = \cos \theta \)

In 8–11, find the exact value.
8. \( \sin 315^\circ \)
9. \( \cos 240^\circ \)
10. \( \cos (-150^\circ) \)
11. \( \sin 135^\circ \)

12. Copy and complete with “positive” or “negative.” If \( \angle B \) is obtuse, then \( \cos B \) is ___ and \( \sin B \) is ___.

13. Refer to the graph at the right. Find \( \theta \) to the nearest degree.

14. Find \( \theta \) such that \( \cos \theta = -\frac{1}{2} \) and \( \sin \theta = -\frac{\sqrt{3}}{2} \), for \( 0^\circ < \theta < 360^\circ \).

15. Suppose \( \sin \theta = -\frac{\sqrt{2}}{2} \). Find two possible values for \( \cos \theta \).

16. Find \( C \) such that \( 0^\circ < C < 180^\circ \) and \( \cos C = -0.251 \).

17. Find two values of \( \theta \) such that \( \sin \theta = \cos \theta \). (Hint: sketch the unit circle.)

18. Refer to the figure at the right.
   a. Find the area of \( \triangle ABC \).
   b. Find \( \sin B \).
   c. By substituting your answer to part b for \( \sin B \), prove that the area of \( \triangle ABC \) equals \( \frac{1}{2} ac \sin B \).
   d. Use the result of part c to estimate the area of \( \triangle DEF \) at the right.

19. Give exact coordinates of the image of \((1, 0)\) under \( R_{135^\circ} \). (Lesson 10-4)

20. Evaluate \( \cos 270^\circ \) without a calculator. (Lesson 10-4)

21. If \( 0^\circ < \theta < 90^\circ \) and \( \sin 83.5^\circ = \cos \theta \), find \( \theta \). (Lesson 10-3)

22. Refer to the triangle at the right \( (m \angle OFX = 63^\circ) \).
   a. Measure \( OX \) and \( FX \), then calculate \( \sin 63^\circ \).
   b. Check your answer to part a by finding \( \sin 63^\circ \) on a calculator. (Lesson 10-1)
23. Solve \(3(x - 4)^5 = 215\) for \(x\). (Lesson 8-11)

24. Which of the following are congruent due to ASA triangle congruence? (Previous course)

(a) 

(b) 

(c) 

(d) 

25. Copy and complete the following chart using your calculator to estimate each value to the nearest thousandth.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(\sin \theta)</th>
<th>(\tan \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2°</td>
<td></td>
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<tr>
<td>1°</td>
<td></td>
<td></td>
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<tr>
<td>0.5°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1°</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Generalize what the table shows.

b. Explain your generalization. (You may wish to draw a series of triangles with these angles.)
In the previous lessons you have learned to use the trigonometric ratios to find unknown sides or angles of right triangles. This lesson will give you the means to determine some unknown sides or angles in any triangle.

The captain of a clipper ship C spots two other ships on the ocean. Ship A is about 5 miles away and ship B is about 5.2 miles away. The angle between the two sightings is 20°. How far apart are ships A and B? The problem is illustrated below, and c is the required distance.

The captain knows the measures of two sides and the included angle. This is the SAS condition and all other sides and angles can be determined. Because \( \triangle ABC \) is not a right triangle, the solution cannot be found using only what you have already learned. However, the unique measure of the third side can be found using the Law of Cosines.

In this theorem and throughout the rest of this chapter, we follow a standard convention that in triangle \( ABC \), \( a \) is the length of the side opposite \( \angle A \), \( b \) is the length of the side opposite \( \angle B \), and \( c \) is the length of the side opposite \( \angle C \). That is, lower-case letters stand for sides opposite the points named by the corresponding capital letters.

### Theorem (Law of Cosines)

In any triangle \( ABC \),

\[
c^2 = a^2 + b^2 - 2ab \cos C.
\]

This theorem works for any two sides and the included angle of a triangle. Thus it is also true that

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

\[
b^2 = a^2 + c^2 - 2ac \cos B.
\]

Before proving this theorem, we provide an example.
Example 1  Use the Law of Cosines to find the distance between ships A and B above.

Solution  The unknown side is \( c \) and the two known sides are \( a = 5.2 \) miles and \( b = 5 \) miles. The included angle is \( 20^\circ \). Substituting into the Law of Cosines gives

\[
c^2 = (5.2)^2 + 5^2 - 2(5.2)(5) \cos 20^\circ \\
c^2 \approx 27.04 + 25 - 52(0.940) \\
c^2 \approx 3.16 \\
c \approx \pm \sqrt{3.16} \\
c \approx \pm 1.78.
\]

Because \( c \) is a distance, \( c > 0 \) and only the positive solution is acceptable. The two ships are about 1.8 miles apart.

Though the Law of Cosines is called a "law," it is also a theorem. That means it can be proved from definitions, other theorems, and postulates. Here is a proof that, in any \( \triangle ABC \),

\[
c^2 = a^2 + b^2 - 2ab \cos C.
\]

Proof

Set up \( \triangle ABC \) on a coordinate plane so that \( C = (0, 0) \) and \( A = (b, 0) \). To find the coordinates of \( B \), notice that \( B \) is \( a \) times farther from the origin than the intersection of the unit circle and \( CB \). Since that intersection has coordinates \((\cos C, \sin C)\), \( B = (a \cos C, a \sin C) \).

All that remains is to find \( c^2 \) by using the distance formula. In general,

\[
\begin{align*}
\quad c & = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
\quad c^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\
\quad c^2 &= (a \cos C - b)^2 + (a \sin C - 0)^2 \\
\quad c^2 &= a^2(\cos C)^2 - 2ab \cos C + b^2 + a^2(\sin C)^2 \\
\quad c^2 &= a^2 \cos^2 C + a^2(\sin C)^2 + b^2 - 2ab \cos C \\
\quad c^2 &= a^2 (\cos C)^2 + (\sin C)^2 + b^2 - 2ab \cos C \\
\quad c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

Distance Formula  squaring both sides
substitution  expansion
Commutative Property of +
Distributive Postulate  Pythagorean Identity

LESSON 10-6  The Law of Cosines  571
Example 1 shows how the Law of Cosines can be used to solve problems where two sides and their included angle are known (SAS). If the lengths of all three sides are known (SSS), the Law of Cosines can be used to find the measure of any angle of the triangle.

**Example 2**  A triangle has sides of length 4, 5, and 8.5. What is the measure of its largest angle?

**Solution**  Draw a figure. The largest angle is opposite the longest side. We call that angle \( A \). Then \( a = 8.5 \). Let \( b = 5 \) and \( c = 4 \).

\[
\begin{align*}
\triangle ABC & \quad a = 8.5 \quad b = 5 \\
\quad & \quad c = 4
\end{align*}
\]

By the Law of Cosines,

\[
a^2 = b^2 + c^2 - 2bc \cos A \\
(8.5)^2 = 5^2 + 4^2 - 2(4)(5) \cos A \\
72.25 = 25 + 16 - 40 \cos A \\
31.25 = -40 \cos A \\
-.781 = \cos A.
\]

To find \( \angle A \), press \( .781 \pm [\text{INV}] \cos \).

Your display should read 141.352 ... . Thus \( \angle A \approx 141^\circ \).

**Check**  Draw such a triangle. The largest angle seems to be about 141°.

**Questions**

1. **True or false**  In \( \triangle ABC \),
\[
a^2 = b^2 + c^2 - 2ab \cos A.
\]

2. According to the Law of Cosines, in \( \triangle WXY \), \( y^2 = \_\_\_\_. \)
3. **Multiple choice.** Which of the following verbally describes the Law of Cosines?
   (a) The third side of a triangle equals the sum of the squares of the other two sides minus the product of the two sides and the included angle.
   (b) The square of the third side of a triangle equals the sum of the squares of the other two sides minus the product of the two sides and the cosine of the included angle.
   (c) The square of the third side of a triangle equals the sum of the squares of the other two sides minus twice the product of the two sides and the cosine of the included angle.
   (d) none of these

4. In the proof of the Law of Cosines, why does $a^2 = ((\cos C)^2 + (\sin C)^2) = a^2$?

5. Refer to the ships in the lesson. Suppose the clipper ship is 1.1 miles from ship A and 2.4 miles from ship B and the angle between the two sightings is 135°. How far apart are ships A and B?

6. Refer to the figure of Example 2. Find the measure of $\angle B$.

7. The water molecule $\text{H}_2\text{O}$ can be modeled by the figure below. The angle between the oxygen-hydrogen bonds is 105°. If the average distance between the oxygen and hydrogen nuclei is $p$ units, how far apart (on average) are the two hydrogen nuclei?

![Water molecule diagram]

8. Refer to the diagram below. Two planes leave from Dallas, one toward Bismarck, and the other toward Chicago. By approximately what angle do their headings differ?

![Plane diagram]

9. Refer to the triangle below.
   a. Find the value of $b$.
   b. Use your answer from part a to find the measure of $\theta$.

![Triangle diagram]
10. The distance from Earth to the star Sirius is about 8.8 light years. The distance from Earth to Alpha Centauri is about 4.3 light years (a light year is the distance light travels in one year). The angle between these stars, with Earth as a vertex, is about 44°. What is the approximate distance (in light years) between Sirius and Alpha Centauri?

11. Solve the Law of Cosines to get a formula for \( \cos C \) in terms of \( a \), \( b \), and \( c \).

12. At a criminal trial, the witness gave the following testimony: "The defendant was 20 ft from the victim. I was 50 ft from the defendant and 75 ft from the victim when the shooting occurred. I saw the whole thing."
   a. Use the Law of Cosines to show that the testimony has errors.
   b. How else could you know that the testimony has errors?

In 13–16, give an exact value. Do not use a calculator. (Lessons 10-3, 10-4, 10-5)

13. \( \cos 30° \)  
14. \( 3(\sin 17°)^2 + 3(\cos 17°)^2 \)
15. \( \sin 150° \)  
16. \( \cos (-45°) \)

17. If \( \sin 160° = \sin \theta \) and \( 0° < \theta < 90° \), what is \( \theta \)? (Lesson 10-5)

18. Solve \( \sin \alpha = \sin 18° \) if \( 90° < \alpha < 180° \). (Lesson 10-5)

19. Use the function \( f(x) = x^2 - 1225 \). (Lessons 6-2, 6-3, 6-4)
   a. How many x-intercepts does the function have?
   b. Find the x-intercepts.
   c. Graph the function.
   d. Name the curve of part c.

20. To the nearest tenth, find the real number \( b \) such that \( 3.2(2 - b)^3 = 8 \). (Lesson 8-11)

21. During the investigation of a shooting, the police found a bullet imbedded in the wall 7 ft above the floor. Investigation revealed that the bullet was fired from a height of 4 ft at a distance of 3 ft from the wall. At what angle of elevation was the bullet fired? (Lesson 10-2)

22. If \( \log x = 3.5 \), find \( x \) to the nearest tenth. (Lesson 9-4)

23. The Law of Cosines is sometimes described as "the Pythagorean Theorem with a correction term." Explain why this is an appropriate description.
Two forest rangers are in stations 25 miles apart at locations $S$ and $T$. On a certain day, the ranger at $S$ sees a fire $F$ at an angle of $40^\circ$ with the line connecting the stations. The ranger at $T$ sees the fire at an angle of $60^\circ$. How far is the fire from each ranger’s station?

The given information here is the ASA condition; only one side is known. As a result, if you try to use the Law of Cosines to solve this problem, you will find that there are two unknowns. Since the Law of Cosines involves the three sides and only one angle of a triangle, it is not useful when only one side is known.

However, the missing distance can be found using an extraordinarily beautiful, simple theorem called the Law of Sines. In a triangle, the ratios of the sine of an angle to the length of its opposite side are equal. Here is a symbolic statement and proof.

**Law of Sines Theorem:**

In any triangle $ABC$,

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
\]
Proof:

Draw the altitude $h$ to side $AC$. Because $\sin C = \frac{h}{a}$, $h = a \sin C$.

So the area of $\triangle ABC$ is $\frac{1}{2}bh = \frac{1}{2}ba \sin C$. Similarly, by drawing the altitudes to the other sides, the area can be shown to equal $\frac{1}{2}ac \sin B$ and $\frac{1}{2}bc \sin A$.

Because the area of the triangle is a constant,

$$\frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$$

So

$$ab \sin C = ac \sin B = bc \sin A.$$

Now divide all three expressions by $abc$.

$$\frac{ab \sin C}{abc} = \frac{ac \sin B}{abc} = \frac{bc \sin A}{abc}$$

Simplify.

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

The Law of Sines is useful whenever two angles and a side of a triangle are known and you wish to find a second side.

---

Example 1

In the situation described at the beginning of this lesson, find the distance from the ranger at station $T$ to the fire.

Solution

The desired length is $s$. The angle opposite $s$ is $\angle S$, with measure $40^\circ$. To use the Law of Sines you need the values of another angle and its opposite side. Because the sum of the measures of the angles in a triangle is $180^\circ$, $\angle F$ has measure $80^\circ$. Since you know $m\angle S = 40^\circ$, $m\angle F = 80^\circ$, and $f = 25$ miles, use the two ratios for $S$ and $F$.

$$\frac{\sin S}{s} = \frac{\sin F}{f} \quad \text{or} \quad \frac{\sin 40^\circ}{s} = \frac{\sin 80^\circ}{25}$$

$$s = \frac{25 \sin 40^\circ}{\sin 80^\circ} \quad \text{or} \quad s = \frac{25 \cdot 0.643}{0.985} = 16.3$$

The fire is about 16 miles from the ranger at $T$. 

Brush fire in Napa, California
Example 1 illustrates how to use the Law of Sines when you are given two angles and the included side (ASA) and want to find the length of another side. The Law of Sines can also be used in an AAS situation: that is, in a triangle in which two angles and a non-included side are given.

Example 2 In \( \triangle XYZ \), \( m\angle X = 25^\circ \), \( m\angle Y = 75^\circ \), and \( x = 4 \). Find \( y \).

Solution

\[
\frac{\sin X}{x} = \frac{\sin Y}{y}
\]

\[
\frac{\sin 25^\circ}{4} = \frac{\sin 75^\circ}{y}
\]

\[
y = \frac{4 \sin 75^\circ}{\sin 25^\circ}
\]

\[
y \approx \frac{4(.966)}{.423} \approx 9.1
\]

The Law of Sines was known to Ptolemy in the 2nd century A.D. A theorem equivalent to the Law of Cosines is in Euclid’s *Elements* written four centuries earlier. The Greeks used these theorems as the forest ranger used them in Example 1, to locate landmarks. This made it possible for reasonably accurate maps of parts of the Earth to be drawn well before the days of man-made satellites.

Questions

1. State the Law of Sines.
2. With information satisfying the given condition, which theorem is more useful for finding other parts of a triangle, the Law of Cosines or the Law of Sines?
   a. SAS  
   b. ASA  
   c. AAS  
   d. SSS
3. What does the expression \( \frac{1}{2}ab \sin C \) represent for triangle \( ABC \)?
4. *Multiple choice* Which of the following verbally describes the Law of Sines? In a triangle:
   (a) the ratio of an angle to the length of a side is a constant.
   (b) the ratio of the sine of an angle to the length of the adjacent side is a constant.
   (c) the ratio of the sine of an angle to the length of the opposite side is a constant.
   (d) None of (a) to (c) describes the Law of Sines.
5. Refer to the forest fire in the lesson. Find the distance of the fire from the ranger at \( S \).
In 6 and 7, find \( y \).

6. \[
\begin{array}{c}
\text{In triangle } X \triangle Y \text{, find } y.
\end{array}
\]

7. \[
\begin{array}{c}
\text{In triangle } O \triangle T \text{, find } y.
\end{array}
\]

8. In \( \triangle ABC \), suppose you are given \( m \angle A = 45^\circ \), \( m \angle B = 60^\circ \), and \( a = 24 \). Find the exact value of \( b \).

9. Refer to \( \triangle PQR \) at the right.
   a. Find \( RQ \).
   b. Use your answer to part a and the Law of Sines to find \( m \angle R \).
   c. Use the Law of Cosines and your answer to part a to find \( m \angle R \).
   d. **True or false.** In an SAS situation, once you find the third side, you can use either the Law of Sines or the Law of Cosines to find a second angle.

10. When a beam of light in air strikes the surface of water, it is refracted or bent as shown at the right. The relationship between \( \alpha \) and \( \theta \) is given by Snell’s Law,

\[
\frac{\sin \alpha}{\text{speed of light in air}} = \frac{\sin \theta}{\text{speed of light in water}}.
\]

The speed of light in air is about \( 3.00 \times 10^8 \) m/sec. If \( \alpha = 45^\circ \) and \( \theta = 32^\circ \), find the speed of light in water.

11. Because surveyors cannot get to the inside center of a mountain, its height must be measured in a more indirect way as shown in this problem. Refer to the diagram below.

\[
\begin{array}{c}
\text{a. Find the measures of } \angle ABD \text{ and } \angle ADB. \text{ (Hint: you only need to use your basic knowledge from geometry.)}
\end{array}
\]

b. Find \( BD \).

c. Find \( DC \), the height of the mountain.
12. a. Write the Law of Sines for the triangle at the right when \( m \angle F = 90^\circ \).
   
   b. How do the ratios from your answer to part a compare to the trigonometric ratios?

13. In using the Law of Sines, Katrina came up with \( \sin A = 1.234 \). What can you tell Katrina about her solution?

14. Kate has $800 to invest in an account which compounds interest continuously at a rate of 7\%\%.
   
   a. How long will it take until the account doubles in value?
   
   b. How long will it take until the account doubles a second time?

   (Lesson 9-7)

15. Multiple choice. Which of the following is the Law of Cosines?
   
   (Lesson 10-6)
   
   (a) \( a^2 = b^2 + c^2 + 2bc \cos A \)
   
   (b) \( a^2 = b^2 + c^2 - bc \cos A \)
   
   (c) \( a^2 = b^2 + c^2 - 2 \cos A \)
   
   (d) \( a^2 = b^2 + c^2 - 2bc \cos A \)

16. Give the coordinates of \( R_{60}(1, 0) \) (a) exactly; (b) to the nearest thousandth. (Lesson 10-4)

17. If \( \sin \theta = \frac{4}{5} \): (a) find the two possible values of \( \cos \theta \) and (b) graph the two points \( (\cos \theta, \sin \theta) \). (Lesson 10-5)

In 18 and 19, true or false. (Lessons 10-4, 10-5)

18. \( \cos 180^\circ = -1 \) 
19. \( \sin 225^\circ = \frac{\sqrt{2}}{2} \)

20. To the nearest tenth of a degree, find the measure of an acute angle and an obtuse angle whose sine is 0.921. (Lessons 10-2, 10-5)

In 21 and 22, consider \( A = \begin{bmatrix} -100 & 5 \\ -80 & 4 \end{bmatrix} \).

21. a. Find \( \det A \).
   
   b. Does \( A^{-1} \) exist? If so, find it. (Lesson 5-5)

22. a. Find an equation for the line through the two points in matrix \( A \). (Lessons 3-5, 4-1)
   
   b. What kind of variation is described by part a? (Lesson 2-1)

23. A rock is thrown upward with an initial velocity of 30 \( \frac{ft}{\text{sec}} \) from a height of 12 ft. (Lessons 6-1, 6-5, 6-6)
   
   a. Write an equation to describe the rock’s height \( h \) (in feet) with respect to time \( t \) (in seconds).
   
   b. Graph the equation.
   
   c. What is the maximum height of the rock?
   
   d. When does the rock hit the ground?
24. Refer to the triangle shown below.
   a. Measure the sides of this triangle in centimeters and the angles in degrees.
   b. Find the sines of the angles.
   c. Substitute the values you get into the Law of Sines.
   d. How nearly equal are the fractions?
Every angle $\theta$ in a triangle has a measure between $0^\circ$ and $180^\circ$. Each value of $\cos \theta$ corresponds to a vertical line $x = \cos \theta$ intersecting the unit circle, and there is only one intersection point for $0^\circ < \theta < 180^\circ$. Thus if you know $\cos \theta$ is positive, then $0^\circ < \theta < 90^\circ$ and the angle must be acute. If $\cos \theta$ is negative, then $90^\circ < \theta < 180^\circ$ and the angle is obtuse.

The situation is different for $\sin \theta$. Each value of $\sin \theta$ between 0 and 1 corresponds to a horizontal line $y = \sin \theta$, which intersects the unit circle in two points. One point is in the first quadrant; one is in the second quadrant. Consequently, the equation $\sin \theta = k$ has two solutions between $0^\circ$ and $180^\circ$, one where $\theta$ is acute and one where $\theta$ is obtuse.

**Example 1** If $\sin \theta = .624$, find $\theta$.

**Solution** For one solution, use a calculator.

The key sequence .624 INV sin yields $\theta \approx 38.6^\circ$. In the drawing below, $\sin \angle POA = .624$ and $\sin \angle QOA = .624$, where $Q$ is the reflection image of $P$ over the $y$-axis. We have found $m\angle POA \approx 38.6^\circ$. Since $m\angle QOB = m\angle POA \approx 38.6^\circ$, $m\angle QOA = 180^\circ - 38.6^\circ = 141.4^\circ$. Thus $\theta \approx 141.4^\circ$.

Notice that $\angle QOB$ and $\angle QOA$ are supplementary. Thus $\angle POA$ and $\angle QOA$ are also supplementary. In general, when $0^\circ < \theta < 180^\circ$, the two solutions to $\sin \theta = k$ are supplementary angles. This result follows from the following theorem.
Supplements Theorem:

For all $\theta$ in degrees

$$\sin \theta = \sin (180^\circ - \theta).$$

The Supplements Theorem is critical when using the Law of Sines to find measures of angles.

**Example 2** In a triangle $ABC$, $a = 13$, $c = 20$, and $m_\angle A = 35^\circ$. Find the measure of $\angle C$.

**Solution** Sketch a picture. It is natural to use the Law of Sines.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 35^\circ}{13} = \frac{\sin C}{20}$$

$$\frac{20 \sin 35^\circ}{13} = \sin C$$

$$0.882 = \sin C$$

Because values of the sine are positive in both the first and second quadrants, there are two possible values for $m_\angle C$ in $\triangle ABC$. One angle is acute (in the first quadrant) and the other is obtuse (in the second quadrant).

A calculator shows $m_\angle C = 61.9^\circ$. Thus, triangle $ABC$ can look like the one at the right.

The obtuse angle whose sine equals 0.882 is $180^\circ - 62^\circ = 118^\circ$. The triangle at the right represents a second solution to the problem.

The situation in Example 2 illustrates the SSA condition, when two sides and a nonincluded angle are given. With SSA, there may be more than one solution. As the next example shows, it is not true that there is always more than one solution.
Example 3  In triangle SPX, \(m \angle S = 75^\circ\), \(s = 11\), and \(x = 9\). Find the measure of \(\angle X\).

**Solution** Since you have the SSA situation, use the Law of Sines.

\[
\frac{\sin S}{s} = \frac{\sin X}{x} = \frac{\sin 75^\circ}{11} = \frac{9 \cdot \sin 75^\circ}{11} \\
.790 = \sin X
\]

A calculator shows \(m \angle X \approx 52^\circ\). The triangle is pictured at the right.

A second angle with sine equal to \(.790\) is the supplement of the first, with measure \(180^\circ - 52^\circ = 128^\circ\). However, \(75^\circ + 128^\circ > 180^\circ\), so these two angles cannot be parts of a triangle.

Thus, in this case, there is only one solution to the problem.

The use of trigonometry to find all the missing measures of sides and angles of a triangle is called **solving the triangle**. When enough information is given, the Law of Cosines and the Law of Sines are all that is needed to solve any triangle.

**Questions**

1. a. According to the figure at the right, \(\cos 59^\circ \approx \ldots\).
   
   b. Give a value of \(\theta\) different from \(59^\circ\) with \(\cos \theta = \cos 59^\circ\).
   
   c. According to the figure, \(\sin 59^\circ \approx \ldots\).
   
   d. Give a value of \(\theta\) different from \(59^\circ\) with \(\sin \theta = \sin 59^\circ\).
2. Solve for $\theta$, where $0^\circ < \theta < 180^\circ$: $\sin \theta = \frac{1}{3}$.

3. In a triangle $ABC$, $m\angle B = 42^\circ$, $c = 13$, and $b = 18$.
   a. Use the Law of Sines to find the measure of angle $C$.
   b. Explain why there is only one solution to part a.

In 4–7, use the Law of Sines or the Law of Cosines to find $x$.

4. 

5. 

6. 

7. 

8. Solve $\triangle JKL$ at the right.
   (Approximate each value to the nearest tenth.)

9. Consider the triangle $RST$, where $\angle R = 52.8^\circ$, $r = 20.1$, and $s = 23.1$.
   a. Find all possible values of the measure of $\angle S$.
   b. For each solution in part a find the length of the third side.

10. A surveyor marks off points $D$, $E$, and $F$ and records that $m\angle D = 40.2^\circ$,
    $d = 100$ m, and $f = 500$ m. Show that there is a problem with the surveyor’s measurements by trying to find $m\angle F$.

11. There is the SSA Triangle Congruence Theorem: If, in two triangles, two sides and the angle opposite the larger side of one are congruent respectively to two sides and the angle opposite the larger side of the other, then the triangles are congruent.

    Take as given: $AB = DE$, $AC = DF$, $\angle C \equiv \angle F$, and $AB > AC$. Use the Law of Sines to show that $\angle B \equiv \angle E$ and thus that the triangles are congruent.
12. To design a map, a cartographer needs to find the distances between city $C$ on one side of a river and cities $A$ and $B$ on the other side. He measured $AB$ to be 130 mi, $m\angle A = 110^\circ$, and $m\angle B = 40^\circ$. Find $AC$ and $BC$. (Lesson 10-7)

In 13–16, give the exact values without using a calculator. (Lessons 10-3, 10-4, 10-5)

13. $\sin(-90^\circ)$
14. $\cos(-60^\circ)$
15. $\sin(390^\circ)$
16. $\cos(720^\circ)$

17. a. Graph $f(x) = x^2$.
   b. On the same set of axes, graph the reflection image of $f$ about $y = x$.
   c. Is $f^{-1}$ a function? (Lessons 7-6, 2-5)

18. Solve $-12 + \sqrt{3}A + 10 = -10$. (Lesson 8-10)

In 19 and 20 refer to the relation graphed below.

19. Is the relation graphed below a function? Why or why not? (Lesson 7-2)

20. Graph the image of the relation above under the translation $T_{10,0}$. (Lesson 4-9)

21. Draw a circle and a triangle $ABC$ with vertices on that circle.
   a. Measure $\angle A$ and side $a$. Verify that the ratio $\frac{a}{\sin A}$ equals the diameter of the circle.
   b. What does $\frac{b}{\sin B}$ equal?
The changing height of each swinging seat can be described as a sine or cosine function.

When \((1, 0)\) is rotated \(\theta\) degrees around the origin its image is the point \((\cos \theta, \sin \theta)\). We can set up a correspondence \(\theta \to \cos \theta\), associating the magnitude of this rotation with the \(x\)-coordinate of the image of \((1, 0)\). This correspondence is a function, because for each \(\theta\) there is only one value for \(\cos \theta\). Similarly, the correspondence \(\theta \to \sin \theta\) is a function that associates \(\theta\) with the \(y\)-coordinate of the image of \((1, 0)\) under \(R_0\).

\[
f: \theta \to \cos \theta \text{ is called the } \textbf{cosine function}.
g: \theta \to \sin \theta \text{ is called the } \textbf{sine function}.
\]

Some ordered pairs of the function \(g(\theta) = \sin \theta\) are given and graphed below. The exact values that you have learned are shown in the table. For instance, since \(\sin 30^\circ = \frac{1}{2}\), the point \((30^\circ, \frac{1}{2})\) is graphed.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>165</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin \theta^\ast)</td>
<td>0</td>
<td>.26</td>
<td>.50</td>
<td>.71</td>
<td>.87</td>
<td>.97</td>
<td>1</td>
<td>.97</td>
<td>.87</td>
<td>.71</td>
<td>.50</td>
<td>.26</td>
</tr>
<tr>
<td>(\sin \theta^{**})</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\sqrt{2})</td>
<td>1</td>
<td>(\sqrt{3})</td>
<td>(\sqrt{2})</td>
<td>(\frac{1}{2})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>180</th>
<th>195</th>
<th>210</th>
<th>225</th>
<th>240</th>
<th>255</th>
<th>270</th>
<th>285</th>
<th>300</th>
<th>315</th>
<th>330</th>
<th>345</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin \theta^\ast)</td>
<td>.26</td>
<td>-.5</td>
<td>-.71</td>
<td>-.87</td>
<td>-.97</td>
<td>-1</td>
<td>-.97</td>
<td>-.87</td>
<td>-.71</td>
<td>-.5</td>
<td>-.26</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(\sin \theta^{**})</td>
<td>-1</td>
<td>(\frac{-\sqrt{2}}{2})</td>
<td>(\frac{-\sqrt{3}}{2})</td>
<td>-1</td>
<td>(\frac{-\sqrt{3}}{2})</td>
<td>(\frac{-\sqrt{2}}{2})</td>
<td>(\frac{-1}{2})</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*decimal approximation
**exact value
As $\theta$ continues to increase beyond $360^\circ$, the rotation images of $(1, 0)$ coincide with previous ones. The value of the $y$-coordinate, $\sin \theta$, repeats itself every $360^\circ$. As a result, the ordered pairs of the function $g(\theta) = \sin \theta$ repeat every $360^\circ$. A more complete graph is below.

The graph of $f(\theta) = \cos \theta$ shown below was constructed by doing a similar analysis of the first coordinate of the rotation image of $(1, 0)$. For instance, because $\cos 30^\circ = \frac{\sqrt{3}}{2} \approx .87$, the point $(30^\circ, .87)$ is graphed.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>165</th>
</tr>
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<tbody>
<tr>
<td>$\cos \theta^*$</td>
<td>1</td>
<td>.97</td>
<td>.87</td>
<td>.71</td>
<td>.50</td>
<td>.26</td>
<td>0</td>
<td>-.26</td>
<td>-.5</td>
<td>-.71</td>
<td>-.87</td>
<td>-.97</td>
</tr>
<tr>
<td>$\cos \theta^{**}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>180</th>
<th>195</th>
<th>210</th>
<th>225</th>
<th>240</th>
<th>255</th>
<th>270</th>
<th>285</th>
<th>300</th>
<th>315</th>
<th>330</th>
<th>345</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \theta^*$</td>
<td>-1</td>
<td>-.97</td>
<td>-.87</td>
<td>-.71</td>
<td>-.5</td>
<td>-.26</td>
<td>0</td>
<td>-.26</td>
<td>-.5</td>
<td>-.71</td>
<td>-.87</td>
<td>-.97</td>
<td>1</td>
</tr>
<tr>
<td>$\cos \theta^{**}$</td>
<td>-1</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*decimal approximation
**exact value
The graphs of the sine and cosine functions have several properties.

1. Because the measure of the angle of rotation, $\theta$, may be any real number, the domain of both the sine and cosine functions is the set of real numbers. Because $\cos \theta$ and $\sin \theta$ are coordinates of points on the unit circle, the range of these functions is the set of real numbers between -1 and 1, inclusive.

2. The sine graph has a y-intercept of 0 and x-intercepts of the even multiples of $90^\circ$; that is, ... , $-180^\circ$, 0, $180^\circ$, $360^\circ$, $540^\circ$, .... The cosine graph has a y-intercept of 1 and x-intercepts of the odd multiples of $90^\circ$, that is, ... , $-270^\circ$, $-90^\circ$, $90^\circ$, $270^\circ$, $450^\circ$, ....

3. A function is periodic if its graph can be mapped to itself under a horizontal translation. Both the sine and cosine functions are periodic. For each function the period is $360^\circ$. This means that $\sin (\theta \pm 360n^\circ) = \sin \theta$ and $\cos (\theta \pm 360n^\circ) = \cos \theta$ where $n$ is any integer.

4. The graphs of $f(\theta) = \cos \theta$ and $g(\theta) = \sin \theta$ are congruent. Each can be mapped to the other with a horizontal translation of $90^\circ$. For this reason both are often called sine waves. They are said to be sinusoidal.

**Definition:**

A sine wave is a graph which can be mapped onto the graph of $g(\theta) = \sin \theta$ by any composite of reflections, translations, and scale changes.

Sine waves have many applications. Pure sound tones travel in sine waves; these can be pictured on an oscilloscope, as shown at the left. The path of a satellite as it travels around the earth is an approximate sine wave, and the time of sunrise for a given location over the year also shows sinusoidal behavior.

**Questions**

1. The function $f: \theta \rightarrow \cos \theta$ maps $\theta$ onto the $-\text{-coordinate of the image of (1, 0) under } R_\theta$.

2. The function $g: \theta \rightarrow \sin \theta$ maps $\theta$ onto the $-\text{-coordinate of the image of (1, 0) under } R_\theta$.

In 3 and 4, for each function: (a) name two points on the function; (b) give its domain; (c) give its range.

3. $f(\theta) = \cos \theta$

4. $g(\theta) = \sin \theta$
5. As $\theta$ increases from $0^\circ$ to $90^\circ$, $\sin \theta$ increases from $\frac{1}{2}$ to $-1$.

6. As $\theta$ increases from $90^\circ$ to $180^\circ$, $\sin \theta$ decreases from $-1$ to $\frac{1}{2}$.

7. As $\theta$ increases from $180^\circ$ to $270^\circ$, does the value of $g(\theta) = \sin \theta$ increase or decrease?

8. As $\theta$ increases from $270^\circ$ to $360^\circ$, how do the values of $\sin \theta$ change?


In 10–12, true or false.

10. The function $f(\theta) = \cos \theta$ is periodic.

11. The cosine function intersects the $x$-axis at $-720^\circ$.

12. $\cos \theta = \cos (360^\circ + \theta)$ for all $\theta$.

13. a. On the same set of axes, graph $f(\theta) = \cos \theta$ and $g(\theta) = \sin \theta$ over the interval $-360^\circ \leq \theta \leq 360^\circ$.
   b. Find all values of $\theta$ between $-360^\circ$ and $360^\circ$ such that $\cos \theta = \sin \theta$.

14. Let $h(\theta) = 2 + \sin \theta$.
   a. Use the Graph Translation Theorem to predict where the graph of $h$ will lie on the coordinate plane, and what its maximum and minimum values will be.
   b. Graph this function.
   c. Is this function a sine wave? Justify your answer.

15. Multiple choice Which choice completes a symbolic definition of “periodic function”?
   $f$ is periodic if and only if there is a number $p$ such that for all $x$:
   (a) $f(x + p) = f(x)$    (c) $f(x) + f(p) = f(x + p)$
   (b) $p \cdot f(x) = f(px)$    (d) $f(x) + p = f(x)$

In 16–19, part of a function is graphed. (a) Is the function periodic? (b) If so, what is the period?

16. 

17. 

18. 

19. 

LESSON 10-9 The Cosine and Sine Functions 589
In 20 and 21, complete each statement with a trigonometric expression to make the equation true.

20. \((\sin \theta)^2 + (__)^2 = 1\) (Lesson 10-3)

21. \(\sin (90^\circ - \theta) = \)? (Lesson 10-3)

22. At what angle \(\theta\) must each side of a 200-foot drawbridge be raised to create a 100-foot gap? (Lesson 10-2)

![Diagram of drawbridge with a 100 ft gap and \(\theta\) angle]

23. In a triangle \(GHI\), \(g = 15\), \(i = 21\), and \(m\angle G = 42^\circ\). Find the measure of \(\angle I\). (Lesson 10-8)

24. In a triangle \(JKL\), \(m\angle L = 81^\circ\), \(l = 20\), \(k = 21\). Find the measure of \(\angle K\). (Lesson 10-8)

25. The circle at the right is tangent to the axes at \((8, 0)\) and \((0, 8)\). Find its area and circumference. (Previous course)

26. An observer in a lighthouse on the shore sees a ship in distress. The ship is 15 miles away at an angle of 20\(^\circ\) to the shoreline. A Coast Guard station is on the shoreline 30 miles away from the lighthouse.

![Diagram of lighthouse, ship, and station]

a. How far will a Coast Guard rescue ship have to travel from the station to reach the ship? (Lesson 10-6)

b. The path of the rescue vessel should be at what angle to the shoreline? (Hint: use your answer from part a.) (Lesson 10-7)
27. The following table lists the time of sunset each Sunday of 1966 for Denver, Colorado. (Daylight savings time has been ignored.)

a. Accurately graph an appropriate function.

b. Is the graph of sunset times periodic? If so, what is its period?

<table>
<thead>
<tr>
<th>Date</th>
<th>Sunset Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>5:46</td>
</tr>
<tr>
<td>1/9</td>
<td>5:33</td>
</tr>
<tr>
<td>1/16</td>
<td>6:33</td>
</tr>
<tr>
<td>1/23</td>
<td>7:32</td>
</tr>
<tr>
<td>1/30</td>
<td>8:26</td>
</tr>
<tr>
<td>2/6</td>
<td>9:10</td>
</tr>
<tr>
<td>2/13</td>
<td>10:01</td>
</tr>
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<td>2/20</td>
<td>11:04</td>
</tr>
<tr>
<td>2/27</td>
<td>12:07</td>
</tr>
<tr>
<td>3/6</td>
<td>1:00</td>
</tr>
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<td>3/20</td>
<td>3:00</td>
</tr>
<tr>
<td>3/27</td>
<td>4:00</td>
</tr>
</tbody>
</table>

591
So far in this chapter you have learned to evaluate \( \sin x \), \( \cos x \), and \( \tan x \) when \( x \) has been given in degrees. Another unit, called the **radian**, is widely used when measuring angles or magnitudes of rotation. In fact, in some later mathematics courses radians are used more than degrees. Here is the idea behind the radian.

Since the radius of a unit circle is the number 1, the circumference of the unit circle is \( 2\pi \). Thus, on a unit circle, a \( 360^\circ \) arc has a length of \( 2\pi \). Similarly, a \( 180^\circ \) arc has a length of \( \pi \), and a \( 90^\circ \) arc has a length of

\[
\frac{1}{4}(2\pi) \text{ or } \frac{\pi}{2}.
\]

The radian is a unit created so that the arc measure and the arc length use the same number.

**Definition:**

The radian is a unit of angle, arc, or rotation measure such that

\[
\pi \text{ radians} = 180^\circ.
\]

The definition indicates that a \( 180^\circ \) angle has measure \( \pi \) radians, and its arc has length \( \pi \). Thus a \( 90^\circ \) angle has measure \( \frac{\pi}{2} \) radians, and its arc has length \( \frac{\pi}{2} \). To repeat, **the measure of an angle in radians equals the length of its arc on the unit circle.** This is one major reason for using radians.

The definition can be transformed to give the two conversion factors for changing degrees into radians and vice versa, without a calculator. Begin with

\[
\pi \text{ radians} = 180^\circ.
\]

Divide both sides by \( \pi \) radians.

\[
1 = \frac{180 \text{ degrees}}{\pi \text{ radians}}
\]

Divide both sides by 180 degrees.

\[
\frac{\pi \text{ radians}}{180 \text{ degrees}} = 1
\]
So to convert radians to degrees, multiply by $\frac{180\text{ degrees}}{\pi\text{ radians}}$; to convert degrees to radians, multiply by $\frac{\pi\text{ radians}}{180\text{ degrees}}$.

**Example 1** Convert $45^\circ$ to radians.

**Solution 1** Multiply by one of the conversion factors. Because you want radians, choose the ratio with radians in the numerator.

$$45^\circ \cdot \frac{\pi\text{ radians}}{180^\circ} = \frac{45^\circ}{180^\circ} \pi\text{ radians}$$

$$= \frac{\pi}{4}\text{ radians}$$

**Solution 2** A scientific calculator usually has a $\text{DRG}$ key to convert from one unit to another. Key in $45\text{ INV DRG}$. You should see $\text{RAD}$ on your display along with $0.7853981\ldots$. This approximates $\frac{\pi}{4}$.

The answer to Example 1 should make sense to you. If $180^\circ$ equals $\pi$ radians, then $\frac{\pi}{4}$ is $\frac{1}{4}$ of $\pi$, and $\frac{1}{4}$ of $180^\circ$ is $45^\circ$. The diagram below shows other common equivalences.

Radian expressions are usually left in terms of $\pi$ because this form gives an exact value. Since magnitudes of rotation and angle measures are often given in radians, the word radian or abbreviation $rad$ is usually omitted.

Many calculators will not easily convert radians to degrees, so you may need to do the conversion by hand.
Example 2  Convert 2 radians to degrees.

\[
\text{Solution 1} \quad 2 \text{ radians} = 2 \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}}.
\]
\[
= \frac{360^\circ}{\pi}
\]
\[
\approx 114.6^\circ
\]

Notice that one radian is much larger than one degree. (1 rad \approx 57^\circ)

Solution 2  Use a calculator.
Press \( \text{drg} \) until the screen of your calculator displays RAD. Then key in \(2 \text{ inv} \ \text{drg} \), repeating \( \text{inv} \ \text{drg} \) until the screen displays DEG. You should see 114.591...

The multiples of \( \pi \) and the simplest divisors of \( \pi \) (e.g., \( \frac{\pi}{2}, \frac{\pi}{3}, \) and \( \frac{\pi}{4} \)) correspond to those angle measures which give exact values of cosines and sines.

Example 3  Evaluate \( \sin \left( \frac{\pi}{4} \right) \) on your calculator.

Solution  Since there is no degree symbol, we assume \( \frac{\pi}{4} \) means \( \frac{\pi}{4} \) radians. Put your calculator in radian mode by pressing \( \text{drg} \) until RAD appears on the screen. Now press \( ( \frac{\pi}{4} ) \ \sin \) The result is about .707.

Check 1  From Example 1, \( \frac{\pi}{4} = 45^\circ \). You know \( \sin 45^\circ = \frac{\sqrt{2}}{2} \), so \( \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \) which is about .707.

Check 2  Return the calculator to the DEG setting. Press 45 \( \sin \). You should get approximately .707 again.
The cosine and sine functions are graphed below with radians instead of degrees. Notice that the scales on the x-axis and y-axis can be equal and the periods are \(2\pi\), a simpler number (in some ways) than \(360^\circ\).

![Graph of cosine and sine functions](image)

Questions

1. A circle has a radius of 1 unit. Give the length of an arc whose measure is:
   a. \(360^\circ\)   b. \(180^\circ\)   c. \(90^\circ\).

2. On a circle of radius 1 meter, find the length of a \(45^\circ\) arc.

3. State how radians and degrees are related.

In 4–7, convert to radians. Give your answer as a number times \(\pi\).

4. \(90^\circ\)   5. \(60^\circ\)
6. \(225^\circ\)   7. \(30^\circ\)

In 8 and 9, convert the radian measure to degrees.

8. \(\frac{\pi}{6}\)   9. \(-\frac{5\pi}{4}\)

In 10–12, evaluate.

10. a. Evaluate \(\sin \left(\frac{3\pi}{2}\right)\) on your calculator.
    
    b. Check your answer to part a, using degrees.

11. \(\cos \frac{\pi}{3}\)

12. \(\tan \frac{\pi}{6}\)

13. In radians, what is the period of the sine function?
14. How far apart are the hands of a clock at 2:00,
a. in degrees?
b. in radians?

15. Copy and complete this unit circle for equivalent degrees and radians.
(All lines which appear to be straight are diameters.)

In 16 and 17, find the exact values.

16. \( \sin \left( \frac{7\pi}{6} \right) \)  
17. \( \cos \left( \frac{15\pi}{4} \right) \)

In 18 and 19, use this relationship between radian measure and arc length.
In a circle of radius \( r \), an angle of \( x \) radians has an arc of length \( rx \).

18. a. How long is the arc of a \( \frac{\pi}{4} \) radian angle in a circle of radius 20?
b. How long is a 45° arc in a circle of radius 20?

19. How long is the arc of a \( \frac{2\pi}{3} \) radian angle in a circle of radius 6 feet?

20. The gradient (abbreviated grad) is another unit for measuring angles.
   It is based on one quarter of a circle having 100 grads.
a. How many grads are in a full circle?
b. How many grads equal 45°?
c. \( \sin 45° \approx .707 \). Use your answer to part b and the \( \Box \) key to verify this value for grads.

21. State (a) the domain and (b) the range of the function \( y = \sin x \).
   (Lesson 10-9)
22. The newspaper article at the right is from the Detroit Free Press, January 29, 1985. Explain why the construction technique leads to an angle of 26.5° using the drawing below. (Lesson 10-2)

Astronomer Solves Riddle of Pyramid

United Press International

A Navy astronomer has devised a surprisingly simple explanation for the angle of a descending passageway in the Great Pyramid of Cheops in Egypt.

In the early 19th Century, English astronomer John Herschel suggested the 377-foot-long passageway was built at its angle of 26.5230° to point at the North Star, making the pyramid an astronomical observatory as well as a tomb for Cheops.

Richard Walker, a U.S. Naval Observatory astronomer based at Flagstaff, Ariz., checked Herschel's idea and found that, because of the wobble of Earth's axis in its orbit around the sun, no prominent star could have been seen from the base of the passageway in 2600 BC when the pyramid was built.

Then why was the passageway inclined at an angle of 26.5 degrees?

According to Walker's report, the angle merely was the result of the construction technique.

By placing three stones of equal length horizontally and then placing a fourth stone of equal size on the top of the third horizontal stone, Walker determined that the angle from the top stone to bottom stone at the other end is 26.5 degrees.

23. One of Murphy's Laws states that the time a committee spends debating a budget item is inversely proportional to the number of dollars involved. Suppose the function is \( t = \frac{1}{d} \), where \( t \) is measured in minutes and \( d \) is in hundreds of dollars.

a. How much time is spent on a $300 item?

b. Graph the equation with \( d \) as the independent variable. (Lesson 2-2)

In 24 and 25 sketch a graph. (Lessons 6-1, 3-2)

24. \( y = 2x^2 + 32x + 128 \)

25. \( y = \frac{x}{2} + 100 \)

26. When \( x \) is measured in radians, \( \sin x \) can be estimated by the expression \( \sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \).

a. How close is the value of this expression to \( \sin x \) when \( x = \frac{\pi}{4} \)?

b. To get greater accuracy, you can add \( \frac{x^9}{362880} \) to the value you got in part a. Where does the denominator come from?
Summary

Trigonometry is the study of relations between sides and angles in triangles. In a right triangle, three important trigonometric ratios are the following:

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} \\
\cos \theta &= \frac{\text{adj}}{\text{hyp}} \\
\tan \theta &= \frac{\text{opp}}{\text{adj}}
\end{align*}
\]

The sine, cosine, and tangent ratios are frequently used to find lengths or angle measures in situations that are modeled by right triangles.

Some of the trigonometric ratios may be calculated exactly.

\[
\begin{align*}
\sin 30^\circ &= \cos 60^\circ = \frac{1}{2} \\
\sin 45^\circ &= \cos 45^\circ = \frac{\sqrt{2}}{2} \\
\sin 60^\circ &= \cos 30^\circ = \frac{\sqrt{3}}{2}
\end{align*}
\]

Others are given as decimal approximations by a calculator or computer.

Lengths of sides or angle measures in nonright triangles may be determined using either the Law of Cosines or the Law of Sines. In any triangle \(ABC\),

\[
\begin{align*}
c^2 &= a^2 + b^2 - 2ab \cos C \quad \text{(Law of Cosines)} \\
\frac{\sin A}{a} &= \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{(Law of Sines)}
\end{align*}
\]

The Law of Cosines is most useful when an SAS or SSS condition is given; the Law of Sines is used in all other situations that determine triangles. When the Law of Sines is used to find an angle in an SSA condition, two solutions may be possible.

By considering rotations of magnitude \(\theta\) of the point \((1, 0)\) around the origin, the trigonometric ratios can be generalized to find sines and cosines for any real number \(\theta\). On the unit circle,

\[
\begin{align*}
\cos \theta &= \text{the } x\text{-coordinate of the image of } (1, 0) \text{ under } R_\theta \\
\sin \theta &= \text{the } y\text{-coordinate of the image of } (1, 0) \text{ under } R_\theta.
\end{align*}
\]

The correspondences

\[
\begin{align*}
f: \theta &\to \sin \theta \\
g: \theta &\to \cos \theta
\end{align*}
\]

are functions whose domains are the set of real numbers and whose ranges are \(\{y : -1 \leq y \leq 1\}\).
Whether considered as ratios in right triangles, coordinates on a unit circle, or values of functions, \( \cos \theta \) and \( \sin \theta \) satisfy several properties for all \( \theta \).

\[
\begin{align*}
\sin \theta &= \cos (90^\circ - \theta) & \text{(Complements Theorem)} \\
\cos \theta &= \sin (90^\circ - \theta) \\
\sin \theta &= \sin (180^\circ - \theta) & \text{(Supplements Theorem)} \\
(\cos \theta)^2 + (\sin \theta)^2 &= 1 & \text{(Pythagorean Identity)} \\
\sin \theta &= \sin (\theta \pm 360n^\circ) & \text{(Periodicity)} \\
\cos \theta &= \cos (\theta \pm 360n^\circ)
\end{align*}
\]

For all these properties, \( \theta \) may be in radians; if so, the degree measure should be replaced by its radian equivalent. For example, the Supplements Theorem would be \( \sin \theta = \sin (\pi - \theta) \).

**Vocabulary**

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with an *.
For all other terms you should be able to give a general description and a specific example of each.

**Lesson 10-1**
- trigonometric ratios
  - *sine of \( \theta \), \( \sin \theta \)
  - *cosine of \( \theta \), \( \cos \theta \)
  - *tangent of \( \theta \), \( \tan \theta \)

**Lesson 10-2**
- angle of elevation
- angle of depression

**Lesson 10-3**
- Complements Theorem
- Exact Value Theorem
- Pythagorean Identity

**Lesson 10-4**
- unit circle

**Lesson 10-6**
- Law of Cosines Theorem

**Lesson 10-7**
- Law of Sines Theorem

**Lesson 10-8**
- Supplements Theorem
- solving a triangle

**Lesson 10-9**
- *cosine function
- *sine function
- periodic
- sine wave
- sinusoidal function

**Lesson 10-10**
- radian, rad
Take this test as you would take a test in class. You will need a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

In 1 and 2, use the triangle below. Round answers to the nearest thousandth. Find:

1. \( \cos \theta \)  
2. \( \sin \theta \)
3. If \( \tan \theta = 1 \) and \( 0 \leq \theta \leq 90^\circ \), what is \( \theta \)?
4. Name all points which are images of \((1, 0)\) under \( R_{120} \).  
   (a) \((\cos 120^\circ, \sin 120^\circ)\)  
   (b) \((-0.5, 0.866)\)  
   (c) \((-\frac{1}{2}, \frac{\sqrt{3}}{2})\)

In 5 and 6, a 14-foot ladder is leaning against a wall. The base of the ladder is 7 feet from the wall.

5. Find the angle of elevation of the ladder.
6. To the nearest inch, how high up the wall does the ladder reach?
7. For what value of \( x \) such that \( 0^\circ < x \leq 360^\circ \) and \( x \neq 57 \) does \( \cos 57^\circ = \cos x^\circ \)?
8. Find an exact value for \( \cos 210^\circ \).

In 9–11, use the graph below.

9. The period of the function is \( ? \).

10. As \( \theta \) increases from \( 90^\circ \) to \( 180^\circ \), the value of \( \sin \theta \) decreases from \( ? \) to \( ? \).
11. What is the range of the sine function?
12. Graph \( y = \cos x \) for \( -90^\circ \leq x \leq 270^\circ \).
13. An observer of a road race estimates that runner A is 110 meters away and runner B is 85 meters away from the observation post. The angle between the sightings is \( 40^\circ \). How far apart are the runners from each other?

14. In 14 and 15, find \( x \).
15.

16. In \( \triangle SLM \), \( m\angle S = 110^\circ \), \( s = 525 \), and \( l = 421 \). Find \( m\angle L \).
17. A 2-foot tall eagle is perched on a 6-foot high road sign. It flies 130 feet directly to its nest on the side of a cliff. The angle of elevation of the flight path (from the eagle's beak) is about \( 70^\circ \). About how high off the ground is the nest?
18. Convert \( \frac{\pi}{3} \) radians to degrees.
19. Find the exact value of \( \sin \frac{7\pi}{6} \).
20. Multiple choice. Which of the following statements is not true?  
   (a) \( \sin (90^\circ - \theta) = \cos \theta \)  
   (b) \( \cos 690^\circ = \frac{1}{2} \)  
   (c) \( (\sin \theta)^2 + (\cos \theta)^2 = 1 \)  
   (d) \( \sin \frac{\pi}{2} = -1 \)
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

**Objective A.** Approximate values of trigonometric functions using a calculator. (Lessons 10-1, 10-5, 10-10)

In 1–6, evaluate. Round your answer to the nearest hundredth.

1. \( \sin 17^\circ \)
2. \( \cos 143^\circ \)
3. \( \sin (-50^\circ) \)
4. \( \cos \frac{\pi}{3} \)
5. \( \sin \frac{11\pi}{6} \)
6. \( \tan \frac{\pi}{12} \)

In 7–9, use the triangle at the right. Approximate each trigonometric value to the nearest thousandth.

7. \( \sin \theta \)
8. \( \cos \theta \)
9. \( \tan \theta \)

**Objective B.** Find exact values of trigonometric functions of certain angles. (Lessons 10-3, 10-4, 10-5, 10-10)

In 10–15, give exact values.

10. \( \cos 45^\circ \)
11. \( \sin 405^\circ \)
12. \( \tan 30^\circ \)
13. \( \cos \left(-\frac{\pi}{6}\right) \)
14. \( \sin \frac{3\pi}{2} \)
15. \( \tan \frac{\pi}{4} \)

**Objective C.** Determine the measure of an angle given its trigonometric values. (Lessons 10-2, 10-8)

In 16–18, find all \( \theta \) between 0° and 180° with the given trigonometric value.

16. \( \cos \theta = .5 \)
17. \( \sin \theta = \frac{\sqrt{2}}{2} \)
18. \( \sin \theta = 1 \)

In 19–21, solve for all \( \theta \) between 0 and \( \frac{\pi}{2} \).

19. \( \tan \theta \approx .466 \)
20. \( \cos \theta \approx .309 \)
21. \( \sin \theta = \frac{2}{3} \)

**Objective D.** Convert angle measures from radians to degrees or degrees to radians. (Lesson 10-10)

In 22–25, convert to radians.

22. 30°
23. 105°
24. 360°
25. 540°

In 26–29, convert to degrees.

26. \( \pi \)
27. \( \frac{9\pi}{4} \)
28. \( \frac{5\pi}{3} \)
29. \( \frac{-\pi}{8} \)

**Objective E.** Find missing parts of a triangle using the Law of Sines or the Law of Cosines. (Lessons 10-6, 10-7, 10-8)

30. Find \( BC \).
31. Find \( m\angle E \).
32. Find \( GH \).
33. Find \( JK \).

---

CHAPTER 10  Chapter Review  601
34. In \( \triangle WET \), \( m\angle W = 112^\circ \), \( w = 9 \), and \( e = 7 \). Find the measure of \( \angle E \).

35. In \( \triangle JHS \), \( j = 2 \), \( s = 3 \), and \( m\angle J = 25^\circ \). Find \( m\angle S \).

**PROPERTIES** deal with the principles behind the mathematics.

- **Objective F.** Identify and use definitions and theorems relating sines and cosines. (Lesson 10-8)
  
  In 36–39, true or false. If false, change it so that it is true.
  
  36. \( (\sin \theta)^2 + (\cos \theta)^2 = 1 \)
  37. \( \cos (90^\circ - \theta) = \sin \theta \)
  38. \( \sin (180^\circ - \theta) = \cos \theta \)
  39. \( \cos \theta = \sin (90^\circ - \theta) \)

**USES** deal with applications of mathematics in real situations.

- **Objective G.** Solve real-world problems using the trigonometry of right triangles. (Lessons 10-1, 10-2)
  
  44. How tall is the building pictured below if a person 6 feet tall sights the top of the building at 49\(^\circ\) while standing 53 feet away?

  ![Building Diagram]

  45. A ship sails 695 kilometers on a bearing of 75\(^\circ\). How far east of its original position is the ship?

  46. A wheelchair ramp must be built so that it has a slope of \( \frac{1}{12} \). What angle will the ramp make with the horizontal?

  47. An airplane begins a smooth final descent to the runway from an altitude of 5,000 feet when it is 30,000 horizontal feet away. At what angle of depression will the plane descend?

  48. The ancient Greeks carved amphitheaters out of the sides of hills. Suppose one amphitheater went 200 feet vertically down while covering 300 feet horizontally. At what angle of depression did they dig?

  ![Amphitheater Diagram]

  Theater at Epidaurus, Greece, from about 325 B.C.
**Objective H.** Solve real-world problems using the Law of Sines or Law of Cosines. (Lessons 10-6, 10-7)

49. An ocean liner pilot spots a freighter and a schooner on the horizon. The freighter is 7.5 miles away and the schooner is 8 miles away. The angle between the two sightings is 16°. How far from the schooner is the freighter?

50. The observers in the lookout towers of two ships 8 miles apart spot land ahead. The observer in ship A spots land at an angle of 44° to the line between the two ships, while the observer in ship B spots land at an angle of 105° to the same line. How far is the land from ship A?

51. Two observers are in lighthouses 75 miles apart, as shown below. The observer in the northern lighthouse spots a ship in distress at an angle of 15° to the line between the lighthouses. The other observer spots the ship at an angle of 35°. How far is the ship from each lighthouse?
CHAPTER 11

Polynomials

11-1 Polynomial Models
11-2 Polynomials and Geometry
11-3 Factoring Polynomials
11-4 The Factor Theorem
11-5 Estimating Zeros of Polynomial Functions
11-6 Solving All Polynomial Equations
11-7 Finite Differences
11-8 Modeling Data with Polynomials

The twin towers of the World Trade Center district of Manhattan Island.
The population of Manhattan Island (part of New York City) has gone up and down over the past 100 years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1890</td>
<td>1,441,216</td>
</tr>
<tr>
<td>1910</td>
<td>2,331,542</td>
</tr>
<tr>
<td>1930</td>
<td>1,867,312</td>
</tr>
<tr>
<td>1950</td>
<td>1,960,101</td>
</tr>
<tr>
<td>1970</td>
<td>1,539,233</td>
</tr>
</tbody>
</table>

None of the kinds of functions you have studied fits these points very well. However, if \( x = \) the number of 20-year periods since 1890, then the population \( P(x) \) of Manhattan (in ten thousands) is closely approximated by the equation

\[
P(x) = \frac{-37}{3} x^4 + \frac{317}{3} x^3 - \frac{1789}{6} x^2 + \frac{1763}{6} x + 144.
\]

The above equation is a polynomial equation, and the function \( P \) is a polynomial function. Although the formula for \( P(x) \) is quite complicated, mathematicians would not be surprised by it. For any finite set of points, no two of which are on the same vertical line, there is a polynomial function whose graph contains those points. A graph of \( y = P(x) \) on the domain \(-1 \leq x \leq 5\) is shown below.

In this chapter, you will study situations that lead to polynomial functions. You will see how to graph and analyze such functions, learn when data can be described by a polynomial model, and learn how to find the model for specific data points. Along the way, you will encounter various properties of polynomials, some of which you have studied before.
The expression \( \frac{37}{3}x^4 + \frac{317}{3}x^3 - \frac{1789}{6}x^2 + \frac{1763}{6}x + 144 \) from the previous page is a polynomial in the variable \( x \). When the polynomial is in only one variable, the largest exponent is the degree of the polynomial. This polynomial has degree 4. The expressions \( \frac{37}{3}x^4 \), \( \frac{317}{3}x^3 \), \( -\frac{1789}{6}x^2 \), \( \frac{1763}{6}x \), and 144 are terms of the polynomial. We have written the terms in decreasing order of the exponents. Polynomials are commonly written in this order to make them easier to read.

**Definition:**

A polynomial in \( x \) is an expression of the form

\[
a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0,
\]

where \( n \) is a positive integer and \( a_n \neq 0 \).

The definition displays the general form of a polynomial. The number \( n \) is the degree of the polynomial and the numbers \( a_n, a_{n-1}, a_{n-2}, \ldots, a_0 \) are its coefficients. The number \( a_n \) is called the leading coefficient of the polynomial. For instance, when \( n = 4 \), the degree is 4 and the subscript for the leading variable is also 4.

\[
anx^n + an-1x^{n-1} + an-2x^{n-2} + \ldots + a1x + a0
\]

We can say that on page 605, the population of Manhattan for twenty-year periods from 1890 to 1970 has been modeled by a 4th degree polynomial. The leading coefficient of this polynomial is \( \frac{37}{3} \). Polynomials of the first degree, such as \( mx + b \), are called linear polynomials. Those of the second degree, such as \( ax^2 + bx + c \), are called quadratic polynomials, and those of the third degree, such as \( ax^3 + bx^2 + cx + d \), are cubic polynomials.

A polynomial function is a function whose rule can be written as a polynomial. You can evaluate polynomial functions in the same way that you evaluate other functions. For instance, consider the function \( P(x) = 6x^5 - 3x^4 + 4x^2 - 2x - 7 \). The value of this function when \( x = 2 \) is represented by \( P(2) \), and \( P(2) = 6(2)^5 - 3(2)^4 + 4(2)^2 - 2(2) - 7 = 149 \).

**Example 1**

Consider the polynomial function \( P \) modeling the population of Manhattan.

a. Find \( P(1) \).

b. \( P(1) \) approximates the population for which year?

**Solution**

a. \( P(1) = \frac{-37}{3}(1)^4 + \frac{317}{3}(1)^3 - \frac{1789}{6}(1)^2 + \frac{1763}{6}(1) + 144 \)

\[= 233 \]
b. \( P(1) \) gives the population in ten thousands in one 20-year period after 1890, that is, the population in 1910.

Check \( 233 \cdot 10,000 = 2,330,000 \), which is close to the 1910 population listed in the table.

The calculation of compound interest can involve polynomial functions of any degree. You have learned that the compound interest formula \( A = P(1 + r)^t \) gives you the value of \( P \) dollars invested at an annual interest rate \( r \) after \( t \) years. If amounts of money are deposited for different periods of time, then this formula must be applied to each amount separately. Example 2 illustrates such a situation.

Example 2 Starting with the summer after her senior year in high school, Yolanda Fish worked to earn money for medical school. At the end of each summer she put her money in a savings account with an annual yield of 6%. How much will be in her account when she goes to medical school, if no other money is added or withdrawn? (Assume Yolanda goes to medical school in the fall following her 4th year in college.)

<table>
<thead>
<tr>
<th>summer</th>
<th>earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>after senior year</td>
<td>$1000</td>
</tr>
<tr>
<td>after 1st year of college</td>
<td>1500</td>
</tr>
<tr>
<td>after 2nd year of college</td>
<td>1400</td>
</tr>
<tr>
<td>after 3rd year of college</td>
<td>2000</td>
</tr>
<tr>
<td>after 4th year of college</td>
<td>2200</td>
</tr>
</tbody>
</table>

Solution The money put in the bank after her senior year earns interest for 4 years. It is worth \( 1000(1.06)^4 \) when Yolanda goes to medical school. Similarly, the amount saved at the end of her first year of college is worth \( 1500(1.06)^3 \). When the values from each summer are added together, we find the total amount that will be in Yolanda's account.

\[
1000(1.06)^4 + 1500(1.06)^3 + 1400(1.06)^2 + 2000 (1.06) + 2200
\]

from summer after senior year from summer after 1st year college from summer after 2nd year college from summer after 3rd year college from senior after 4th year college

Evaluating this expression shows that Yolanda will have about $8942 in her account.
In Example 2, if you don’t know the interest rate, it’s reasonable to replace 1.06 with $x$. Then when she goes to medical school Yolanda will have

$$1000x^4 + 1500x^3 + 1400x^2 + 2000x + 2200.$$ 

This expression gives the amount in the account for any interest rate compounded annually. If the interest rate is $r$, just substitute $1 + r$ for $x$ and find the new total. Be careful when constructing an expression in a situation like this. Check whether or not the last amount saved earns interest. You can see that if the first deposit has earned interest for $n$ years, the result is a polynomial of degree $n$.

Questions

In 1–3, tell whether or not the expression is a polynomial. If it is, state its degree and leading coefficient.

1. $4x + 7$  
2. $7x^4 - 12x^2 + 100$  
3. $x^2 + x^1 + 1$

In 4–6, write the general form of:

4. a cubic polynomial in the variable $x$  
5. a fifth degree polynomial in the variable $y$  
6. an $n$th degree polynomial in the variable $x$

7. Refer to the definition of an $n$th degree polynomial and the polynomial  
   $5x^7 + 4x^6 - 8x^3 + 1.3x^2 - x$.

   State the value of each of the following.
   a. $n$  
   b. $a_n$  
   c. $a_{n-1}$  
   d. $a_0$  
   e. $a_1$  
   f. $a_2$  
   g. $a_5$

In 8 and 9, refer to the population function $P$ for Manhattan Island.

8. What is the leading coefficient of $P$?

9. a. Evaluate $P(3)$.
   b. How close is $P(3)$ to the 1950 population?

10. Refer to Example 2. Suppose that in successive summers beginning after eighth grade, Yolanda saved $200, $500, $1475, $1600, and $1300.
   a. If the bank pays 6% interest compounded yearly and no other money is added or withdrawn, how much is in her account when she goes to college? (Assume Yolanda goes to college in the fall immediately after finishing high school.)
   b. Let $x = 1.06$. Write a polynomial in $x$ to give the amount in Yolanda’s account.
   c. What is the degree of the polynomial in part b?
11. On her first birthday, Jennifer got $25. On each successive birthday she got twice as much money. The money was put into a bank paying 7% interest, compounded annually. No additional money was added or withdrawn.

a. Write a polynomial expression to give the total amount in Jennifer’s account on her sixth birthday. (The money from her sixth birthday earns no interest.) Do not calculate the total.

b. Replace 1.07 by $x$ and write a polynomial in $x$ that gives the total amount in the account.

c. What is the degree of the polynomial in part b?

In 12 and 13, recall the formula for the height of an object thrown upward:

$$h(t) = -\frac{1}{2} gt^2 + v_0t + h_0$$

where $t$ is the number of seconds after being thrown, $h_0$ the initial height, $v_0$ the initial velocity and $g$ the acceleration due to gravity ($32$ ft/sec$^2$).

This formula describes a polynomial function in $t$.

12. What is the degree of this polynomial?

13. Suppose a ball is thrown upward from the ground with initial velocity 45 ft/sec. Find its height after .9 seconds.

14. Consider the sequences $A$ and $B$ below. (Lessons 8-5, 3-7, 3-6)

A: 16, 4, 1, $\frac{1}{4}$, ...  
B: 53, 41, 29, 17, ...

a. Write the next two terms of each sequence.

b. Write an explicit formula for the geometric sequence.

c. Write a recursive formula for the arithmetic sequence.

d. Which sequence might model the consecutive heights of a bouncing ball?

15. A cheetah trots along at 5 mph for a minute, then spies a small deer and speeds up to 60 mph in just 6 seconds. After chasing the deer at this speed for 30 seconds, the cheetah gives up and, over the next 20 seconds, slows to a stop. Graph this situation, plotting time on the horizontal axis and speed on the vertical axis. (Lesson 3-8)

16. Graph $y = 2x^2 - 4x + 1$. (Lessons 6-5)

17. Multiply $5(7x + 2)(3x - 1)$. (Lessons 1-5)

18. Find an expression for the volume of the rectangular solid below. (Previous course)
In 19 and 20, refer to the population of Manhattan given on the opening page of the chapter. (Lesson 2-4)

19. Find the rate of change in population per year for the period 1890 to 1910.

20. What was the average yearly change in population between 1950 and 1970

21. What did the keeper say to the parrot who needed to go on a diet?

22. Use the Manhattan Island population function given on page 605.
   a. Graph P using a function plotter.
   b. Use the graph to estimate the population in 1900, 1920, 1940, and 1960.
   c. Find the actual population of Manhattan Island in 1900, 1920, 1940, and 1960.
   d. How close are the estimates in part b to the actual values?
   e. According to the function P, when did Manhattan’s population peak?
   f. How good is P(x) in approximating the population of Manhattan for other dates?
Polynomials are sometimes classified according to the number of terms they have. For instance, a monomial is a polynomial with one term; a binomial is a polynomial with two terms; and a trinomial is a polynomial with three terms. Below are some examples.

- **Monomial**: $-7, x^2, 3y^4$
- **Binomial**: $x^2 - 11, 3y^4 + y, 12a^5 + 4a^3$
- **Trinomial**: $x^2 - 5x + 6, 10y^6 - 9y^5 + 17y^2$

Notice that monomials, binomials, and trinomials can be of any degree.

When a polynomial in one variable is added to or multiplied by a polynomial in another variable, the result is called a polynomial in several variables. The degree is the largest sum of the exponents of the variables in any term. For instance, $x^3 + 8x^2y^3 + y^4$ is a trinomial of degree 5 in two variables, $x$ and $y$.

Some applications of polynomials arise from geometry, particularly from the study of area.

---

**Example 1**  The widths of the town houses at the right are $x$, $y$, and $z$. Each has height $f + s$. Find a polynomial for $A$, the surface area of the fronts of the three town houses.

![Diagram of town houses](image)

**Solution 1**  Think of the surface area of the front of the town houses as the sum of the areas of the six smaller rectangles (three first floors, three second floors).

$$ A = fx + fy + fz + sx + sy + sz $$

**Solution 2**  Think of the surface area of the front of the town houses as the area of one large rectangle with base $x + y + z$ and height $f + s$. Thus,

$$ A = (x + y + z)(f + s). $$

Use the Distributive Property, considering $x + y + z$ as a chunk, to rewrite this as

$$ A = (x + y + z)f + (x + y + z)s. $$

Further applications of the Distributive Property lead to the same results obtained in Solution 1.

$$ A = xf + yf + zf + xs + ys + zs $$
Because of the multiple use of the Distributive Property, we call this an instance of the \textit{Extended Distributive Property}.

\begin{center}
\textbf{The Extended Distributive Property:}
\end{center}

To multiply two polynomials, multiply each term in the first polynomial by each term in the second.

If one polynomial has \( m \) terms and the second \( n \) terms, there will at first be \( mn \) terms in their product. When possible, you should simplify the product by adding like terms.

\begin{center}
\textbf{Example 2} Multiply \((5x^2 - 4x + 3)(x - 7)\).
\end{center}

\begin{center}
\textbf{Solution} Multiply each term in the first polynomial by each in the second. There will be six terms in the product.
\end{center}

\begin{align*}
&= 5x^2 \cdot x + 5x^2 \cdot (-7) + (-4x) \cdot x + (-4x) \cdot (-7) + 3 \cdot x + 3 \cdot (-7) \\
&= 5x^3 - 35x^2 - 4x^2 + 28x + 3x - 21
\end{align*}

Now simplify by adding or subtracting like terms.

\begin{align*}
&= 5x^3 - 39x^2 + 31x - 21
\end{align*}

Other applications of polynomials arise from volume. Consider a piece of metal 20" by 24" which is to be folded into a box after cutting out a square from each corner. What will be the volume of the box?

The volume depends on the length \( x \) of the side of the square cut out. Suppose \( x = 2" \). Then the box will be 2" high and its other dimensions will be 20" and 16". Its volume is then 2" \( \times \) 20" \( \times \) 16", or 640 cubic inches.

Is this the largest possible volume? To answer that question, we need the volume in terms of \( x \).
Example 3  A piece of metal 20 inches by 24 inches is made into a box by cutting out squares of side \( x \) from each corner. Let \( V(x) \) be the volume of the box. Find a polynomial formula for \( V(x) \).

**Solution**  The volume of a rectangular box is the product of the dimensions, that is, \( V = lwh \). When the metal is folded up, the dimensions of the box will be \((24 - 2x)\) in. by \((20 - 2x)\) in. by \( x \) in. high.

So \( V(x) = (24 - 2x)(20 - 2x) \)

\[
V(x) = (24x - 2x^2)(20 - 2x)
\]

\[
V(x) = 480x - 48x^2 - 40x^2 + 4x^3
\]

\[
V(x) = 4x^3 - 88x^2 + 480x
\]

Distribute the \( x \).

FOIL Theorem

Collect like terms; write with decreasing exponents.

**Check**  Because volume is 3-dimensional, you should expect a volume formula to involve a third power. Choose a particular value of \( x \), and calculate the volume using the formula.

When \( x = 2'' \), \( V(x) = 4 \cdot 2^3 - 88 \cdot 2^2 + 480 \cdot 2 \)

\[
= 32 - 352 + 960 = 640 \text{ cubic inches.}
\]

This agrees with the value found using dimensions 2", 20", and 16" on page 612.

With a formula known for \( V(x) \), the function \( V \) can be graphed. The graph below shows that \( x = 2 \) does not give the largest volume. A slightly larger volume occurs when \( x = 3 \), and the largest volume occurs when \( x \) is a little less than 4. You are asked to estimate this value in Question 11.
In 1–6, *multiple choice.* State whether the polynomial is (a) a monomial, (b) a binomial, or (c) a trinomial, and give its degree.

1. \( x^9 - x \)
2. \( 3x^5 + x^2 \)
3. \( a^3 - b^3 \)
4. \( 5x + x \)
5. \( x^2 + 7xy - 8 \)
6. \( 173x^2y^3z \)

7. Find the area \( A \) of the largest rectangle below:

   a. summing the areas of the six small rectangles.

   b. using the formula \( A = lw \) directly and applying the Extended Distributive Property.

8. \( (3x^2 + x - 5)(x + 2) \)
9. \( (4a^2 + 2a + 1)(2a - 1) \)

In 8 and 9, multiply and simplify.

10. a. Use the equation \( V(x) = 4x^3 - 88x^2 + 480x \) to complete the table below for \( x = 1 \) to 5.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
  x  & 1  & 2  & 3  & 4  & 5  & 6  & 7  & 8  \\
  \hline
  V(x) & 640 &  &  &  &  &  &  & 108 \\
\end{array}
\]

b. Use the factored form \( V(x) = x(24 - 2x)(20 - 2x) \) to check the value of the polynomial for \( x = 9 \).

c. Use the factored form to calculate the value of the polynomial for the other integer values of \( x \) from \( x = 6 \) to 11.

d. For what integer value of \( x \) between 1 and 11 is \( V(x) \) greatest? Least?

e. Interpret your results for \( x = 10 \) in terms of the box.

f. What is a reasonable domain for the function \( V \)?
11. a. Use a function grapher to estimate to the nearest tenth the value of \( x \) at which \( V \) achieves its maximum value.
   b. Use this value to find the dimensions of the box with the largest possible volume.

12. a. Find a polynomial formula in terms of \( x \) for the surface area \( S(x) \) of the open box. (Hint: There are five sides.)
   b. Calculate \( S(3) \).
   c. Explain why you can tell that \( S(5) < S(4) \) without calculating these values.

13. Suppose a piece of metal is 18 inches by 10 inches and squares of side \( x \) are cut out of the corners. An open box is formed from the remaining metal.
   a. Find a polynomial formula in terms of \( x \) for the volume \( V(x) \) of the box.
   b. Calculate \( V(2) \).
   c. Find a value of \( x \) such that \( V(x) > V(2) \).

In 14–16, multiply and simplify.

14. \((x^2 - 2x + 2)(x^2 + 2x + 2)\)

15. \((a + b - c)(a - b + c)\)

16. \((2a - 1)^3\)

17. The slant height of a right circular cone is 15 cm. Show that the volume of the cone can be expressed as a polynomial in \( h \) by following the steps below.

![Diagram of right circular cone]

a. Write the volume in terms of \( r \) and \( h \).
   b. Use the Pythagorean Theorem to express \( r \) in terms of \( h \).
   c. Substitute the result of part b into the formula in part a.

18. For 6 years, after each birthday Devin invested his money in an account which compounded interest annually at rate \( r \). He saved

\[
56x^5 + 32x^4 + 40x^3 + 47x^2 + 61x + 59 \text{ dollars,}
\]

where \( x = 1 + r \).

a. What is the degree of this polynomial?
   b. How much money would Devin have if the money were invested at a rate of 7.25%? (Lesson 11-1)
In 19–21, consider that during the early part of the twentieth century, the deer population of the Kaibab Plateau in Arizona grew rapidly, because hunters had reduced the number of natural predators. Later, the increase in population depleted the food supply and the deer population declined quickly. From 1905 to 1930 the number \( N(t) \) of deer was approximated by

\[
N(t) = -0.125t^2 + 3.125t^3 + 4000
\]

where \( t \) is the time in years after 1905. This function is graphed below.

(Lesson 11-1)

19. To the nearest thousand, what was the deer population in 1905?

20. To the nearest thousand, what was the deer population in 1930?

21. a. Over what time period (between 1905 and 1930) was the deer population increasing?
   b. In about what year did the deer population start to decline?

In 22 and 23, consider that the price of a diamond tends to vary directly as the square of its weight. Suppose a half-carat diamond costs $640. (Lessons 2-1, 2-3)

22. Estimate the cost of a 1\( \frac{1}{2} \) carat diamond of similar quality.

23. How many times more will a 2-carat diamond cost than a half-carat diamond of similar quality?

24. In Question 14, the product of two trinomials simplifies to be a binomial. Find another two trinomials whose product is a binomial.
Polynomials are usually written in the general form as a sum:

\[ P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 \]

When evaluating or graphing a polynomial, it may help to rewrite the polynomial as a product of factors. For instance, the polynomial \(4x^3 - 88x^2 + 480x\) in the last lesson was originally in the factored form \(x(24 - 2x)(20 - 2x)\). This can be further factored into \(4x(12 - x)(10 - x)\).

There are four common ways of factoring:
1. factoring the largest common monomial factor
2. factoring following a pattern
3. quadratic trinomial factoring
4. using the Factor Theorem

You may have seen these in earlier courses. We review three types here and one type in the next lesson.

**Example 1** Factor \(6x^5 - 18x^3 + 42x^2\).

**Solution** First we look for the largest common monomial factor of the terms. The greatest common factor of 6, -18, and 42 is 6. The highest power of \(x\) that divides each term is \(x^2\). So \(6x^2\) is the largest common monomial factor of \(6x^5\), -18\(x^3\), and \(42x^2\). Now apply the Distributive Property.

\[
6x^5 - 18x^3 + 42x^2 = 6x^2(\ ? - \ ? + \ ?)
= 6x^2(x^3 - 3x + 7)
\]

This cannot be factored further.

**Check:** Test a special case. Let \(x = 2\). Do the first and last expressions have the same values?

\[
6x^5 - 18x^3 + 42x^2 = 6 \cdot 32 - 18 \cdot 8 + 42 \cdot 4 = 216
\]

\[
6x^2(x^3 - 3x + 7) = 6 \cdot 4(8 - 6 + 7) = 216
\]

Yes.
There are five factoring patterns you should memorize. You learned two of them in Chapter 6.

### Perfect Square Trinomial Patterns:

For all \( a \) and \( b \),

\[
\begin{align*}
    a^2 + 2ab + b^2 &= (a + b)^2 \\
    a^2 - 2ab + b^2 &= (a - b)^2.
\end{align*}
\]

The sum of two squares, \( a^2 + b^2 \), cannot be factored without using complex numbers. However, the difference of two squares, \( a^2 - b^2 \), has a well-known factorization.

### Difference of Squares Pattern:

For all \( a \) and \( b \),

\[
a^2 - b^2 = (a + b)(a - b).
\]

Example 2 shows the use of this pattern to factor a polynomial of the form \( a^2 - b^2 \).

---

**Example 2** Factor \( 9m^2n^2 - 49 \).

**Solution** Both terms of the polynomial are perfect squares.

\[
\begin{align*}
    9m^2n^2 &= (3mn)^2 \\
    49 &= 7^2
\end{align*}
\]

Thus the difference of squares pattern can be applied using \( a = 3mn \) and \( b = 7 \).

\[
9m^2n^2 - 49 = (3mn)^2 - (7)^2 = (3mn + 7)(3mn - 7)
\]

**Check:** Use the FOIL Theorem.

\[
(3mn + 7)(3mn - 7) = 9m^2n^2 - 21mn + 21mn - 49 = 9m^2n^2 - 49
\]
The final two patterns involve cubes. They are easily proved using the Distributive Property. The proof of the sum of cubes pattern is below. You are asked to complete the proof of the difference of cubes pattern in Question 26.

For all \(a\) and \(b\),

**Sum of Cubes Pattern:** \[ a^3 + b^3 = (a + b)(a^2 - ab + b^2) \]

**Difference of Cubes Pattern:** \[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]

**Proof**

\[
(a + b)(a^2 - ab + b^2)
= a(a^2 - ab + b^2) + b(a^2 - ab + b^2)
= a(a^2) - a(ab) + a(b^2) + b(a^2) - b(ab) + b(b^2)
= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3
= a^3 - ab^2 + ab^2 + ab^2 + b^3
= a^3 + b^3
\]

---

**Example 3**  Factor \(x^3 + 125\).

**Solution**  This polynomial is the sum of two cubes, so use that pattern with \(a = x\) and \(b = 5\).

\[
x^3 + 5^3 = (x + 5)(x^2 - x \cdot 5 + 5^2)
= (x + 5)(x^2 - 5x + 25)
\]

**Example 4**  Factor \(8 - s^3p^6\).

**Solution**  Notice that \(8\) is the cube of \(2\), and \(s^3p^6\) is the cube of \(sp^2\). So the pattern for the difference of two cubes can be used.

Thus, \[ 8 - s^3p^6 = (2)^3 - (sp^2)^3 = (2 - sp^2)(4 + 2(sp^2) + (sp^2)^2) = (2 - sp^2)(4 + 2sp^2 + s^2p^4). \]

Usually we consider only polynomials with integer coefficients. Then the quadratic trinomial \(ax^2 + bx + c\) can be factored into linear factors if and only if \(b^2 - 4ac\) is a perfect square. Once you know a trinomial can be factored, the usual method for finding the factors is trial and error.
Example 5 Factor $6y^2 - 7y - 5$ if possible.

Solution Here $a = 6$, $b = -7$, and $c = -5$. Since $b^2 - 4ac = (-7)^2 - 4(6)(-5) = 169$, a perfect square, this quadratic polynomial is factorable. First write the form of two linear polynomials in $y$.

$$6y^2 - 7y - 5 = (\_\_\_ y + \_\_\_)(\_\_\_ y + \_\_\_\_)$$

The coefficients of $y$ will multiply to 6. Thus either they are $3y$ and $2y$, or $y$ and $6y$. The constant terms will multiply to -5, so they are either 1 and -5, or -1 and 5.

Here are all the possibilities with $3y$ and $2y$.

$$(3y + 1)(2y - 5)$$
$$(3y - 1)(2y + 5)$$
$$(3y - 5)(2y + 1)$$
$$(3y + 5)(2y - 1)$$

Here are all the possibilities with $y$ and $6y$.

$$(y + 1)(6y - 5)$$
$$(y - 1)(6y + 5)$$
$$(y - 5)(6y + 1)$$
$$(y + 5)(6y - 1)$$

At most, you need to do these eight multiplications. If one of them gives $6y^2 - 7y - 5$, then that is the correct factoring.

We show all eight products. You can see that the desired one is third.

$$(3y + 1)(2y - 5) = 6y^2 - 13y - 5$$
$$(3y - 1)(2y + 5) = 6y^2 + 13y - 5$$
$$(3y - 5)(2y + 1) = 6y^2 - 7y - 5$$
$$(3y + 5)(2y - 1) = 6y^2 + 7y - 5$$
$$(y + 1)(6y - 5) = 6y^2 + y - 5$$
$$(y - 1)(6y + 5) = 6y^2 - y - 5$$
$$(y - 5)(6y + 1) = 6y^2 - 29y - 5$$
$$(y + 5)(6y - 1) = 6y^2 + 29y - 5$$

So $6y^2 - 7y - 5 = (3y - 5)(2y + 1)$.

In the lessons that follow you will learn how factoring a polynomial can help you graph and analyze polynomial functions.
Covering the Reading

Applying the Mathematics

Questions


In 2 and 3, factor.

2. $-62x^2y^2 + 124x^4y^3$  
3. $21x^3 - 28x + 35x^4$

In 4 and 5, choose from the following:

- $a^2 - b^2$
- $a^2 + b^2$
- $a^3 - b^3$
- $a^3 + b^3$

4. List all expressions that have $a - b$ as a factor.

5. List all expressions that have $a + b$ as a factor.

In 6–14, (a) describe the polynomial as a perfect square, difference of squares or cubes, or sum of squares or cubes, and (b) factor.

6. $x^2 - y^2$  
7. $a^2 - 2ab + b^2$  
8. $m^3 + n^3$

9. $x^2 - 256$  
10. $25a^2 - 36b^2$  
11. $64 - 27c^3$

12. $64 + 27c^3$  
13. $49a^2 - 42ab + 9b^2$  
14. $x^3 + 27$

15. If $a$, $b$, and $c$ are integers, when is $ax^2 + bx + c$ factorable into factors with integer coefficients?

16. One factor of $6x^2 + 7x - 10$ is $(x + 2)$. Find the other factor.

In 17–19, (a) determine whether the trinomial is factorable, and (b) if so, factor.

17. $5x^2 + 8x - 4$  
18. $y^2 + 3y + 4$  
19. $7z^2 - z - 8$

20. Multiple choice  
Which of the following is a perfect square trinomial?  
(a) $9x^2 + 60x + 25$  
(b) $a^4 + 24a^3 + 144$  
(c) $4q^2 + r^2 - 4qr$

21. a. Write $16x^4 - 81$ as the product of two binomials.  
b. Write $16x^4 - 81$ as the product of three binomials.

In 22 and 23, first factor out the common monomial factor. Then complete the factorization.

22. $4x^3 - 88x^2 + 480x$  
23. $1000x^3 + 216x^3y^3$

24. a. Multiply $(x - \sqrt{7})(x + \sqrt{7})$.  
b. Use your answer to part a to help factor $x^2 - 19$ over the set of real numbers.

25. Multiple choice  
Which is a factorization of $x^2 + y^2$ over the complex numbers?  
(a) $(x + y)(x + y)$  
(b) $(x + iy)(x + iy)$  
(c) $(x - iy)(x - iy)$  
(d) $(x + iy)(x - iy)$

26. Prove that for all numbers $a$ and $b$, $(a - b)(a^2 + ab + b^2) = a^3 - b^3$. 

LESSON 11-3 Factoring Polynomials 621
In 27 and 28, consider a closed rectangular box with dimensions $h$, $h + 2$, and $h + 5$.

Write a polynomial in standard form for

27. $S(h)$, the surface area of the box;

28. $V(h)$, the volume of the box.

(Lesson 11-2)

29. A graphic designer works with sheets of paper 11 in. by 17 in. Suppose the designer lays out a rectangular design in the center of the sheet with a border of $x$ in. on each side. (Lesson 11-2)

a. Find the area of the design in the center if $x = 3$.

b. Write an expression for $A(x)$, the area of the design in the center.

30. a. Let $f(x) = x^3 - x^2 - 12x$. Construct a table using integer values of $x$ on the domain $-5 \leq x \leq 5$.

b. Plot the eleven points in part a. Estimate what the graph of $y = f(x)$ looks like by drawing a smooth curve through the points.

c. Check your work in part a by using a function grapher. (Lesson 11-1)

31. A function $P(x)$ is graphed at the right. (Lessons 6-3, 6-4, 6-8)

a. Which word best describes the function: constant, linear, quadratic, or exponential? Explain how you know.

b. For what value(s) of $x$ does $P(x) = 0$?

32. a. $x^2 - 1$ is the product of $x - 1$ and $x + 1$.

b. $x^3 - 1$ is the product of $x - 1$ and $x^2 + x + 1$.

c. $x^4 - 1$ is the product of $x - 1$ and $x^3 + x^2 + x + 1$.

d. Generalize the above pattern.
A product equals 0 if and only if one of the factors equals 0. This result is called the Zero Product Theorem.

Zero Product Theorem:
For all $a$ and $b$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

This theorem is true for any expressions $a$ and $b$, so it holds for polynomials. Example 1 uses a polynomial from Lesson 11-2.

Example 1 Let $V(x) = 4x^3 - 88x^2 + 480x$. Solve $V(x) = 0$ for $x$.

Solution We want to know when $4x^3 - 88x^2 + 480x = 0$. The Zero Product Theorem can be applied if the polynomial is factored. Factoring $V(x)$ we have

$$4x(x - 12)(x - 10) = 0.$$  

Now, applying the Theorem,

$$4x = 0 \text{ or } x - 12 = 0 \text{ or } x - 10 = 0.$$  

Thus $x = 0$ or $x = 12$ or $x = 10$.

Solution 2 Graph the function $V$. $V(x) = 0$ at the $x$-intercepts of $V$, that is, when the function intersects the $x$-axis. A graph verifies that these intersections are at 0, 10, and 12.

Solution 3 The equation $V(x) = 4x^3 - 88x^2 + 480x$ gives the volume of a box with sides of dimensions $x$, $24 - 2x$, and $20 - 2x$. The volume of the box will be 0 exactly when any one side has length 0, leading to the same values as in the other two solutions.
The \( x \)-intercepts of the graph of a function are called the **zeros** of the function. The function \( V \) has zeros at 0, 10, and 12. It has factors \( x, x - 10, \) and \( x - 12 \). The general relationship, which holds for any polynomial function, is surprisingly simple.

**Factor Theorem:**

\[ x - r \text{ is a factor of a polynomial } P(x) \text{ if and only if } P(r) = 0. \]

**Proof**

The general polynomial is \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \). If \( x \) is a factor of \( P(x) \), then \( a_0 = 0 \) and so \( P(0) = 0 \). And, if \( P(0) = 0 \), then \( a_0 = 0 \), so \( x \) is a factor. This means \( x \) is a factor of \( P(x) \) if and only if \( P(0) = 0 \). The graph will then go through the origin.

Now apply the Graph Translation Theorem. Replace \( x \) by \( x - r \). This translates the graph \( r \) units to the right. The graph will contain \( (r, 0) \). Then \( r \) is a zero of the function and \( x - r \) is a factor of the polynomial.

The words ‘‘if and only if’’ in the Factor Theorem mean that the theorem has two parts. If \( P(r) = 0 \), then \( x - r \) is a factor of \( P(x) \). And, if \( x - r \) is a factor of \( P(x) \), then \( P(r) = 0 \). So zeros determine factors and vice-versa.

**Example 2** Find the zeros of \( P(x) = 3x^4 - 28x^3 - 20x^2 \) by factoring.

**Solution**

\[
P(x) = x^2(3x^2 - 28x - 20) \\
= x^2(3x + 2)(x - 10) \\
= 3x^2(x + \frac{2}{3})(x - 10)
\]

The factors are \( x, x, x + \frac{2}{3}, \) and \( x - 10 \), so the zeros are 0, 0, -\( \frac{2}{3} \), and 10.

**Check** Graph \( P \) using a function plotter. The graph shows zeros at approximately 10, 0, and -\( \frac{2}{3} \). You might wish to zoom in on the origin to verify the last two values and show that \( P(x) \) crosses the x-axis at \( x = -\frac{2}{3} \).
Example 3 reverses the process of Example 2.

Example 3  Factor \( f(x) = 2x^3 - 5x^2 - 28x + 15 \) by graphing.

**Solution**  The graph below was drawn with a function plotter. It shows two zeros at approximately 5 and -3 and a third zero between 0 and 1.

We check 5 by calculation: \( f(5) = 2 \cdot 5^3 - 5 \cdot 5^2 - 28 \cdot 5 + 15 = 0 \).
Thus \( x - 5 \) is a factor. Similarly, \( f(-3) = 2(-3)^3 - 5(-3)^2 - 28(-3) + 15 = 0 \), and \( x + 3 \) is a factor. Thus

\[
f(x) = 2x^3 - 5x^2 - 28x + 15 = (x - 5)(x + 3)(?)
= (x^2 - 2x - 15)(?)
\]

The third factor must be linear of the form \( ax + b \) because \( f(x) \) has degree 3. To find \( a \), we know \( 2x^3 = x^2(ax) \), so \( a = 2 \). To find \( b \), \( 15 = (-15)b \), so \( b = -1 \). Thus the third factor is \( 2x - 1 \) and

\[
2x^3 - 5x^2 - 28x + 15 = (x^2 - 2x - 15)(2x - 1).
\]

If \( 2x - 1 = 0 \), then \( x = \frac{1}{2} \).
Check by calculation: \( f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 5(\frac{1}{2})^2 - 28(\frac{1}{2}) + 15 = 0 \).
The Factor Theorem can be used to find equations of polynomials given their zeros.

Example 4  Find an equation for a polynomial function with zeros -4, \( \frac{7}{2} \), and \( \frac{5}{3} \).

Solution  Call the polynomial \( p(x) \). It is given that \( p(-4) = 0 \), \( p\left(\frac{7}{2}\right) = 0 \), and \( p\left(\frac{5}{3}\right) = 0 \). So the zeros of \( p \) are -4, \( \frac{7}{2} \), and \( \frac{5}{3} \). By the Factor Theorem, \((x - -4)\), \((x - \frac{7}{2})\), and \((x - \frac{5}{3})\) must be factors of \( p(x) \). Thus

\[
p(x) = k(x + 4)(x - \frac{7}{2})(x - \frac{5}{3})
\]

where \( k \) is any nonzero constant or polynomial in \( x \).

Check  Substitute -4 for \( x \). Is \( p(-4) = 0 \)?

\[
p(-4) = k(-4 + 4)(-4 - \frac{7}{2})(-4 - \frac{5}{3})
\]

So,

\[
p(-4) = 0.
\]

Similarly, \( p\left(\frac{7}{2}\right) = 0 \) and \( p\left(\frac{5}{3}\right) = 0 \).

Notice that the degree of \( p(x) \) in Example 4 is at least 3. However, from the given information we cannot conclude the value of \( k \), nor even whether \( k \) is a constant or a variable. Thus, we cannot be sure of the degree of \( p(x) \). Many polynomials go through the points (-4, 0), \( \left(\frac{7}{2}, 0\right) \), and \( \left(\frac{5}{3}, 0\right) \). Three examples are

\[
f(x) = 6(x + 4)(x - \frac{7}{2})(x - \frac{5}{3}) = 6x^{3} - 7x^{2} - 89x + 140,
g(x) = (x + 4)(x - \frac{7}{2})(x - \frac{5}{3}),
and h(x) = x^{2}(x + 4)(x - \frac{7}{2})(x - \frac{5}{3}).
\]

Graphs of these three functions are shown below.
1. State the Zero Product Theorem.

In 2 and 3, solve.
2. \((x + 9)(3x + 4) = 0\)
3. \((-\frac{5}{2}k + 2)(k - 2)(k - 9) = 0\)

4. If \(f(x) = x(x + 7)(x - 3)\), solve \(f(x) = 0\).

5. Suppose that \(P(x)\) is a polynomial and \(x - 4\) is a factor of \(P(x)\). According to the Factor Theorem, what can you conclude?

In 6 and 7, factor each polynomial and find its zeros.
6. \(j(x) = x^2 - 10x - 24\)
7. \(k(x) = 2x^3 - 17x^2 + 8x\)

8. True or false If the graph of a polynomial function crosses the x-axis at \((3, 0)\) and \((-4, 0)\), then \((x + 3)\) and \((x - 4)\) are factors of the polynomial.

In 9 and 10, find the zeros of the polynomial function by graphing. Use this information to factor the polynomial.
9. \(y = x^3 - 5x^2 - 28x + 32\)
10. \(P(n) = 6n^3 + 5n^2 - 24n - 20\)

In 11 and 12, refer to Example 4.
11. What is the form of the equation for a polynomial function with zeros equal to \(-4\), \(\frac{7}{2}\), and \(\frac{5}{3}\)?
12. Name three specific polynomial functions whose zeros are \(-4\), 4, and .5.

13. Find a general equation for a polynomial function whose zeros are 8, -10, and 2.4.

14. At the right is the graph of a third degree polynomial function with zeros at 0, 3, and 6 and leading coefficient 1. What is an equation for this function?

In 15 and 16: (a) factor each polynomial; (b) find the zeros of each function; and (c) check your answers by drawing a graph using a function plotter.
15. \(g(x) = 8x^4 - 125x\)
16. \(f(x) = 3x^4 - 108x^2\)
17. A horizontal beam has its left end built into a wall and its right end rests on a support, as shown in the figure below. The weight of the beam is distributed uniformly along its length. As a result, the beam sags downward according to the equation

\[ y = -x^4 + 24x^3 - 135x^2 \]

where \( x \) is the distance (in meters) from the wall to a point on the beam, and \( y \) is the distance (in hundredths of a millimeter) from the \( x \)-axis to the beam caused by the sag.

\[ \text{wall} \]
\[ \text{uniform load} \]
\[ \text{beam} \]
\[ \text{support} \]

\[ y \]

**a.** What is an appropriate domain for \( x \) if the beam is 9 meters long?

**b.** Find the zeros of this function and tell what they represent in the real world.

18. Cassandra has a piece of construction paper 9 in. by 12 in. Suppose she cuts squares with side \( x \) inches from each corner, and folds the paper as in Lesson 11-2 to make a box. Let \( V(x) \) = the volume of the box and \( S(x) \) = the surface area of the open box. Find polynomial formulas for **a.** \( V(x) \) and **b.** \( S(x) \). (Lesson 11-2)

In 19–21, factor if possible. (Lesson 11-3)

19. \( 27x^3 - 1 \)

20. \( a^2 + 14ab + 49b^2 \)

21. \( 3x^2 - 3y^2 \)

In 22–24, complete the expression to form a perfect square. (Lessons 11-3, 6-5)

22. \( x^2 + \_ + 100 \)

23. \( n^2 - 18n + \_ \)

24. \( y^2 + 5y + \_ \)

25. Multiply and simplify: \((x + 2)(x + 3)(x + 4)\). (Lesson 11-2)
26. Brianna, a traffic engineer, wanted to know how much force $F$ would be needed to keep a car of weight $w$ traveling at $S$ mph from skidding on a curve of radius $r$. She determined that the force varied jointly as the weight and the square of the speed. But she still needed to find the relationship between the force and the radius. *(Lessons 2-7, 2-8)*

**a.** With a $2000$ lb car traveling at $30$ mph, she obtained the following data.

<table>
<thead>
<tr>
<th>radius of curve (ft)</th>
<th>125</th>
<th>250</th>
<th>375</th>
<th>500</th>
<th>625</th>
</tr>
</thead>
<tbody>
<tr>
<td>force (lb)</td>
<td>12000</td>
<td>6000</td>
<td>4000</td>
<td>3000</td>
<td>2400</td>
</tr>
</tbody>
</table>

Graph these data points.

**b.** How does $F$ vary with $r$?

**c.** Write an equation relating $F$, $r$, $S$, and $w$. Do not find the constant of variation $k$.

27. Graphs of cubic functions may have any one of the four kinds of shapes below.

![Graphs of cubic functions](image)

**a.** Using your function plotter or the Factor Theorem, find an equation for a cubic function other than those given in this chapter

(i) with $3$ $x$-intercepts whose graph looks like I.
(ii) with $3$ $x$-intercepts whose graph looks like II.
(iii) with $1$ $x$-intercept whose graph looks like III.
(iv) with $1$ $x$-intercept whose graph looks like IV.

**b.** Can the graph of a cubic polynomial function ever have $2$ $x$-intercepts? If so, give an equation for such a function. If not, explain why not.
You already know how to find exact zeros of any linear or quadratic polynomial function. You can find zeros of any higher degree polynomial function if it can be factored. You may also be fortunate and find zeros by substitution. But for higher degree polynomial functions which cannot be factored or are difficult to factor, the best that can be done is estimate the zeros.

There are two basic ways to estimate zeros.
1. Draw a graph.
2. Make a table of values.

Either way can be done by hand, but today’s function plotters and computer programs allow graphs and tables to be made quickly.

Example 1

At the right is a graph of \( P(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78 \) for the interval \(-3 \leq x \leq 3\). From it you can see that \( P \) has a zero between -3 and -2 and another zero between 1 and 2. Using a function plotter, estimate to the nearest hundredth the \( x \)-intercept between -3 and -2.
Solution  Use the zoom or rescale feature on your function plotter to examine the graph on a smaller domain. Here is a graph of P when x is between -3.1 and -1.9.

A zero of P looks to be between -2.2 and -2.1. Zoom or rescale again to examine the graph between these two values. If possible, adjust the y-values on the window. Our function plotter shows the following with a viewing window of -2.2 ≤ x ≤ -2.09 and -5 ≤ y ≤ 7.

From this graph you can see that one x-intercept of y = P(x) occurs closer to -2.11 than it does to -2.12. Thus, to the nearest hundredth, x = -2.11 is a zero of P(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78.

Check
P(-2.11) = 10(-2.11)^4 + 15(-2.11)^3 + 14(-2.11)^2 + 20(-2.11) - 78 ≈ -0.57
P(-2.12) = 10(-2.12)^4 + 15(-2.12)^3 + 14(-2.12)^2 + 20(-2.12) - 78 ≈ 1.60
P(-2.11) is close to 0.

Many function graphers also make tables of values. If your function plotter does not make tables, you can use a computer program to do so.
Here is a BASIC program that lists values of any function \( y = f(x) \) for \( x = A \) to \( x = B \) in increments of \( C \).

```
10 REM PROGRAM TO PRINT TABLE OF FUNCTIONAL VALUES
20 INPUT "ENDPOINTS A AND B OF DOMAIN", A, B
30 INPUT "STEP SIZE"; C
40 PRINT "X", "Y"
50 FOR X = A TO B STEP C
60 REM TYPE IN YOUR OWN FUNCTION AT LINE 70
70 Y =
80 PRINT X, Y
90 NEXT X
100 END
```

To use this program, first type the formula for \( y \) as a function of \( x \) in line 70. Then each time you run the program, enter values for \( A, B, \) and \( C \) when prompted.

**Example 2**  From the graph in Example 1 you know that the polynomial

\[
P(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78
\]

has a zero between \( x = 1 \) and \( x = 2 \). Use the program above to estimate this zero to the nearest hundredth.

**Solution**  Load the program above, and for line 70 type

\[
Y = 10*X^4 + 15*X^3 + 14*X^2 + 20*X - 78
\]

Run the program using \( A = 1 \) and \( B = 2 \). A step size of 0.1 will locate the zero to the tenths place. Our output is as follows:

```
ENDPOINTS A AND B OF DOMAIN? 1, 2
STEP SIZE? 0.1
X  Y
  1 -19
  1.1 -4.454
  1.2 12.816
  1.3 33.176
  1.4 57.016
  1.5 84.75
  1.6 116.816
  1.7 153.676
  1.8 195.816
  1.9 243.746
  2.0 298
```
Note that the output shows that the y-values change from negative to positive as x changes from 1.1 to 1.2. This means that there is a zero of P between x = 1.1 and 1.2. Run the program again using these as the values of A and B. Make the step size 0.01. The output is as follows:

ENDPOINTS A AND B OF DOMAIN? 1.1, 1.2
STEP SIZE? 0.01

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>-4.454</td>
</tr>
<tr>
<td>1.11</td>
<td>-2.85543</td>
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<tr>
<td>1.12</td>
<td>-1.22929</td>
</tr>
<tr>
<td>1.13</td>
<td>.424791</td>
</tr>
<tr>
<td>1.14</td>
<td>2.10716</td>
</tr>
<tr>
<td>1.15</td>
<td>3.81819</td>
</tr>
<tr>
<td>1.16</td>
<td>5.55823</td>
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<td>7.32767</td>
</tr>
<tr>
<td>1.18</td>
<td>9.12686</td>
</tr>
<tr>
<td>1.19</td>
<td>10.9562</td>
</tr>
<tr>
<td>1.20</td>
<td>12.816</td>
</tr>
</tbody>
</table>

The output shows that the function has a zero between 1.12 (where the value of y is negative) and 1.13 (where y is positive). It appears to be closer to 1.13 than to 1.12. To achieve more accuracy, enter 1.12 for A, 1.13 for B, and .001 for C. The output is

ENDPOINTS A AND B OF DOMAIN? 1.12, 1.13
STEP SIZE? 0.001

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.258116</td>
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<tr>
<td>1.13</td>
<td>.424791</td>
</tr>
</tbody>
</table>

This shows there is a zero between 1.127 and 1.128. Thus, to the nearest hundredth, x = 1.13 is a zero of \( P(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78 \).

The techniques illustrated in Examples 1 and 2 for estimating zeros of functions apply to all polynomial functions. In fact they can be used with any function that is continuous on the domain \( A \leq x \leq B \).
Estimating zeros of polynomials has many applications. Recall Yolanda Fish's savings (Example 2 of Lesson 11-1). With a 6% annual rate she was able to save $8942 for medical school. Suppose you wondered: What rate would Yolanda need to save $10,000? Then you would have to solve

\[1000x^4 + 1500x^3 + 1400x^2 + 2000x + 2200 = 10,000.\]

Here \(x = 1 + r\), where \(r\) is the annual rate of interest.

**Example 3**

a. Solve the equation above for \(x\).

b. Use the solution(s) to find a rate \(r\) at which Yolanda would have to invest in order to save $10,000.

**Solution**

a. Rewrite the equation so it is in the form \(f(x) = 0\). That is, subtract 10,000 from each side.

\[1000x^4 + 1500x^3 + 1400x^2 + 2000x - 7800 = 0\]

Divide each side by 100 to simplify.

\[10x^4 + 15x^3 + 14x^2 + 20x - 78 = 0\]

Now consider the equation \(y = f(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78\). This is the polynomial of Examples 1 and 2. In Example 1 we found that one zero is about \(-2.11\); in Example 2 we found that another zero is about 1.13. Thus, \(x \approx -2.11\) or \(x \approx 1.13\) are solutions to Yolanda's equation

\[1000x^4 + 1500x^3 + 1400x^2 + 2000x + 2200 = 10,000.\]

b. \(x = 1 + r\), so solve

\[1 + r \approx -2.11 \quad \text{or} \quad 1 + r \approx 1.13.\]

\[r \approx -3.11 \quad \text{or} \quad r \approx 0.13\]

Only \(r \approx 0.13 = 13\%\) makes sense for an interest rate. Yolanda would need a yield of almost 13% compounded annually to save $10,000. (From the last part of Example 2 you can conclude even more specifically that \(0.127 < r < 0.128\).)

In a similar way, any polynomial equation can be written in a form which asks for the zeros of a polynomial function. This is why finding zeros is so important.
1. Refer to the polynomial function from Examples 1 and 2:
   \[ P(x) = 10x^4 + 15x^3 + 14x^2 + 20x - 78 \]
   a. How many zeros does the function have?
   b. Estimate them to the nearest hundredth.

2. Describe how to estimate a zero of a function from a graph.

3. Describe how to estimate a zero of a function from a table of values.

4. A graph of a polynomial function \( f \) is shown below.
   a. What is the minimum number of zeros \( f \) has?
   b. Between which pairs of consecutive integers must they occur?

5. Sketch a graph of \( f \).

6. Between which pairs of consecutive even integers must the zeros of \( f \) occur?

7. Estimate the largest zero to the nearest tenth.

8. Estimate the smallest zero to the nearest tenth.

9. The solutions to the equation \(-2x^3 + 4x^2 = 1\) are equal to the zeros of which function?

10. Refer to Example 3. Using either a graph or a table of values, determine \( r \), the rate of interest Yolanda Fish would need to save $9500. Give your answer to the nearest whole percent.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>3102</td>
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<td>-8</td>
<td>1728</td>
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<tr>
<td>-6</td>
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</tr>
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<td>-4</td>
<td>300</td>
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<tr>
<td>-2</td>
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<tr>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
</tr>
<tr>
<td>6</td>
<td>-50</td>
</tr>
<tr>
<td>8</td>
<td>-336</td>
</tr>
<tr>
<td>10</td>
<td>-918</td>
</tr>
</tbody>
</table>
11. As shown below, there are two values of \( x \) between -2 and 4 at which the graphs of \( y = 5 \) and 
\[ y = -x^4 + 3x^3 - 3x^2 + x + 10 \]
intersect. Use a function plotter to estimate these values to the nearest tenth.

![Graph](image)

12. The polynomial function \( A \) defined by
\[ A(x) = -.015x^3 + 1.058x \]
gives the approximate alcohol concentration (in tenths of a percent) in an average person’s bloodstream \( x \) hr after drinking about 8 oz of 100 proof whiskey.

a. Graph \( y = A(x) \).

b. From the graph estimate the number of hours necessary for the alcohol concentration to revert back to 0.

c. Check your answer to part b by solving \( A(x) = 0 \). (The function \( A \) is approximately valid for \( x \) between 0 and 8, so be careful how you interpret this answer.)

d. Using the graph, estimate the time at which the alcohol concentration was the greatest.

e. In some countries a person is legally drunk if the blood alcohol concentration exceeds 0.07%. Use your graph to estimate the length of time in which this average person is legally drunk.

In 13 and 14, multiply and simplify. (Lesson 11.2)

13. \((x^2 - y^2)(x^2 + y^2)\)  
14. \((a + b + c)(a - b)\)

In 15–17, factor. (Lesson 11.3)

15. \(4x^2 - 12x + 9\)  
16. \(x^3 + 8y^6\)  
17. \(100n^4 - 100\)
18. a. Graph \( f(x) = \sin x \), \( 0^\circ \leq x \leq 360^\circ \).

b. For what value(s) of \( x \) in this domain does \( f(x) = 0? \) \( (\text{Lesson 10-9}) \)

19. Consider square MATH shown below. Show using slope that the diagonals of MATH are perpendicular to each other. \( (\text{Lesson 4-7}) \)

![Diagram of square MATH with coordinates labeled]

20. Suppose that the lateral height of a cone is 6 cm and its height is \( h \).

![Diagram of a cone with lateral height and height labeled]

a. Express the radius \( r \) of the cone in terms of \( h \). \( (\text{Previous course}) \)

b. Write a formula for \( V(h) \), the volume of the cone as a function of \( h \). \( (\text{Lesson 11-2}) \)

21. Which is largest, \( \frac{1}{3} \), \( \frac{1}{4} \), \( \frac{1}{5} \), or \( \frac{1}{2} \)?

22. The end behavior of a function refers to the values of a function \( f: x \rightarrow y \) when \( |x| \) is very large. Values may get larger and larger, that is, more and more positive, or get smaller and smaller, that is, more and more negative. (For instance, in Example 2, when \( x \) is very large, \( h(x) \) gets larger and larger. When \( x \) is very small, \( h(x) \) gets smaller and smaller.) Describe the end behavior of each function.

a. the function \( f \) of Questions 5–8

b. \( g(x) = 3x^4 - 100x + 600 \)

c. \( y = 2x^7 \)

d. \( m(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \)
LESSON

11-6

Solving All Polynomial Equations

You know a way to find an exact solution to any linear equation with real coefficients. To solve any quadratic equation exactly, there is the Quadratic Formula. But that formula requires new numbers, the complex numbers. The formula works even with those quadratic equations having complex coefficients. In the last lesson you learned that all polynomial equations can be solved using either graphs or tables of values. It is natural to wonder whether all polynomial equations can be solved exactly and whether any new types of numbers are needed to solve them.

These questions occupied mathematicians even before today’s notation for polynomials was invented. In the sixteenth century a number of Italian mathematicians began to answer these questions. Scipione del Ferro (1465–1526) discovered how to solve all cubic equations exactly. (His method is too complicated to be discussed in this book.) Perhaps independently, by about 1541 Niccolo Tartaglia (1500–1557) learned to solve cubic equations by the same method. Girolamo Cardano, whom we have mentioned in Lessons 6–8 and 6–10, published del Ferro’s method in 1545 in his book Ars magna. About the same time, Cardano’s secretary, Ludovico Ferrari (1522–1565), discovered how to solve any quartic (fourth degree polynomial) equation. Cardano also published this result in Ars magna. The amazing thing was that you didn’t need to learn any new types of numbers to solve cubic or quartic equations. Then, for the next 250 years mathematicians tried unsuccessfully to find a formula for solving any quintic (fifth degree polynomial) equation. Perhaps new numbers were needed.
However, new numbers are not needed. In 1797, at the age of 20, the great German mathematician Karl Gauss proved the following theorem.

**The Fundamental Theorem of Algebra:**

Every polynomial equation \( P(x) = 0 \) of any degree with complex number coefficients has at least one complex number solution.

(Remember that complex numbers include the reals.) From the Fundamental Theorem of Algebra, and the Factor Theorem, it is possible to prove that every solution to a polynomial equation is a complex number. Thus, no new type of number is needed to solve higher degree polynomials. So, for instance, the solutions to \( x^5 + 3x^3 - ix^2 + 4 - 3i = 0 \) are complex numbers.

How many complex solutions does a given polynomial have? Recall that the linear equation \( ax + b = 0 \) has one root: \( x = -\frac{b}{a} \). The quadratic equation \( ax^2 + bx + c = 0 \) generally has two roots: 

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

However, when the discriminant is 0, the two roots are equal. When this happens this root is considered to be a *double root* or a *root of multiplicity 2*. For instance, when \( x^2 - 8x + 16 = 0 \), then 

\[
x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} = \frac{8 \pm \sqrt{0}}{2} = 4.
\]

So \( x = 4 \) is the only root of \( x^2 - 8x + 16 = 0 \), but the number 4 is said to be a double root.

Notice that \( x^2 - 8x + 16 = (x - 4)^2 \); that is, \( x - 4 \) appears twice as a factor. The *multiplicity of a root* \( r \) is the highest power of \( x - r \) that appears as a factor of the polynomial. For instance, the equation \( (x - 3)^10 = 0 \) is an equation with only one root, 3, but the root has multiplicity 10.

Recall the Factor Theorem: if \( r \) is a root of a polynomial equation \( P(x) = 0 \), then \( (x - r) \) is a factor of \( P(x) \). Another way to state this is to say that if \( r \) is a root of the polynomial equation \( P(x) = 0 \), there is some polynomial \( Q(x) \) such that \( P(x) = (x - r) \cdot Q(x) \) and the degree of \( Q(x) \) is one less than the degree of \( P(x) \). For instance, when \( P(x) \) is cubic, then \( Q(x) \) is quadratic. This implies that when \( P(x) \) is cubic, \( P(x) = 0 \) has three roots: one from the linear factor \( (x - r) \), and two from the quadratic factor \( Q(x) \). Of course, one of these might be a multiple root.
Similarly, 4th degree polynomials can be rewritten as the product of a linear and a cubic polynomial, or of two quadratic polynomials. Thus, 4th degree polynomial equations have 4 complex roots. By extending this pattern we know we can express any higher degree polynomial equation as a product of lower degree polynomials.

**Example 1**

a. Find all solutions of \(x^5 - 8x^2 = 0\).

b. Are any multiple roots? If so, which?

**Solution**

a. Factor out the common factor \(x^2\).

\[x^5 - 8x^2 = x^2(x^3 - 8)\]

Now factor the cubic into a linear and quadratic factor.

\[x^5 - 8x^2 = x^2(x - 2)(x^2 + 2x + 4)\]

Thus, \(x^5 - 8x^2 = 0\) if and only if \(x = 0, x = 2,\) or \(x^2 + 2x + 4 = 0\). The solutions to \(x^2 + 2x + 4 = 0\) can be found by the Quadratic Formula. They are \(-1 \pm i\sqrt{3}\). Thus, \(x^5 - 8x^2 = 0\) has five roots—three real and two nonreal: \(0, 0, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\).

b. The number 0 is a double root.

These instances can be summarized in the following theorem.

**The Number of Roots of a Polynomial Equation Theorem:**

Every polynomial equation of degree \(n\) has exactly \(n\) roots, provided that multiple roots are counted as separate roots.

**Example 2**

How many roots does each of the following equations have?

a. \(x^5 - 7x^3 + 15x^2 + 3 = 10\)

b. \(-2ix^4 - ex^2 + 
\pi x - 12 = 0\)

**Solution**

a. The degree is 5, so the equation has 5 roots.

b. The degree is 4, so the equation has 4 roots.

The question of whether a formula exists for solving all quintic equations was finally settled in 1799 by an Italian mathematician, Paolo Ruffini (1767–1822). He proved that the general quintic equation cannot be solved by formulas. A Norwegian mathematician, Niels Henrik Abel (1802–1829), made the same discovery independently in 1824.

Perhaps the most important discovery about solving polynomial equations was made by a Frenchman, Évariste Galois (1811–1832). The night before he was killed in a duel, Galois wrote a letter which
included some important mathematical discoveries. In it, Galois showed a method to determine which polynomial equations of degree five or more can be solved exactly using formulas. We now know that there are no methods for finding exact solutions of all polynomials of degree higher than four.

The two theorems of this Lesson tell you how many roots a polynomial equation \( P(x) = k \) has, and that all roots can be expressed as complex numbers. They do not tell you how to find the roots, nor do they tell you how many of the roots are real. To answer these questions, you can apply the methods studied in the last two lessons for finding zeros of polynomial functions.

**Example 3**
Consider the equation from Example 2(a),

\[
x^5 - 7x^3 + 15x^2 + 3 = 10.
\]

How many of its solutions are real?

**Solution**
Set one side of the equation equal to zero, by subtracting 10 from each side.

\[
x^5 - 7x^3 + 15x^2 - 7 = 0
\]

The solutions to this equation are the zeros of the function \( f(x) = x^5 - 7x^3 + 15x^2 - 7 \). Use approximation methods from Lesson 11-5. First, look at the behavior of the function over a large domain. We show both a table and a graph below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50</td>
<td>-311587507</td>
</tr>
<tr>
<td>-40</td>
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<td>50</td>
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</table>

Observe the sign changes in \( y \)-values. The graph and table together show that a zero or zeros occur between \( x = -10 \) and \( x = 10 \). Also, when \( x < -10 \) or \( x > 10 \), the value of \( x^5 \) dominates the value of \( f(x) \), so there are no zeros outside \(-10 < x < 10\). Another search shows that zeros occur on the intervals \(-4 < x < -3\), \(-1 < x < 0\), and \(0 < x < 1\).
Thus, there are three real roots. The other two roots are nonreal and are not indicated on this graph.

Further use of tables, graphs, calculators, or computers shows that, to the nearest tenth, \( f(x) = x^3 - 7x^2 + 15x - 7 \) has zeros at -3.4, -0.6, and 0.8.

**Questions**

1. Name three 16th-century mathematicians who worked on solving cubic or quartic equations.

2. The Fundamental Theorem of Algebra states that ___.

3. Who proved the Fundamental Theorem of Algebra?

In 4 and 5, \( a \neq 0 \). Solve for \( x \).

4. \( ax + b = 0 \)

5. \( ax^2 + bx + c = 0 \)

In 6–8: a. solve for \( x \) by factoring the polynomial; and b. identify any multiple roots.

6. \( x^2 - 10x + 25 = 0 \)

7. \( x^3 - 25x = 0 \)

8. \( x^3 - 8 = 0 \)

9. Every polynomial equation of degree \( n \) has exactly ___ roots, provided that ___.

10. *True or false* All solutions to polynomial equations are complex numbers.

In 11 and 12, state the number of roots each equation has. Do not solve.

11. \( x^5 + x^3 + x = 0 \)

12. \( 17y^2 + 7y + iy^3 = 12 \)

13. State one result about polynomials discovered by Galois.

14. Consider the equation \( 2x^5 - 3x^3 - x = 1 \).
   a. How many solutions does it have?
   b. Approximate the real solutions to the nearest tenth by using tables, graphs, calculators, or computers.
In 15 and 16, solve.

15. \(-3x + 7i = 0\)  
16. \(2ix^2 + 8x + 5i = 0\)

17. The equation \(x^3 - 1 = 0\) is equivalent to \(x^3 = 1\). Thus, the roots of \(x^3 - 1 = 0\) can be considered cube roots of 1.
   a. Find all cube roots of 1.
   b. In Question 8, you are finding the cube roots of \(\text{?}\).

18. One root of \(x^3 - 8x^2 + 22x - 20 = 0\) is 2. Find all the other roots.

19. Find all the roots of \(z^4 - 1 = 0\) by factoring and solving the resulting equations.

20. A table of values for a polynomial function \(g\) is given at the right.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
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</thead>
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<td>4</td>
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<tr>
<td>5</td>
<td>164</td>
</tr>
</tbody>
</table>

   a. What is the minimum number of zeros \(g\) may have?
   b. Between which pairs of consecutive integers must they occur? (Lesson 11-5)

21. The sum of the cube and the square of a number is 1.
   a. Let \(n\) equal this number. Write a polynomial equation that can be used to find \(n\).
      (Lesson 11-1)
   b. To the nearest hundredth, what is that number? (Hint: Draw a graph or make a table.) (Lesson 11-5)

22. Graph and state an equation of the image of \(y = 3x^2\) under \(T_{-3,4}\).
   (Lessons 2-5, 4-9, 6-1)

23. A person sights a pier directly across a river, then walks 100' and sights the pier at an angle of 70°. How wide is the river at the pier? (Lesson 10-1)

   In 24–27, solve.

24. \(\log x = 5\) (Lesson 9-4)  
25. \(e^x = 5\) (Lessons 9-7, 9-8)

26. \(\sin x = 5\) (Lesson 10-2)  
27. \(\sqrt{x} = 5\) (Lesson 6-1)

28. A theorem called the “Fundamental Theorem of Algebra” is discussed in this lesson. There is also a theorem called the “Fundamental Theorem of Arithmetic.” Look in a dictionary to find out what theorem this is. (You probably know it but didn’t know it has this name.)
You have seen many instances in this book where mathematics is used to model real life situations. In some cases it helps to graph data points. Graphs can often be used to find an equation to describe the situation.

In Chapter 2 you used this idea in the context of variation models. Consider the following data points:

<table>
<thead>
<tr>
<th>W</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>0</td>
<td>50</td>
<td>200</td>
<td>450</td>
<td>800</td>
</tr>
</tbody>
</table>

If you graphed them as below, you might say that $N$ varies directly as the square of $W$ because the graph looks like a parabola through the origin. This would give the equation $N = kW^2$, which is a quadratic polynomial function.

As you have seen, other equations may have graphs which resemble the one above. Is it possible to determine in a conclusive way which equation fits? When the function is a polynomial function, the answer is yes.

To see why this is true, we need to look at some polynomial functions. First, consider a linear polynomial function like $y = 4x + 5$. Choose consecutive integer values of $x$, such as 1, 2, 3, 4, 5, ... . Find the corresponding values of the polynomial.

$$
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & 5 & \ldots \\
  y = 4x + 5 & 9 & 13 & 17 & 21 & 25 & \ldots \\
\end{array}
$$

Now take the differences of the consecutive $y$-values (right minus left).

$$
\begin{array}{cccccc}
  9 & 13 & 17 & 21 & 25 & \ldots \\
  4 & 4 & 4 & 4 & & \ldots \\
\end{array}
$$

Note that the differences are all equal or constant.
Next consider a quadratic polynomial function like \( y = 5x^2 \). For \( x \) use the consecutive integer values 1, 2, 3, 4, 5, 6, \ldots. As before, find the values of the polynomial and find the differences of consecutive terms.

\[
\begin{array}{ccccccc}
  x & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
  y = 5x^2 & 5 & 20 & 45 & 80 & 125 & 180 & \ldots \\
\end{array}
\]

1st differences \( \begin{array}{ccccccc} 15 & 25 & 35 & 45 & 55 & \ldots \end{array} \)

The differences are not all equal, but notice what happens if differences are taken a second time.

\[
\begin{array}{ccccccc} 15 & 25 & 35 & 45 & 55 & \ldots \\
2nd differences & 10 & 10 & 10 & 10 & \ldots \end{array}
\]

The 2nd differences are all equal.

Consider the cubic polynomial function \( h(x) = x^3 - 5x^2 + 10x + 50 \). Let us try the method of differences again, using -1, 0, 1, 2, 3, 4, and 5 for \( x \) and the corresponding values of \( h(x) \).

\[
\begin{array}{cccccccc}
  x & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
  h(x) & 34 & 50 & 56 & 58 & 62 & 74 & 100 & \ldots \\
\end{array}
\]

1st differences \( \begin{array}{cccccccc} 16 & 6 & 2 & 4 & 12 & 26 & \ldots \end{array} \)

2nd differences \( \begin{array}{cccccccc} -10 & -4 & 2 & 8 & 14 & \ldots \end{array} \)

3rd differences \( \begin{array}{cccccccc} -6 & 6 & 6 & 6 \ldots \end{array} \)

In this case the 3rd differences are equal.

Do all polynomial functions eventually lead to constant differences? The answer is yes, if the \( x \)-values form an arithmetic sequence. Consider again the linear function of \( y = 4x + 5 \) from the previous page. For \( x \)-values, use the arithmetic sequence -2, 1, 4, 7, 10, \ldots .

\[
\begin{array}{cccccccc}
  x & -2 & 1 & 4 & 7 & 10 & \ldots \\
  y = 4x + 5 & -3 & 9 & 21 & 33 & 45 & \ldots \\
\end{array}
\]

1st differences \( \begin{array}{cccccccc} 12 & 12 & 12 & 12 & \ldots \end{array} \)

Again the 1st differences are all equal.
Each of these examples is an instance of the following theorem. Its proof requires ideas from calculus beyond the scope of this book, and so is omitted.

**Polynomial Difference Theorem:**

\[ y = f(x) \text{ is a polynomial function of degree } n \text{ if and only if, for any set of } x\text{-values that forms an arithmetic sequence, the } n\text{th differences of corresponding } y\text{-values are equal and non-zero.} \]

The Polynomial Difference Theorem is important for at least two reasons. First, it indicates that functions known to be polynomial eventually yield differences which are all equal. Second, it provides a technique to determine whether a function expressed as data points is a polynomial. The technique suggested by this theorem is called **finite differences**. That is, from a table of \( y \)-values corresponding to an arithmetic sequence of \( x \)-values, take differences of consecutive \( y \)-values. Only if those differences are eventually constant is the function polynomial.

**Example 1** Consider the data points at the beginning of the lesson. Use the method of finite differences to show that \( N \) is a polynomial function of \( W \) with degree 2.

**Solution** Notice that the values of the independent variable \( W \) form an arithmetic sequence, so the Polynomial Difference Theorem applies.

<table>
<thead>
<tr>
<th>( W )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>0</td>
<td>50</td>
<td>200</td>
<td>450</td>
<td>800</td>
<td>...</td>
</tr>
</tbody>
</table>

1st differences | 50 | 150 | 250 | 350 | ... |

2nd differences | 100 | 100 | 100 | ... |

\( N \) is a polynomial function of \( W \) because the differences eventually are all equal. Because the 2nd differences are equal, the degree of the polynomial is 2. Notice that all differences after the second differences will be zero.

Calculating differences can also be used to tell when a sequence cannot be described with an explicit polynomial formula.
Example 2  The recursive formula

\[
\begin{align*}
a_1 &= 4 \\
a_{n+1} &= 2a_n - 1, \quad n > 1
\end{align*}
\]

generates the sequence

\[4, 7, 13, 25, 49, 97, 193, \ldots\]

Is there an explicit polynomial formula for this sequence?

**Solution**  Take differences between consecutive terms.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>4</td>
<td>7</td>
<td>13</td>
<td>25</td>
<td>49</td>
<td>97</td>
<td>193</td>
<td>...</td>
</tr>
</tbody>
</table>

1st differences 3 6 12 24 48 96 ...  
2nd differences 3 6 12 24 48 ...  

The pattern will continue to repeat and will never yield constant differences. So there is no polynomial formula.

In the next lesson you will learn how to find the equation for a function given as data points once you determine it to be polynomial.

**Questions**

In 1–3, refer to the Polynomial Difference Theorem.

1. If the \(y\)-values are all equal and nonzero for the 10th set of differences of consecutive \(x\)-values, what is the degree of the polynomial?

2. **True or false**  The technique of finite differences takes only the differences of consecutive \(x\)-values.

3. State two reasons why this theorem is important.

In 4 and 5, refer to Example 1.

4. How do we know that \(N\) is a polynomial function of \(W\)?

5. Why must the degree of the polynomial be 2?

6. Refer to Example 2. Write the values of the 3rd differences.
Applying the Mathematics

In 7–9, use the data points listed in each table below.

a. Determine if \( y \) is a polynomial function of \( x \) of degree less than 6.
b. Find the degree, if the function is a polynomial.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3</td>
<td>11</td>
<td>31</td>
<td>69</td>
<td>131</td>
<td>223</td>
<td>351</td>
<td>521</td>
<td>739</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>31</td>
<td>63</td>
<td>127</td>
<td>255</td>
<td>511</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

10. a. How many times will it take to get equal differences for the polynomial function \( y = x^4 + x^2 \)?
b. Construct a table of \( x \) and \( y \) values for the above function for integer values of \( x \) between -3 and 4.
c. Use the technique of finite differences to justify your response to part a.

In 11 and 12, (a) generate the first seven terms of the sequence. (b) Tell whether the sequence can be described explicitly by a polynomial function. (c) If it can, state its degree.

11. \( a_1 = 7; \quad a_{n+1} = a_n + 3 \) for \( n \geq 1 \)
12. \( a_1 = 4; \quad a_{n+1} = 2a_n + 1 \) for \( n \geq 1 \)

13. a. Find the values of the first differences of the function represented by the data points below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>11</td>
<td>17</td>
<td>23</td>
<td>29</td>
</tr>
</tbody>
</table>
b. Find the degree of the function.
c. Plot the data points.
d. Find an equation for the line passing through these points.
e. Make a generalization about what first differences represent on this graph. Does your generalization apply to other linear functions?

14. a. If \( f(x) = ax^2 + bx + c \), find \( f(1), f(2), f(3), f(4), \) and \( f(5) \).
b. Prove that the 2nd differences of these values are constant.

15. Consider the following pattern:

\[
\begin{align*}
f(1) &= 1^2 = 1 \\
f(2) &= 1^2 + 2^2 = 5 \\
f(3) &= 1^2 + 2^2 + 3^2 = 14 \\
f(4) &= 1^2 + 2^2 + 3^2 + 4^2 = 30
\end{align*}
\]
and so on.

a. Find \( f(5) \) and \( f(6) \).
b. Using the Polynomial Difference Theorem, what is the degree of the polynomial \( f(n) \)?
16. Consider the polynomial sequence
   4, 15, 38, 79, 144, 239, ….
   By using finite differences, predict the next term.

In 17–20, consider the function \( y = x^3 + x^2 - 144x - 144 \). Computer-generated coordinate values for it are shown at the right.

(Lessons 11-5, 11-4, 11-3)

17. According to the Fundamental Theorem of Algebra, how many zeros does the function have? Justify your answer.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>396</td>
</tr>
<tr>
<td>-8</td>
<td>560</td>
</tr>
<tr>
<td>-6</td>
<td>540</td>
</tr>
<tr>
<td>-4</td>
<td>384</td>
</tr>
<tr>
<td>-2</td>
<td>140</td>
</tr>
<tr>
<td>0</td>
<td>-144</td>
</tr>
<tr>
<td>2</td>
<td>-420</td>
</tr>
<tr>
<td>4</td>
<td>-640</td>
</tr>
<tr>
<td>6</td>
<td>-756</td>
</tr>
<tr>
<td>8</td>
<td>-720</td>
</tr>
<tr>
<td>10</td>
<td>-484</td>
</tr>
</tbody>
</table>

18. According to the table, between which two even numbers must a zero occur?
19. Sketch a graph of this function. Number the \( x \)-axis by ones and the \( y \)-axis by hundreds.

20. Find a zero of this function to the nearest tenth.

21. Consider the equation \( x^4 - x^3 + 5x^2 = 200\pi \).
   a. How many solutions does it have?
   b. Approximate its real solutions to the nearest tenth. (Lesson 11-5)

22. Find all roots of \( z^3 + 125 = 0 \). (Lessons 11-3, 11-6)

23. Below is the graph of the polynomial function
   \( y = (x - 1)(-x^2 + 2x + 2) \).
   
   a. Rewrite the function in the standard form of a polynomial.
   b. How many \( x \)-intercepts does the function have? Find all of them.
   (Lessons 11-4, 11-1)
24. Solve the system. (Lesson 5-3)
\[
\begin{align*}
  x + 4y - 3z &= 6 \\
  2y + z &= 9 \\
  z &= 8
\end{align*}
\]

25. Multiple choice

Which of the following equations could describe the relationships graphed above, where k is a constant? (Lesson 2-8)
(a) \( y = \frac{kwz}{x^2} \)  
(b) \( y = kwz \)  
(c) \( y = \frac{kwz^2}{x} \)  
(d) \( y = \frac{kwx^2}{z} \)

26. A window sill of a building is twelve feet above the ground. How long must a ladder be if it is to reach from the sill to the ground and form less than an \(80^\circ\) angle with the ground? (Lesson 10-1)

27. Find a sequence of at least 6 terms in which the 3rd differences are all 30.
The employees at Primo's Pizzeria liked to cut pizza into odd shaped pieces. In so doing they noticed that there is a maximum number of pieces that can be formed from a given number of cuts.

<table>
<thead>
<tr>
<th>number of cuts</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum number of pieces</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

Primo's employees wondered if there was a formula relating \( p \), the maximum number of pieces that could be obtained, from \( x \), a given number of cuts. As in the last lesson, they found differences between consecutive terms.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>1st differences</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2nd differences</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Because it took two times to get equal differences, they knew that a quadratic polynomial could be used to model these points. That is, they knew that

\[
p = ax^2 + bx + c
\]

but they did not know \( a, b, \) or \( c \).
It is possible to find the values of \( a, b, \) and \( c \) by solving a system of equations. Because there are three variables, you need three equations. Solutions of the equation
\[
p = ax^2 + bx + c
\]
are ordered pairs of the form \((x, p)\). Thus, to determine a system of equations substitute any data point into the equation to get a true statement. It is usually easiest to use three small values of \( x \) in an arithmetic sequence.

When \( x = 0, p = 1: \)
\[
1 = a(0)^2 + b(0) + c = c
\]
\( x = 1, p = 2: \)
\[
2 = a(1)^2 + b(1) + c = a + b + c
\]
\( x = 2, p = 4: \)
\[
4 = a(2)^2 + b(2) + c = 4a + 2b + c
\]

This system of three equations with three variables may look hard to solve. However, the systems encountered using the technique of this lesson can be solved easily using subtraction and substitution. Here’s how you can organize the work. Reorder the equations so that the largest coefficients are on the top line. You will always subtract an equation from the one above it.

\[
\begin{align*}
4a + 2b + c &= 4 \\
   a + b + c &= 2 \\
\quad c &= 1
\end{align*}
\]

From the equation \( 2a = 1, a = \frac{1}{2} \). Now substitute \( a = \frac{1}{2} \) and \( c = 1 \) into \( a + b + c = 2 \) to get \( \frac{1}{2} + b + 1 = 2 \). Thus, \( b = \frac{1}{2} \).

Therefore, the equation
\[
p = \frac{1}{2}x^2 + \frac{1}{2}x + 1
\]
models the data from Primo’s Pizzeria.

**Example 1**

a. Show that the formula \( p = \frac{1}{2}x^2 + \frac{1}{2}x + 1 \) correctly describes the relation between number of cuts and maximum number of pieces for \( x = 3 \).

b. Predict the maximum number of pieces that can result from 5 cuts to a pizza. Check your answer with a drawing.

**Solution**

a. Substitute \( x = 3 \) into the formula. Verify that \( p = 7 \), the value in the table on the previous page.

\[
p = \frac{1}{2}(3)^2 + \frac{1}{2}(3) + 1
\]
\[
= \frac{9}{2} + \frac{3}{2} + 1
\]
\[
= \frac{12}{2} + 1
\]
\[
p = 7
\]
b. Substitute \( x = 5 \).

\[
\begin{align*}
\rho &= \frac{1}{2}(5)^2 + \frac{1}{2}(5) + 1 \\
&= \frac{25}{2} + \frac{5}{2} + 1 \\
\rho &= 16
\end{align*}
\]

The figure below shows how to get 16 pieces of pizza with 5 cuts.

The following example shows how to use finite differences and systems of equations to handle a polynomial of higher degree.

**Example 2**

A display of oranges can be stacked in a square-based pyramid in the following way: 1 orange is in the top level, 4 oranges in the second level, 9 in the third level, 16 in the fourth level, etc. How many oranges are needed for a display with \( n \) rows?

**Solution**

First, list some values showing how the number of oranges depends on the number of rows. The total number of oranges in the display is the following:

<table>
<thead>
<tr>
<th>Number of Rows</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Oranges</td>
<td>1</td>
<td>5</td>
<td>14</td>
<td>30</td>
<td>55</td>
<td>91</td>
</tr>
</tbody>
</table>
| 1st Differences | 4 | 9 | 16 | 25 | 36 | ...
| 2nd Differences | 5 | 7 | 9 | 11 | ... |
| 3rd Differences | 2 | 2 | 2 | ... |

Second, use the method of finite differences to determine whether a polynomial model fits.

The 3rd differences are constant. Thus the situation can be represented by a polynomial function of degree three.
Third, use a system of equations to find a polynomial model. Let \( n \) be the number of rows and \( f(n) \) the total number of oranges in \( n \) rows. We know the polynomial is of the form

\[
f(n) = an^3 + bn^2 + cn + d.
\]

Substitute \( n = 4, 3, 2, \) and \( 1 \) into the equation and solve the system as before. That is, subtract pairs of equations to eliminate, in this order, \( d, c, \) and \( b. \)

\[
\begin{align*}
\begin{cases}
f(4) = 64a + 16b + 4c + d = 30 \\
f(3) = 27a + 9b + 3c + d = 14 \\
f(2) = 8a + 4b + 2c + d = 5 \\
f(1) = a + b + c + d = 1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
37a + 7b + c = 16 \\
19a + 5b + c = 9 \\
7a + 3b + c = 4
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
18a + 2b = 7 \\
12a + 2b = 5
\end{cases}
\end{align*}
\]

\[
6a = 2
\]

From the equation \( 6a = 2, \) \( a = \frac{1}{3}. \) By substitution into \( 12a + 2b = 5, \) \( b = \frac{1}{2}. \) Then another substitution gives \( c = \frac{1}{6}. \) Finally, using \( a + b + c + d = 1, \) \( d = 0. \) Thus,

\[
f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.
\]

**Check** You should check that this equation fits the data points. For instance, if \( n = 5, \) then

\[
f(n) = \frac{1}{3}(5)^3 + \frac{1}{2}(5)^2 + \frac{1}{6}(5) = \frac{125}{3} + \frac{25}{2} + \frac{5}{6} = 55,
\]

which checks.

When using finite differences, you must have a sufficient number of data points to check the formula you get. For instance, suppose you are given only the data

\[
\begin{array}{cccc}
x & 1 & 2 & 3 \\
y & 1 & 2 & 4
\end{array}
\]

The 1st differences are 1 and 2 and you only have one 2nd difference. So you cannot tell whether the second differences are constant. If the next \( y \)-values are 7 and 11, then the 2nd differences are equal and the polynomial model is

\[
y = \frac{x^3 - x + 2}{2}.
\]
However, if 8 and 15 are the next $y$-values, then the 3rd differences are equal and the polynomial equation modeling the data would be

$$y = \frac{x^3 - 3x^2 + 8x}{6}.$$

These are only two of many polynomial equations fitting the data points (1, 1), (2, 2), (3, 4).

Questions

In 1–3, refer to Primo’s data at the beginning of this lesson.

1. What is the general quadratic polynomial equation which models these data?

2. Primo employees had three variables to find, so they needed to solve a system of $\Box$ equations.

3. Show that the formula $p = \frac{1}{2}x^2 + \frac{1}{2}x + 1$ is correct for $x = 4$.

In 4–6, refer to Example 2.

4. In which equation could you substitute $a = \frac{1}{3}$ and $b = \frac{1}{2}$ to find $c = \frac{1}{6}$?

5. a. Predict the number of oranges in a display with 6 rows.
   b. Justify your answer with a drawing.

6. Predict the number of oranges in a display with 15 rows.

7. Consider the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>3</td>
<td>16</td>
<td>39</td>
<td>72</td>
<td>115</td>
<td>168</td>
</tr>
</tbody>
</table>

   a. Determine the degree of a polynomial function that models these data.
   b. Find a formula for the polynomial model.

8. Suppose that the data in the table below have a formula of the form $y = ax^3 + bx^2 + cx + d$. What four equations are satisfied by $a$, $b$, $c$, and $d$?

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>40</td>
<td>168</td>
<td>432</td>
<td>...</td>
</tr>
</tbody>
</table>

In 9 and 10, solve each system.

9. $x + y + z = -2$
   $4x + 2y + z = 7$
   $9x + 3y + z = 5$

10. $p + q + r + s = 4$
    $8p + 4q + 2r + s = 15$
    $27p + 9q + 3r + s = 40$
    $64p + 16q + 4r + s = 85$
11. Suppose Norma, using the technique of finite differences, determined \( y = n^2 - n + 2 \) to be the formula for the data below.

\[
\begin{array}{c|c|c|c}
 n & 1 & 2 & 3 \\
 y & 2 & 4 & 8 \\
\end{array}
\]

a. Check that these data satisfy Norma's equation.
b. Can Norma be assured that her equation is the correct one? If so, why? If not, find another formula which Norma's data also satisfy.

12. a. Complete the table below which describes the number of diagonals \( d \) for a polygon with \( n \) sides.

\[
\begin{array}{c|c|c|c|c|c|c|c|}
 n & 3 & 4 & 5 & 6 & 7 & \ldots \\
 d & & & & & & \ldots \\
\end{array}
\]

b. Use the technique of finite differences to determine the polynomial function which models the number of diagonals in an \( n \)-gon.

13. When objects are arranged in equilateral triangles, the numbers of objects determine a sequence called triangular numbers.

\[
\begin{array}{c|c|c|c|c}
 n & 1 & 2 & 3 & 4 \\
 t_n & 1 & 3 & 6 & 10 \\
\end{array}
\]

Above are the first five triangular numbers. Find a formula which will generate any triangular number \( t_n \) in terms of \( n \), its position in the sequence.

14. One cross section of a honeycomb is a mosaic of hexagons with three hexagons meeting at each vertex. The mosaic is formed by starting with one hexagon, surrounding it with six more hexagons, and then surrounding these with another “circle” of 12 hexagons. If this pattern were to continue, find:

a. the number of hexagons in the 4th circle
b. the total number of hexagons in the first four circles
c. a polynomial equation which expresses the total number of hexagons \( h \) as a function of the number of circles \( n \)
d. the total number of hexagons in a honeycomb with 10 circles.
15. Consider the data below.

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>...</td>
</tr>
</tbody>
</table>

a. Can the data be modeled by a polynomial function?
b. If so, what is the degree of the polynomial? (Lesson 11-7)

16. Consider the graph below of a polynomial. (Lessons 11-5, 11-4, 7-2)

![Graph of a polynomial function]

a. What are the zeros of the polynomial?
b. Suppose the polynomial has degree 4. Find a polynomial function which fits this graph.
c. What is the range of this function?

In 17 and 18, consider the function $f(x) = 9x^3 - 100x$. (Lessons 11-5, 11-4)

17. a. How many zeros does $f$ have?
    b. Find the zero(s) of the function $f$.

18. Graph the function over the domain $x = -5$ to 5.

19. Sergei earns 1% interest a month on every dollar he saves from working. He receives interest on the interest, paid at the end of each month. His deposits, made on the first of each month, were $77 in March, $51 in April, $37 in May, $86 in June, $39 in July, and $35 in August. On August 1st how much will he have altogether, including the interest? [Hint: $77x^5 + 51x^4 + 37x^3 + 86x^2 + 39x + 35$] (Lesson 11-1)

20. Ben Spender recently had his credit limit reduced by 25%. What percent increase would he need to return to his original amount? (Previous course)
21. Factor \(4x^4y^2 - 16x^2y^4\). (Lesson 11-3)

In 22–24, solve for \(z\). (Lessons 10-2, 9-9, 8-6)

22. \(8^2 = 16\) \hspace{1cm} 23. \(5z^{-1/6} = 5^7\) \hspace{1cm} 24. \(\cos z = -.57\)

25. The strength \(S\) of a wooden beam of rectangular cross section is directly proportional to the width \(w\) of the beam and the square of its depth \(d\).

a. Express the above statement with a variation formula. (Lesson 2-9)

b. Suppose a beam is to be sawed from a round log of diameter 2 ft. Express \(d\) in terms of \(w\). (Previous course)

c. Use the results of part a and part b to express \(S\) as a polynomial function of \(w\). (Lesson 11-1)

d. What is the degree of the polynomial in part c? (Lesson 11-1)

26. In Example 1b, explore how to cut the pizza so that the 16 pieces are closer to the same size.
A polynomial in $x$ is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$. Polynomials arise directly from compound interest situations and questions of surface area and volume. They also can be modeled to fit points on a graph of a function. By solving a system of equations, any $n$ points can be fit by a polynomial whose degree is at most $n$. The degree of the polynomial can be found by the method of finite differences. Polynomial functions include the linear and quadratic functions and the direct variation and power functions you studied in previous chapters.

The polynomial above has degree $n$; when set equal to zero, the equation has $n$ roots. The roots of a polynomial equation $P(x) = 0$ are the zeros of the function $P$. Sometimes they can be found exactly by factoring, by substitution of a (lucky?) value for $x$, or by the Factor Theorem. They can be approximated by making a table or drawing a graph. The most efficient tables and graphs are created with the help of a calculator or computer.

**Vocabulary**

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with *. For all other terms you should be able to give a general description and a specific example of each.

**Lesson 11-1**
- polynomial in $x$
- degree of a polynomial
- coefficients of a polynomial
- leading coefficient
- general form of a polynomial
- linear, quadratic, cubic polynomials

**Lesson 11-2**
- monomial
- binomial
- trinomial
- Extended Distributive Property

**Lesson 11-3**
- Perfect Square Trinomial Patterns
- Difference of Squares Pattern
- Sum of Cubes Pattern
- Difference of Cubes Pattern

**Lesson 11-4**
- Zero Product Theorem
- zero of a function
- Factor Theorem

**Lesson 11-5**
- end behavior of a function

**Lesson 11-6**
- quartic, quintic equations
- Fundamental Theorem of Algebra
- multiplicity of a root
- Number of Roots of a Polynomial Equation Theorem

**Lesson 11-7**
- method of finite differences
- Polynomial Difference Theorem
Take this test as you would take a test in class. You will need a calculator. Then check your work with the solutions in the Selected Answers section in the back of the book.

In 1 and 2, use this information. The summer after Beth turned 16, she began saving money from her summer jobs. After the first summer she put away $750. Following the summer after her 17th birthday she saved $600. After the next summer she saved $925 and after the following two summers she put away $1075 and $800, respectively. Beth invested all this money at an interest rate of \( r \), compounded annually, and did not add or withdraw any other money.

1. If \( x = 1 + r \), write a polynomial in terms of \( x \) which models the final amount of money in her account the summer after her 21st birthday.

2. How much money would she have the summer after her 21st birthday if she had been able to invest at 7% interest?

3. Elysa needs to make a box from cardboard with dimensions 40 in. by 60 in. Find a polynomial formula for the volume \( V(x) \) of the box if she forms the box by cutting out squares of length \( x \) from each corner and folding up the sides.

In 4 and 5, consider the polynomial function \( P \) where \( P(x) = x^4 + 9x^2 - 3 - 8x^5 \).

4. What is the degree of the polynomial?

5. Is \( P(x) \) a monomial, binomial, trinomial, or none of these?

6. Multiply and simplify \((a^2 + 3a - 7)(5a + 2)\).

7. Find the zeros of the polynomial function with equation
   \[
   p(x) = 4x^3(5x - 11)(x + \sqrt{7}).
   \]

In 8 and 9, use the function \( f \), where \( f(x) = 3x^4 - 12x^3 + 9x^2 \).

8. Find its zeros.

9. Graph the function on the domain \(-1 \leq x \leq 4\).

In 10 and 11, consider the table of values below for the polynomial function with equation
   \[
   y = x^3 - 3x^2 - 3x + 9.
   \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-36</td>
</tr>
<tr>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

10. How many zeros does this polynomial have?

11. a. Between what pairs of consecutive integers must the zeros of the polynomial be located?

   b. Estimate the non-integer zeros to the nearest tenth.
12. Find all solutions: \( z^3 - 216 = 0 \).

13 and 14, *Multiple choice.*

13. A polynomial equation of degree 11 has 12 complex roots.
   (a) always  (b) sometimes  (c) never

14. When \( f(x) = x^4 + 3x - 22 \), \( f(2) = 0 \). Which is a factor of \( x^4 + 3x - 22 \)?
   (a) 0  (b) 2  (c) \( x + 2 \)  (d) \( x - 2 \)

15. Write a possible equation for the 4th degree polynomial function with the integer zeros graphed below.

![Graph of a polynomial function with integer zeros](image)

16. \( 10s^7t^2 + 15s^3t^4 \)

17. \( 9z^2 - 196 \)

18. \( 25y^2 + 60y + 36 \)

19. Refer to the table below.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … 
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>20</td>
<td>27</td>
<td>35</td>
<td>44</td>
<td>…</td>
</tr>
</tbody>
</table>

a. Can the data points be modeled by a polynomial function of degree \( \leq 5 \)?

b. If so, what is the smallest possible degree of the polynomial? If not, why not?

20. Find an equation for a polynomial function which is described by the data points below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>12</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>12</td>
<td>24</td>
<td>…</td>
</tr>
</tbody>
</table>

In 16–18, factor.
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations.
The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

Objective A. Multiply polynomials. (Lesson 11-2)
In 1–4, multiply and write in the general form of a polynomial.
1. \((x^2 + x + 3)(x - 1)\)
2. \((a + 6)(a + 7)(a + 8)\)
3. \((2y + 5)^3\)
4. \((2x^2 - x + 4)(3x - 10)\)
In 5 and 6, multiply and simplify.
5. \((2x^2 - y)(3x + y)\)
6. \((p + q + r)(p + q - r)\)

Objective B. Factor polynomials using common monomial factoring, perfect square patterns, trial and error with trinomials, or patterns for the sum or difference of cubes. (Lesson 11-3)
7. Copy and complete:
\[7a^5b^2 - 63a^2b^4 = 7a^2b^2(\_\_\_\_ + \_\_\_\_).\]
8. Copy and complete with the value(s) which will make a perfect square trinomial:
\[w^2 + \_\_\_\_ + 25.\]
In 9–16, factor.
9. \(x^2 - 14x + 49\)
10. \(a^2 - b^2\)
11. \(4x^3 - 12x^2 - 28x\)
12. \(16m^2 - 88m + 121\)
13. \(r^4s^4 - 81\)
14. \(6x^2 + 26x + 8\)
15. \(x^3 - 27\)
16. \(8g^3 + 125h^6\)

Objective C. Calculate or estimate zeros of polynomial functions. (Lessons 11-4, 11-5, 11-6)
In 17 and 18, find the exact zeros of the polynomial function with the given equation.
17. \(f(x) = x^2(x - .5)(3x + 1)\)
18. \(P(x) = x^2 - 36\)
In 19 and 20, estimate to the nearest tenth the zeros of the function by graphing or making a table.
19. \(f(x) = -9x^3 + 5x^2 - 7\)
20. \(y = x^4 + 3x^3 - 20\)
In 21–24, (a) solve. (b) Identify any multiple roots.
21. \(0 = 5x(x + 4)(9x + 7)\)
22. \(0 = (x - 1)^3(x - 2)^2\)
23. \(n^3 + 64 = 0\)
24. \(n^4 - 81 = 0\)

Objective D. Determine an equation for a polynomial function from data points using the Polynomial Difference Theorem. (Lessons 11-7, 11-8)
In 25 and 26, is the function defined below a polynomial? If so, find an equation for the polynomial. If not, why not?
25. The function \((n, a_n)\) where \(a_1 = 5\) and \(a_{n+1} = a_n - 6\) for \(n \geq 1.\)
26. \[
\begin{array}{c|ccccccc}
    x & 1 & 2 & 3 & 4 & 5 & 6 \\
    \hline
    y & 1 & 3 & 7 & 15 & 31 & 63 \\
\end{array}
\]
27. Consider the polynomial function described by the data points below.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>43</td>
</tr>
<tr>
<td>4</td>
<td>77</td>
</tr>
<tr>
<td>5</td>
<td>121</td>
</tr>
<tr>
<td>6</td>
<td>175</td>
</tr>
</tbody>
</table>

a. What is the degree of the polynomial function?

b. Multiple choice Which system of equations could be solved to find the coefficients of the polynomial?

\[
\begin{align*}
\text{(i)} & \quad 9a + 3b + c = 43 \\
\text{(ii)} & \quad 4a + 2b + c = 19 \\
& \quad a + b + c = 5 \\
\text{(iii)} & \quad 3x^2 + 3x + 3 = 43 \\
\text{(iv)} & \quad 2x^2 + 2x + 2 = 19 \\
& \quad x^2 + x + 1 = 5 \\
\text{(v)} & \quad 2a + b = 19 \\
& \quad a + b = 5 \\
\text{(vi)} & \quad \text{none of these}
\end{align*}
\]

c. Determine an equation for the polynomial function.

PROPERTIES deal with principles behind the mathematics.

**Objective E.** Use technical vocabulary to describe polynomials. (Lessons 11-1, 11-2)

In 28 and 29, state (a) the degree and (b) the leading coefficient of the polynomial.

28. \(7x^5 + 3x^2 - 15\)

29. \(1 + x - 12x^2 - 8x^9\)

In 30–33, multiple choice. State whether the polynomial is (a) a monomial, (b) a binomial, (c) a trinomial, or (d) none of (a)–(c).

30. \(x^5 - 6\)

31. \(32x^2y^3\)

32. \(\frac{6}{x^2}\)

33. \(x^2 + x + 7\)

**Objective F.** Apply the Zero Product Theorem, Factor Theorem, and Fundamental Theorem of Algebra. (Lessons 11-4, 11-6)

In 34–36, multiple choice.

34. If \(xyz = 0\), then which of the following is true?

(a) \(x = 0\)

(b) \(x = 0\) or \(y = 0\) or \(z = 0\)

(c) \(x = 0\) and \(y = 0\) and \(z = 0\)

(d) none of these

35. Every polynomial equation of degree \(n\) has exactly \(n\) roots, provided that \(n\) roots are counted separately.

36. Suppose \(p(x)\) is a polynomial, \(p(r) = 0\), \(p(s) = 0\), and \(p(t) = 7\). Which of the following is not true?

(a) \(p(r) \cdot p(s) = 0\)

(b) \(k(x - r)(x - s)(x - t) = p(x)\)

(c) \(r\) and \(s\) are \(x\)-intercepts of \(p(x)\)

(d) \(r\) and \(s\) are roots of the equation \(p(x) = 0\)

37. True or false If \((x - 7)\) is a factor of some polynomial function \(P\), then \(P(7) = 0\).

38. Suppose \(x - r\) and \(x - s\) are factors of a quadratic polynomial \(p(x)\). Which of the following is not true for all \(x\)?

(a) \(p(r) = 0\)

(b) \(p(s) = 0\)

(c) \(k(x - r)(x - s) = p(x)\)

(d) \((x - r)(x - s) = 0\)

39. Find an equation for a quadratic polynomial function whose graph crosses the \(x\)-axis at \((-69, 0)\) and \((-4.5, 0)\).

40. Find equations for two polynomial functions whose zeros are \(-12, 0, \frac{1}{4}\), and \(\frac{1}{6}\).

In 41 and 42, explain why the Zero Product Theorem cannot be used directly on the given equation.

41. \((x - 8)(x + 11) = 10\)

42. \(2(x + 2) - (x + \frac{2}{3}) = 0\)
USES deal with applications of mathematics in real situations.

Objective G. Use polynomials to model real world situations. (Lessons 11-1, 11-2, 11-8)

43. Each birthday from age 9 on, Charles decided to save $150 of his gifts. He put the money in a savings account at an interest rate of r, compounded annually, without withdrawing or adding any other money.
   a. Write a polynomial in x, where 
      \[ x = 1 + r, \] 
      that represents the amount of money he would have after his 16th birthday.
   b. If the bank pays 6% interest annually, calculate how much money Charles would have after his 16th birthday?
   c. At about what rate of interest would Charles have to invest in order to have $2000 on his 16th birthday?

In 44 and 45, suppose that a manufacturer determines that n employees on a certain production line will produce f(n) units per month, where
   \[ f(n) = 80n^2 - 0.1n^4. \]

44. How many units will be produced monthly by
   a. 3 employees?
   b. 10 employees?

45. Use an automatic grapher to sketch a graph of f, and determine a reasonable domain for f in this model.

In 46 and 47, consider that a worker cuts a square out of each corner of a piece of sheet metal which measures 1 m \times 1.5 m. If the length of the side of the square is x meters long, find a polynomial for each quantity:

46. V(x), the volume of the box when folded.
47. S(x), the surface area of the open box.

48. Recall that when a beam of light in air strikes the surface of water it is refracted or bent (see page 578, question 10). Below at the right are the earliest known data on the relation between i, the angle of incidence in degrees, and r, the angle of reflection in degrees. The measurements are recorded in the Optics of Ptolemy, a Greek scientist who lived in the 2nd century A.D.
   a. Can these data be modeled by a polynomial function?
   b. If so, what is the degree of the polynomial function? If not, explain why a polynomial function is not a good model.

   \[
   \begin{array}{c|c}
   i & r \\
   \hline
   10 & 8 \\
   20 & 15.5 \\
   30 & 22.5 \\
   40 & 29 \\
   50 & 35 \\
   60 & 40.5 \\
   70 & 45.5 \\
   80 & 50 \\
   \end{array}
   \]

49. Suppose that Theresa stacks soccer balls in a triangular pyramid display.
   a. Complete the table below where n is the number of rows and \( T \) is the total number of soccer balls.

   \[
   \begin{array}{c|c|c|c|c|c|c}
   \hline
   n & 1 & 2 & 3 & 4 & 5 & 6 \\
   \hline
   T & 1 & 4 & & & & \\
   \hline
   \end{array}
   \]
   b. How many soccer balls are needed for a display with \( n \) rows?
Objective H. Determine properties of a polynomial function from its graph. (Lessons 11-4, 11-5)
In 50–52, refer to the graph below for the general polynomial function
\[ y = P(x) = a_3x^3 + a_2x^2 + a_1x + a_0. \]

![Graph of a cubic function](image)

50. Multiple choice \( a_0 = ? \)
   (a) 24  (b) 2  (c) 0  (d) -3

51. True or false \( P(2) = P(-2) \)

52. Find an equation for the polynomial.

53. Refer to the graph of the polynomial function \( y = f(x) \) below.

![Graph of a quadratic function](image)

a. What are the zeros of this function?
b. What does your answer to part a imply about the degree of \( f \)?
c. Find an equation for this polynomial function.

c. How many roots of the equation \( x^4 + 10x^3 + 5x^2 + 200 = 0 \) are not real?

Objective I. Read or generate computer output to graph or find zeros of polynomials. (Lessons 11-5, 11-6)
In 54–56, use the computer output below for the function
\[ J(x) = -x^3 - x^2 + 7x + 18. \]

54. Sketch a graph of \( y = J(x) \).

55. How many \( x \)-intercepts does this polynomial function have?

56. Use a computer program or a function grapher to estimate each \( x \)-intercept to the nearest tenth.

57. Refer to the graph of the function \( f(x) = x^4 + 10x^3 + 5x^2 + 200 \) below.
   a. Name two pairs of consecutive integers between which a zero of \( f \) must occur.
   b. Estimate each real zero of \( f \) to the nearest hundredth.
A quadratic equation in two variables \( x \) and \( y \) is an equation which can be written in the form

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,
\]

where \( A, B, C, D, E, \) and \( F \) are real numbers, and at least one of \( A, B, \) or \( C \) is not zero. If the relation symbol in the sentence above is the equal symbol = or one of the inequality symbols \( >, <, \geq, \) or \( \leq \), the resulting sentence is called a **quadratic relation in two variables**.

The parabolas you studied in Chapters 2 and 6 and the hyperbolas studied in Chapter 2 are examples of curves that can be described by quadratic relations. Circles, which you have studied since elementary school, can also be described by quadratic equations in two variables.

Quadratic relations have connections with a wide variety of ideas you already know. They describe the paths of thrown objects, the orbits of comets and planets, and the shapes of communication receivers and mirrors used in car headlights.

Quadratic relations may also be interpreted geometrically. Surprisingly, all quadratic equations in two variables can be derived from the intersection of a **double cone**, shown at the left, and a plane. Such cross-sections of a double cone are usually called conic sections, or simply conics.
The four most important conic sections are hyperbolas, parabolas, ellipses, and circles.

In this chapter you will study quadratic relations algebraically and geometrically; that is, as equations or inequalities and as figures with certain properties. You will also learn how to solve systems of quadratic equations.
Circles occur in many situations. You have probably noticed that when you throw a pebble into a calm body of water, concentric circles soon form around the point where the pebble hit the water. Similarly, when an earthquake occurs, energy waves radiate in concentric circles from the epicenter, the point on the earth’s surface above the point where the earthquake began. Using an instrument called a seismograph, scientists can calculate the distance from a recording station to the epicenter. In Lesson 12-10 you will see how seismographs at three stations can determine the exact location of an earthquake’s epicenter.

In this lesson you will learn how to find an equation for any circle. Recall the definition of a circle.

**Definition:**

A circle is the set of all points in a plane at a given distance (its radius) from a fixed point (its center).

From its definition and the Distance Formula, you can find an equation for any circle.

**Example 1** Suppose a seismograph shows the epicenter to be about 60 miles away from Station 1. Find an equation for the set of points satisfying this condition.

**Solution** The given information means that the epicenter is located somewhere on a circle with center at Station 1 and radius 60. Set up a coordinate system with (0, 0) at Station 1. Let (x, y) be any point on the circle. The distance between (x, y) and (0, 0) is 60.

The Distance Formula gives

\[ \sqrt{(x - 0)^2 + (y - 0)^2} = 60. \]

By squaring both sides,

\[ (x - 0)^2 + (y - 0)^2 = 60^2 \]

or

\[ x^2 + y^2 = 3600. \]
Notice that the equation determined in Example 1 is a quadratic equation in $x$ and $y$.

**Example 2**

Suppose Station 2 is 150 miles east and 100 miles north of Station 1; that is, at the point $(150, 100)$. If the seismograph at this station shows the epicenter to be 130 miles away, find an equation for the circle 130 miles from Station 2.

**Solution**

The epicenter is on a circle with center $(150, 100)$ and radius 130.

Let $(x, y)$ be any point on the circle. Using the Distance Formula,

$$
\sqrt{(x - 150)^2 + (y - 100)^2} = 130.
$$

Squaring gives

$$(x - 150)^2 + (y - 100)^2 = 16900.
$$

**Check**

The point $(280, 100)$ is 130 units from $(150, 100)$ and its coordinates should satisfy the equation.

Does

$$(280 - 150)^2 + (100 - 100)^2 = 16900?$$

Does

$$130^2 + 0^2 = 16900? \text{ Yes.}$$

Notice that Examples 1 and 2 together determine that the epicenter is at one of two points, $A$ or $B$. A third circle is needed to determine which is really the epicenter.
Example 2 can be generalized to determine an equation for any circle. Let \((h, k)\) be the center of a circle with radius \(r\), and let \((x, y)\) be any point on the circle. Then from the definition of a circle, the distance between \((x, y)\) and \((h, k)\) equals \(r\).

From the Distance Formula,

\[
\sqrt{(x - h)^2 + (y - k)^2} = r.
\]

Squaring gives

\[
(x - h)^2 + (y - k)^2 = r^2.
\]

This argument proves the following theorem.

**Theorem (Center-Radius Equation for a Circle):**

The circle with center \((h, k)\) and radius \(r\) is the set of points \((x, y)\) that satisfies

\[
(x - h)^2 + (y - k)^2 = r^2.
\]

For a circle centered at the origin, \((h, k) = (0, 0)\). The equation

\[
(x - h)^2 + (y - k)^2 = r^2
\]

becomes

\[
(x - 0)^2 + (y - 0)^2 = r^2
\]

or

\[
x^2 + y^2 = r^2.
\]

This proves the following special case of the Center-Radius Equation for a Circle.

**Theorem:**

The circle with center at the origin and radius \(r\) is the set of points \((x, y)\) that satisfies the equation \(x^2 + y^2 = r^2\).

The following example illustrates how to graph a circle given an equation for it.

**Example 3**

a. Find the center and radius of the circle with equation

\[
(x - 1)^2 + (y + 2)^2 = 9.
\]

b. Graph this circle.
Solution

a. The equation is in the center-radius form for a circle with 
\((h, k) = (1, -2)\) and \(r = 3\).
b. You can sketch this circle by locating the center and then 
four points whose distance from the center is 3, as 
illustrated at the right.

If you know an equation for a circle and one coordinate of a point 
on the circle, you can determine the other coordinate of that point.

Example 4 Refer to the circle in Example 3. There are two points at which \(x = 3\). 
Find the \(y\)-coordinate of each point.

Solution Substitute \(x = 3\) into the equation for the circle, and solve 
for \(y\).

\[
(3 - 1)^2 + (y + 2)^2 = 9 \\
4 + (y + 2)^2 = 9 \\
(y + 2)^2 = 5 \\
y + 2 = \pm \sqrt{5} \\
y = -2 \pm \sqrt{5}
\]

So \(y \approx 0.236\) or \(y \approx -4.236\).

Check Refer to the graph above. Both \((3, 0.236)\) and \((3, -4.236)\) 
seem to be on the circle.

Questions

1. State the general form of a quadratic equation in \(x\) and \(y\).
2. A conic section is the intersection of a(n) \_\_\_ and a(n) \_\_\_.
3. Name four types of conic sections.
4. As conic sections, how do parabolas and hyperbolas differ?
5. What is the epicenter of an earthquake?
In 6 and 7, consider the circle with equation $x^2 + y^2 = 60^2$.

6. **Multiple choice** Find the radius of this circle.
   (a) 30   (b) $\sqrt{60}$   (c) 60   (d) 3600

7. **Multiple choice** Tell which point(s) is (are) on the circle.
   (a) (0, 0)   (b) (0, 60)   (c) (-60, 0)   (d) (30, 30)   (e) (60, 60)

8. The circle with equation $(x - h)^2 + (y - k)^2 = r^2$ has center $\_\_\_\_$
   and radius $\_\_\_\_$.

In 9 and 10, for each circle (a) state its center; (b) state its radius; and (c) sketch it.

9. $x^2 + y^2 = 25$

10. $(x - 5)^2 + (y - 1)^2 = 36$

11. Consider the circle $x^2 + y^2 = 100$.
    a. Find the $y$-coordinates of all points where $x = -6$.
    b. Sketch the circle.

In 12 and 13, find an equation for the circle with the given center $C$ and radius $r$.

12. $C = (0, 0); r = 9$

13. $C = (-3, -2); r = 8$

---

**Applying the Mathematics**

In 14 and 15, use the coordinates of the earthquake recording stations in Example 2.

14. Suppose Station 3 is 40 miles west and 125 miles north of Station 1. If the epicenter of an earthquake is 100 miles from Station 3, give an equation for the circle on which it must lie.

15. Suppose the epicenter is known to be 30 miles east of Station 1, and 60 miles away. Then how far north or south of the station is it?

16. A circle has center at the origin and radius $\sqrt{10}$.
    a. Find an equation for this circle.
    b. Give the coordinates of eight points on the circle with integer coordinates.
    c. Graph the circle.

17. a. Expand the binomials in the equation for the circle of Example 2. Then simplify to get an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
    b. What are the values of $A, B, C, D, E,$ and $F$?
In 18–20, the shape of the light beam from a flashlight is a cone. When that cone of light hits a flat surface, the outline is a conic section. Use an actual flashlight and a wall in a darkened room to tell which conic section is formed when the flashlight is held:

18. perpendicular to the wall
19. at an angle of 75° to the wall
20. touching the wall, parallel to it.

21. a. Graph \(\{(x, y): y = x^2\}\).
   b. Name the line of symmetry. (Lesson 2-5)

22. Skill sequence Expand and simplify. (Lessons 6-1, 11-2)
   a. \((x + 3)^2 + y^2\)
   b. \((x + y)^2 + 3^2\)
   c. \((x + y + 3)^2\)

23. Skill sequence Solve for \(y\). (Lesson 6-1)
   a. \(y^2 = 100\)
   b. \(25 + y^2 = 100\)
   c. \(x^2 + y^2 = 100\)

24. Find an equation for the line containing the origin and \((5, -4)\). (Lesson 3-5)

25. Let \(A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}\) and \(B = \begin{bmatrix} 0.2 & -0.4 \\ 0.1 & 0.3 \end{bmatrix}\).
   a. Find \(AB\).
   b. How are \(A\) and \(B\) related? (Lessons 4-5, 5-5)

26. What are the zeros of the polynomial function graphed at the right? (Lesson 11-4)

27. A lattice point is a point with integer coordinates. If possible, find an equation for a circle that passes through
   a. no lattice points
   b. exactly one lattice point
   c. exactly two lattice points
   d. exactly three lattice points
   e. more than ten lattice points.
Many vertical lines intersect a circle in two points. Consequently, the Vertical-Line Test shows that a circle is a relation but not a function. Thus many automatic function graphers cannot graph a circle directly. With these graphers, you need to think of the circle as the union of two semicircles, each of which is a function. To graph the circle \( x^2 + y^2 = 100 \) with center \((0, 0)\) and radius 10, solve the equation for \( y \) and graph each part separately.

\[
\begin{align*}
  x^2 + y^2 &= 100 \\
  y^2 &= 100 - x^2 \\
  y &= \sqrt{100 - x^2} \text{ or } y = -\sqrt{100 - x^2}
\end{align*}
\]

Semicircles occur often in architecture. The following example shows how to use graphs and equations to determine the height of a semicircular arch at points on the arch.

**Example 1** A semicircular arch over a street has radius 10 feet. How high is it at a point whose ground distance is 4 feet from the center?

**Solution** Make a mathematical model of the situation. Imagine that the arch is on a coordinate system with the \( x \)-axis representing the street and the origin at the center of the arch. Then the circle determined by the arch has center \((0, 0)\) and radius 10, so its equation is \( x^2 + y^2 = 10^2 \). The height of the circle 4 feet from the center equals the \( y \)-coordinate of the point on the graph where \( x = 4 \). Thus, you should evaluate \( x^2 + y^2 = 100 \) at \( x = 4 \).
Substitute 4 for $x$ in the equation and solve for $y$.

\[
16 + y^2 = 100 \\
y^2 = 84 \\
y = \sqrt{84} \text{ or } -\sqrt{84}
\]

It is necessary to reject $-\sqrt{84}$ because $y$, the height above the ground, cannot be negative. Thus, $y = \sqrt{84} \approx 9.17$. The bridge is about 9.17 ft (about 9 ft 2 in.) high at a point whose ground distance is 4 ft from its center.

**Check 1** Examine the graph. This value looks about right.

**Check 2** Substitute $(4, 9.17)$ into $x^2 + y^2 = 100$.
Does $4^2 + (9.17)^2 \approx 100$? Yes.

Every circle separates the plane into three regions. The region inside the circle is called the **interior** of the circle. The region outside the circle is called the **exterior** of the circle. The circle itself is the **boundary** between these two regions.

Concentric circles are often used in target practice. The object can hit the target only in the shaded regions and not on the boundaries. To describe the regions mathematically, place the target on a coordinate system with the center at $(0, 0)$. The region worth 50 points is the interior of the circle with radius 3. All points in this region are less than 3 units from the origin. Thus if $(x, y)$ is a point in this region, from the distance formula you can conclude that $\sqrt{x^2 + y^2} < 3$. In the inequality, the expressions on both sides are positive. Whenever $a$ and $b$ are positive and $a < b$, then $a^2 < b^2$. Thus, when both sides of the inequality $\sqrt{x^2 + y^2} < 3$ are squared, the sentence becomes $x^2 + y^2 < 9$. So $x^2 + y^2 < 9$ describes the region worth 50 points.
Example 2  Write a sentence to describe all points in those regions of the target worth less than 40 points.

Solution  All \((x, y)\) in the region worth less than 40 points are in the exterior of the circle with radius 6. The distance formula gives \(\sqrt{x^2 + y^2} > 6\) or, squaring both sides, \(x^2 + y^2 > 36\).

The two instances above can be generalized in the following theorem.

**Theorem (Interior and Exterior of a Circle):**

Given a circle with center \((h, k)\) and radius \(r\).

The interior of the circle is described by

\[(x - h)^2 + (y - k)^2 < r^2.\]

The exterior of the circle is described by

\[(x - h)^2 + (y - k)^2 > r^2.\]

If \(\geq\) or \(\leq\) is used, the boundary is included.

Example 3  Graph the points satisfying \((x - 3)^2 + (y + 5)^2 \geq 16\).

Solution  The sentence represents the union of a circle, with center at \((3, -5)\) and radius 4, and its exterior.

Example 4  Consider the target on page 675. Write a sentence to describe all points in the region worth 30 points.

Solution  This region is the intersection of the interior of the circle with radius 9 and the exterior of the circle with radius 6.

The interior of the circle with radius 9 is the set \(\{(x, y): x^2 + y^2 < 81}\), and the exterior of the circle with radius 6 is the set \(\{(x, y): x^2 + y^2 > 36}\).

So \(\{(x, y): 36 < x^2 + y^2 < 81\}\) describes the 30-point region.
Questions

1. Sketch by hand on separate sets of axes.
   a. \( x^2 + y^2 = 49 \)  
   b. \( y = \sqrt{49 - x^2} \)  
   c. \( y = -\sqrt{49 - x^2} \)

2. Refer to Example 1. How high is the arch at a point whose ground distance is 2 feet from the center?

3. The region outside a circle is called its \( \square \).

4. Multiple choice On the target shown in the lesson, all \((x, y)\) in the region worth 50 points lie
   (a) in the interior of the circle with radius 3.
   (b) on the circle with radius 3.
   (c) in the exterior of the circle with radius 3.

5. Write a sentence to describe the set of points \((x, y)\) in the 40-point region of the target.

6. Graph the points satisfying \((x - 3)^2 + (y + 5)^2 < 16\).

In 7 and 8, multiple choice. Given a circle with center \((h, k)\) and radius \(r\), state which of the following the given sentence describes:
(a) the interior of the circle  
(b) the exterior of the circle  
(c) the union of the circle and its interior  
(d) the union of the circle and its exterior
7. \((x - h)^2 + (y - k)^2 \geq r^2\)  
8. \((x - h)^2 + (y - k)^2 < r^2\)

In 9 and 10, use your automatic grapher to graph each entire circle.

9. \(x^2 + y^2 = 8\)  
10. \((x + 3)^2 + (y + 4)^2 = 25\)

11. The BASIC program at the left printed the output at the right.

```
10 INPUT "RADIUS"; R
20 PRINT "X" "Y" 
30 FOR X = -R TO R STEP 0.5 
40 Y1 = SQR(R^2 - X^2)
50 PRINT X, Y1
60 NEXT X
70 FOR X = -R TO R STEP 0.5 
80 Y2 = -1 * SQR(R^2 - X^2)
90 PRINT X, Y2
100 NEXT X
110 END
```

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1.5</td>
<td>1.322876</td>
</tr>
<tr>
<td>-1</td>
<td>1.732051</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.936492</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0.5</td>
<td>1.936492</td>
</tr>
<tr>
<td>1</td>
<td>1.732051</td>
</tr>
<tr>
<td>1.5</td>
<td>1.322876</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

a. Plot the coordinates given in the output.

b. Find the value of \(R\) input by the user.

c. Find an equation for the circle you graphed in part a.

d. To print a table of solutions to \(x^2 + y^2 = 169\), what number should be input for \(R\)?

e. What lines in the program generate the points on the semicircle above the \(x\)-axis?
12. A moving van 6 ft wide and 12 ft high is approaching a semicircular tunnel with radius 13 ft.
   a. Explain why the truck cannot pass through the tunnel if it goes on only one side of the median strip.
   b. Can the truck fit through the tunnel if it is allowed to drive anywhere on the roadway? Justify your answer.

13. In sumo wrestling, the participants wrestle in the interior of a circle. A wrestler wins by pushing his opponent out of the circle. Suppose the circle has radius \( r \) and its center is \((0, 0)\).
   a. What sentence describes positions where a wrestler is in bounds?
   b. What sentence describes losing positions?

In 14–16, refer to the graph at the right.

14. What inequality describes the points in the checked region?

15. What inequality describes all \((x, y)\) in the striped region?

16. \textit{Multiple choice}  The region described by \[ \{(x, y): 4 < x^2 + y^2 < 16\} \] is
   (a) the gray region.
   (b) the union of the gray region and its inner boundary.
   (c) the union of the gray region and its outer boundary.
   (d) the union of the gray region and both its boundaries.

In 17 and 18, define the term. \textit{(Lessons 6-3, 12-1)}

17. circle
18. parabola

19. a. Find an equation for the circle with center at \((0, 0)\) and radius 1. \textit{(Lesson 12-1)}
   b. What is this circle called? \textit{(Lesson 10-4)}

20. A circle has center at the origin and passes through the point \((3, -4)\).
   a. Find the radius of the circle.
   b. Find an equation for the circle. \textit{(Lesson 12-1, Previous course)}

In 21 and 22, recall that in football a touchdown is worth 6 points, a field goal 3 points, a safety 2 points, and a point-after-touchdown 1 point.

21. If a team gets \( T \) touchdowns, \( F \) field goals, \( S \) safeties, and \( P \) points-after-touchdown, how many total points does it have? \textit{(Lesson 3-3)}

22. A team has no safeties and no points-after-touchdown and a total of at most 27 points. Graph the set of possible ways this could happen. \textit{(Lesson 3-9)}
23. Semicircular arches were popular with the early Romans. Below is an example. Prepare a brief report on the use of circles and semicircles in architecture. Consider Roman, Renaissance, and modern uses of this form.

24. Can you draw a circle with a ruler? "Of course not," you may think. "A circle is round and a ruler is straight."

a. Try this. Mark a point \( P \) on a sheet of plain paper. Take a ruler and put one edge so that it goes through the point. Then draw a line along the other edge.

b. Repeat this twice using the same point and the same ruler. Perhaps you have something like this:

c. Draw more lines in the same way. You will begin to see something like the following:

d. The lines are said to form an envelope of a circle. No line drawn as suggested above will intersect the interior of this circle. Where is the center of the circle you've formed? What is its radius?

e. If you were to repeat this process using a ruler of a different width, how would the outcome be affected?
The first pages of this chapter used 3-dimensional ideas to describe ellipses and hyperbolas as conic sections. It is also possible to give definitions for the curves using only 2 dimensions.

Recall from Lesson 6-3 that a parabola is determined by a point (its focus) and a line (its directrix). Both ellipses and hyperbolas—although they look totally unlike each other—are determined by two points (their \textit{foci}, pronounced "foe sigh," plural of focus) and a number (the \textit{focal constant}). Consider first an ellipse.

Let the foci be the points $F_1$ and $F_2$ with the distance between them $F_1F_2 = 12$. The focal constant can be any number larger than this distance. Suppose it is 20. A point $P$ is on this ellipse if and only if the sum of its distances from the foci equals 20.

In the drawing below, think of $F_1$ and $F_2$ as the foci. The Triangle Inequality guarantees that the sum of distances from $P$ to the foci must be greater than 12. Here $PF_1 + PF_2 = 20$, which is greater than 12.

Now we look for all points $P$ such that $PF_1 + PF_2 = 20$. These points are on a curve. The drawing below shows the curve and six points on it, $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, and $P_6$. You should verify that $P_nF_1 + P_nF_2 = 20$ for each $n$. By definition, the curve is said to be an \textit{ellipse}, with foci $F_1$ and $F_2$ and focal constant 20.

\textbf{Definition:}

Let $F_1$ and $F_2$ be any two points and $d$ be a constant with $d' > F_1F_2$. Then the \textit{ellipse with foci $F_1$ and $F_2$ and focal constant $d$} is the set of points $P$ in a plane which satisfy $PF_1 + PF_2 = d$. 

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We need \( d > F_1F_2 \) because of the Triangle Inequality. The **vertices** of the ellipse are the points of intersection of the ellipse and the line containing its foci. So in the diagram above, points \( P_3 \) and \( P_5 \) are the vertices of the ellipse. Note that the distance \( P_3P_5 = 20 \), the focal constant.

Graph paper consisting of two intersecting sets of concentric circles makes it easy to draw ellipses. Such graph paper is sometimes called *conic graph paper*. In the conic graph paper below the centers of the two sets of circles are 12 units apart.

**Example 1** Draw the ellipse with \( F_1F_2 = 12 \) and \( PF_1 + PF_2 = 20 \).

**Solution** Use conic graph paper for which \( F_1F_2 = 12 \). Now consider two numbers whose sum is 20, say 14 and 6. Mark the four points that are 14 units from one focus and 6 units from the other. On the figure below we have labeled such points \( P_1, P_2, P_3, \) and \( P_4 \). Find two other numbers whose sum is 20, say 15 and 5. Mark the four points that are 15 units from one focus and 5 from the other. Continue for other pairs of whole numbers whose sum is 20. Draw a smooth curve through the points you have marked. Note that \( V_1 \) and \( V_2 \), which were found using the pair of numbers 4 and 16, are the vertices of the ellipse.
If, in the definition of an ellipse, the distances from $P$ to the foci are subtracted instead of added, then the curve that results is a hyperbola.

**Definition:**

Let $F_1$ and $F_2$ be any two points and $d$ be a constant with $0 < d < F_1F_2$. Then the hyperbola with foci $F_1$ and $F_2$ and focal constant $d$ is the set of points $P$ in a plane which satisfy $|PF_1 - PF_2| = d$.

Consider again points $F_1$ and $F_2$ with $F_1F_2 = 12$, and the set of points $P$ such that $|PF_1 - PF_2| = 10$. Below are three points, $P_1$, $P_2$, and $P_3$, which lie on the resulting hyperbola.

**Example 2**

Use conic graph paper to draw a hyperbola in which the distance between foci is $F_1F_2 = 12$ and the focal constant is $|PF_1 - PF_2| = 10$.

**Solution**

Use graph paper in which the centers of the two sets of circles are 12 units apart. Find a pair of numbers whose difference is 10, say 16 and 6. Mark a point $P$ such that $PF_1 = 16$ and $PF_2 = 6$. (Notice that there are two such points, labeled $P_1$ and $P_2$ in our drawing below.) Now mark points such that $PF_2 = 16$ and $PF_1 = 6$. (We have labeled these $P_3$ and $P_4$.) Continue marking points determined by other pairs of numbers whose difference is 10. Connect these points with two smooth curves.
The two unconnected parts of a hyperbola are called its **branches**. One branch results from $PF_1 - PF_2 = d$; the other from $PF_1 - PF_2 = -d$. The segment joining the foci intersects the branches in the **vertices** of the hyperbola. In the hyperbola in Example 2, the vertices $V_1$ and $V_2$ can be found using the pair of numbers 11 and 1.

At first glance, each branch may seem to be a parabola, but the shape of the curve is different. In particular, as points on a branch get farther from the foci, they approach but do not meet one of the two orange lines. These lines are the **asymptotes** of the hyperbola.

Every ellipse and hyperbola has two symmetry lines, the line through the foci and the perpendicular bisector of the segment joining the foci. So once you have found a point on one of these curves, you can use that point to find three others.

Ellipses and hyperbolas have many applications. In 1609 Johannes Kepler discovered that each planet orbits the sun in an ellipse in which the sun is at one focus. This is known as Kepler's first law of planetary motion. About 60 years later, Sir Isaac Newton used this idea to formulate his theory of universal gravitation. Besides planets, moons and artificial satellites around planets have elliptical orbits. Comets either have elliptical or hyperbolic orbits.
Questions

In 1–3, refer to Example 1.
1. The distance between the foci is ______.
2. The focal constant is ______.
3. \( F_1V_2 + F_2V_2 = ? \).

In 4 and 5, give the singular form of each word.
4. foci  
5. vertices

In 6 and 7, true or false.
6. The focal constant of a hyperbola equals the distance between the foci.
7. If \( F_1 \) and \( F_2 \) are the foci of a hyperbola, then \( F_1F_2 \) is a line of symmetry for the curve.
8. The orbit of a comet is either ______ or ______.

In 9 and 10, use conic graph paper with concentric circles with radii from 1 to 16 units, and with centers 10 units apart.

9. Draw an ellipse with foci \( F_1 \) and \( F_2 \), where \( F_1F_2 = 10 \), and with focal distance 14.
10. Draw a hyperbola with \( F_1F_2 = 10 \) and \( |PF_1 - PF_2| = 2 \).

In 11–13, use this definition. The eccentricity of an ellipse or hyperbola is the ratio of the distance between its foci to its focal constant.
11. What is the eccentricity of the ellipse in Example 1?
12. Sketch an ellipse with eccentricity \( \frac{1}{2} \).
13. Why must the eccentricity of an ellipse be a number between 0 and 1?
14. An elliptical surface has a special reflecting property. When sound, light, or some other object originating at one focus reaches the ellipse, it is reflected in such a way that it passes through the other focus. Suppose you are playing pool at an elliptical table which has only one pocket located at one focus. If the cue ball is placed at the other focus, trace a path the ball would follow if it strikes the cushion.

![Elliptical Surface Diagram]

15. The points (10, 0), (0, 5), (8, 3), and (6, 4) satisfy \(x^2 + 4y^2 = 100\).
   a. Use this information to find 8 other points with integer coordinates which satisfy the equation.
   b. Graph these 12 solutions.
   c. Name the conic section being graphed.
   d. Give equations for the symmetry lines.

16. Consider the equation \(x^2 - y^2 = 144\).
   a. Find 14 points with integer coordinates between -20 and 20 which satisfy this equation.
   b. Graph these points.
   c. Name the vertices.
   d. Conjecture which lines might be the asymptotes for this hyperbola.

17. Use a ruler and compass.
   a. Draw two points, \(F_1\) and \(F_2\), 3 cm apart as shown at the right. Now find five points \(P_n\), \(n = 1, 2, 3, 4, \) and 5, such that \( P_nF_1 + P_nF_2 = 5 \) cm.
   b. The points \(P_n\) in part a lie on a(n) _____ with foci _____ and _____ and focal constant ___.
   c. Sketch the rest of the curve satisfying conditions in parts a and b.

18. The figure below shows a cross-section of a semicircular tunnel with diameter 40 feet. A sign [Entering Tunnel—Do Not Pass] must be hung 16 feet above the roadway. Find the length \(BE\) of the beam that will support the bottom of the sign. (Lesson 12-2)
19. Write a sentence describing the set of all points \((x, y)\) in the shaded region at the right. \((Lesson\ 12-2)\)

20. Find (a) the center and (b) the radius of the circle with the given equation. \((Lesson\ 12-1)\)

21. \(x^2 + y^2 = 2\)

22. The circle at the right is tangent to the axes at \((2, 0)\) and \((0, 2)\).
   a. Write an equation for this circle.
   b. Find two values of \(y\) such that \((1, y)\) is on this circle. \((Lesson\ 12-1)\)

23. Give the distance between \((x, y)\) and \((c, 0)\). \((Previous\ course)\)

24–27. Simplify. \((Lessons\ 6-1,\ 8-5)\)

24. \((\sqrt{x})^2\)

25. \((\sqrt{x} + 3)^2\)

26. \((\sqrt{x} + 3)^2\)

27. \((2a - \sqrt{p})^2\)

28. a. Find the vertices of the image of the square at the right under \(S_{5,1/2}\).
   b. Describe what \(S\) does to the preimage.
   c. Find the area of the square and the area of the image. \((Lesson\ 4-4)\)

29. a. Using two thumbtacks and a piece of string, draw a curve as shown below.
   b. Explain why the curve is an ellipse.
A boat company operates a sightseeing tour and shuttle between two small islands 12 miles apart. Because of fuel restrictions boats cannot travel more than 20 miles in going from one island to the other.

This situation is like that found in the previous lesson. A boat can go no farther than the ellipse determined by the islands as foci and focal constant 20 miles.

To find an equation for this ellipse, consider a coordinate system that locates the origin midway between the islands with the islands on an axis. Then the islands are located at (-6, 0) and (6, 0). This is called the standard position for the ellipse.

If \( P = (x, y) \) is on the ellipse, then by the definition of an ellipse,
\[
PF_1 + PF_2 = 20.
\]

So by the Distance Formula,
\[
\sqrt{(x + 6)^2 + (y - 0)^2} + \sqrt{(x - 6)^2 + (y - 0)^2} = 20
\]
\[
\text{or} \quad \sqrt{(x + 6)^2 + y^2} + \sqrt{(x - 6)^2 + y^2} = 20.
\]

This is an equation for the ellipse and it is quite complicated. Surprisingly, it is easier to begin with a more general case. The resulting simplified equation is well worth the effort it takes to get it.
In the following theorem, the focal constant is called $2a$, rather than $d$, because that simplifies equations in the proof starting with step 6:

**Theorem (Equation for an Ellipse):**

The ellipse with foci $(c, 0)$ and $(-c, 0)$ and focal constant $2a$ has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2 - c^2$.

**Proof**

Let $F_1 = (-c, 0), F_2 = (c, 0)$, and $P = (x, y)$. We number the steps for reference.

1. By the definition of an ellipse,

$$PF_1 + PF_2 = 2a.$$

Using the Distance Formula, this becomes

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

2. Subtracting one of the square roots from both sides,

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}.$$

3. Squaring both sides (the right side is like a binomial),

$$(x - c)^2 + y^2 = 4a^2 - 2a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2.$$

4. Expanding binomials and doing appropriate subtractions,

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx.$$

5. Using the Addition Property of Equality and rearranging terms,

$$4a\sqrt{(x + c)^2 + y^2} = 4a^2 + 4cx.$$

6. Multiplying by $\frac{1}{4}$,

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx.$$

7. Squaring a second time,

$$a^2[(x + c)^2 + y^2] = a^4 + 2a^2cx + c^2x^2.$$
8. Expanding the binomial and subtracting $2a^2cx$ from both sides,
\[ a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2. \]

9. Subtracting $a^2c^2$ and $c^2x^2$ from both sides, then factoring,
\[ (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2). \]

10. Since $c > 0$, $F_1F_2 = 2c$, and $2a > F_1F_2$, we have $2a > 2c > 0$. So $a > c > 0$. Thus $a^2 > c^2$ and $a^2 - c^2$ is not negative. So $a^2 - c^2$ can be considered as the square of some real number, say $b$. Now let $a^2 - c^2 = b^2$ and substitute.
\[ b^2x^2 + a^2y^2 = a^2b^2 \]

11. Dividing both sides by $a^2b^2$, \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]

The equation \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] is in the standard form for an equation of this ellipse. By substitution, it is easy to check that $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$ are on this ellipse. This helps graph it. Note that $(-a, 0)$ and $(a, 0)$ are the vertices of the ellipse.

Since $2a$ is the focal constant, $B_1F_1 = a$. By the Pythagorean theorem, $b^2 + c^2 = a^2$. This confirms step 10 of the proof that $b^2 = a^2 - c^2$.

The segments $\overline{A_1A_2}$ and $\overline{B_1B_2}$ are, respectively, the major and minor axes of the ellipse. (The major axis contains the foci and is always longer.) The two axes lie on the symmetry lines and intersect at the center $O$ of the ellipse. The previous diagram illustrates the following theorem which applies to all ellipses centered at the origin with foci on one of the coordinate axes.

**Theorem:**

In the ellipse with equation \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \], $2a$ is the length of the horizontal axis, and $2b$ is the length of the vertical axis.
The longer axis, on which the foci lie, is the major axis. Its length is the focal constant. Specifically, if \(a > b\), then \((c, 0)\) and \((-c, 0)\) are the foci, the focal constant is \(2a\), and \(b^2 = a^2 - c^2\) as in the ellipse on page 688. If \(b > a\), then the major axis is vertical. So the foci are \((0, c)\) and \((0, -c)\), the focal constant \(2b\), and \(a^2 = b^2 - c^2\).

**Example 1**

Graph the ellipse with equation \(\frac{x^2}{4} + \frac{y^2}{9} = 1\).

**Solution**

Since \(a^2 = 4\) and \(b^2 = 9\), then \(a = 2\) and \(b = 3\). Since \(b > a\), the foci of the ellipse are on the y-axis. The length of the major axis is 6; the minor axis has length 4. Four points on this ellipse are easy to find: \((2, 0)\), \((-2, 0)\), \((0, 3)\), and \((0, -3)\). They are the endpoints of the minor and major axes, respectively.

With these two theorems, the possible locations of the shuttle boat can be described algebraically.

**Example 2**

Find an equation in standard form for the ellipse with foci \((6, 0)\) and \((-6, 0)\) and focal constant 20.

**Solution**

This ellipse is in standard position, so it has an equation of the form

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]

Only the values of \(a^2\) and \(b^2\) are needed. From the given information, \(c = 6\) and \(2a = 20\). So \(a = 10\), and \(a^2 = 100\). Now \(b^2 = a^2 - c^2 = 100 - 6^2 = 64\). Thus an equation is

\[
\frac{x^2}{100} + \frac{y^2}{64} = 1.
\]

A graph is shown at the right. The boats can reach anywhere on or in the interior of this ellipse, which is described by the inequality

\[
\frac{x^2}{100} + \frac{y^2}{64} \leq 1
\]
In general, if the equal sign in the equation for an ellipse is replaced by < or >, the resulting inequality represents either the interior or exterior of the ellipse, respectively.

Questions

1. a. An equation for the ellipse with foci (6, 0) and (-6, 0) and focal constant 20 is \( \sqrt{(x - 6)^2 + y^2} = 20 \).
   b. The equation of part a can be simplified to what equation in standard form?

2. The boat company’s shuttle can reach anywhere in the region described by what inequality?

In 3–6, use the ellipse at the right. The foci are F and G. Name:

3. the major axis
4. the minor axis
5. the center
6. the vertices.

In 7–9, for the ellipse with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), identify:

7. its center
8. the length of the major and minor axes
9. the endpoints of the major and minor axes.

In 10 and 11, graph the ellipse with the given equation.

10. \( \frac{x^2}{4} + \frac{y^2}{25} = 1 \)
11. \( \frac{x^2}{9} + y^2 = 1 \)

12. Find an equation in standard form for the ellipse with focal constant 25 and foci (10, 0) and (-10, 0).

13. If the company’s boat in this lesson has the fuel to travel 30 miles in going from one island to another, give a sentence in standard form for the possible positions of the boat.

14. Refer to the ellipse graphed at the right.
   a. Find an equation for the ellipse.
   b. What is a sentence to describe the interior of this ellipse?
15. **Multiple choice** Which of the following describes the set of points $P$ whose distances from $(7, 2)$ and $(3, 4)$ add up to 12?
(a) $(x - 7)^2 + (y - 2)^2 + (x - 3)^2 + (y - 4)^2 = 12$
(b) $(x + 7)^2 + (y + 2)^2 + (x + 3)^2 + (y + 4)^2 = 12$
(c) $\sqrt{(x - 7)^2 + (y - 2)^2} + \sqrt{(x - 3)^2 + (y - 4)^2} = 12$
(d) $\sqrt{(x + 7)^2 + (y + 2)^2} + \sqrt{(x + 3)^2 + (y + 4)^2} = 12$

16. In the United States Capitol there is an elliptical chamber in which a person whispering while standing at one focus can be easily heard by another person standing at the other focus. The whispering gallery in the Capitol’s Statuary Hall is 46 ft wide and 96 ft long.

![Ellipse Diagram]

a. A politician noted this feature of the chamber because the desk of the opposing party’s floor leader was at one focus. How far from that desk should the politician stand to overhear the floor leader’s whispered conversations?
b. How far from either end of the gallery would they be?
c. Find an equation which could describe the ellipse of the whispering gallery.

17. The orbits of the planets are elliptical with the sun at one focus. Venus’s orbit can be described by the equation

$$\frac{x^2}{5013} + \frac{y^2}{4970} = 1,$$

where $x$ and $y$ are in millions of miles.
a. What is the farthest Venus gets from the sun?
b. What is the closest Venus gets to the sun?
18. Consider $6x^2 + 3y^2 = 36$.
   a. Show that this is the equation for an ellipse by rewriting it in the form
   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
   \]
   b. What is the length of the major axis?

In 19 and 20, use conic graph paper with centers 12 units apart. Draw the set of points $P$ satisfying each equation.

19. $PF_1 + PF_2 = 16$  \hspace{1cm} 20. $|PF_1 - PF_2| = 6$ (Lesson 12-3)

In 21–24, each circle drawn below has radius 4 and its center is on either the $x$- or $y$-axis. (Lessons 12-1, 12-2, Previous course)

21. Write an equation for circle $a$.
22. Find an inequality describing the interior of circle $b$.
23. Find the circumference of each circle.
24. True or false The area of each circle is $4\pi$. 

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25. Solve for $h$: $2\pi r^2 + 2\pi rh = 132$. (Lesson 1-9)

26. a. Draw the triangle $ABC$ with vertices $A = (-2, 3)$, $B = (4, 3)$, and $C = (4, -1)$.
   b. Draw its image $\triangle A'B'C'$ under the scale change $S_{2,3}$.
   c. Is $\triangle ABC \sim \triangle A'B'C'$? Why or why not?
   d. Find the area of $\triangle ABC$.
   e. Find the area of $\triangle A'B'C'$.
   f. The area of $\triangle A'B'C'$ is how many times larger than the area of $\triangle ABC$? (Lesson 4-4, Previous course)

27. Use the triangle at the right.
   a. Find the length of $BC$ to the nearest tenth.
   b. Find the measures of $\angle B$ and $\angle C$ to the nearest degree.
      (Lessons 10-6, 10-7)

28. Look up each word in a dictionary. Explain how each is connected with ellipses. (The last word is relatively new, having first been used in the 1960s.)
   a. aphelion
   b. perihelion
   c. apogee
   d. apses
   e. perilune

29. a. Use the Graph Translation Theorem to predict what the graph of $\frac{(x - 2)^2}{9} + \frac{(y + 6)^2}{25} = 1$ will look like.
   b. Check your conjecture with an automatic grapher.
   c. Graph some other equations of the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.
   d. Write a paragraph summarizing your work.
As you know, in some ellipses the major axis is much longer than the minor axis. In others, the two axes are almost equal. Consider the three cases below.

If the major and minor axes are equal, as in the case below, the ellipse is a circle.

This can be verified in general by looking at the equations for circles and ellipses. Consider the standard form of an equation for an ellipse, \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Suppose the major and minor axes are equal and each of length \( 2r \). Then we may substitute \( r \) for \( a \) and \( r \) for \( b \).

The equation becomes \( \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \). Now, multiplying both sides of the equation by \( r^2 \), the result is \( x^2 + y^2 = r^2 \). This is an equation for a circle with center at the origin and radius \( r \). Thus a circle is a special kind of ellipse whose major and minor axes are equal.

Ellipses and circles are related in other ways. If you look at a circle on an angle, then it appears to be a non-circular ellipse. Notice how the circular hoop at the left appears to be taller than it is wide. Artists who want to draw circles in perspective must actually draw non-circular ellipses.
An ellipse can also be thought of as a stretched circle. The basic transformation which causes stretches and shrinks is the scale change which you studied in Lesson 4-4.

Consider the circle with equation \( x^2 + y^2 = 1 \) under the scale change \( S_{2,3} \). The scale change \( S_{2,3} \) has a horizontal magnitude of 2 and a vertical magnitude of 3. The images of several points on the circle are graphed below at the right.

\[
\begin{align*}
(1,0) & \rightarrow (2, 0) \\
(0, 1) & \rightarrow (0, 3) \\
(-1, 0) & \rightarrow (-2, 0) \\
(0, -1) & \rightarrow (0, -3) \\
(-0.8, 0.6) & \rightarrow (-1.6, 1.8) \\
(0.6, -0.8) & \rightarrow (1.2, -2.4)
\end{align*}
\]

From these six points you can see that the image of the unit circle under this scale change is not another circle.

**Example 1** Find an equation for the image of the circle \( x^2 + y^2 = 1 \) under \( S_{2,3} \).

**Solution** To find an equation of the image of the circle, let \((x', y')\) be the image of \((x, y)\).

Since \( S_{2,3}: (x, y) \rightarrow (2x, 3y) \)

\[
2x = x' \quad \text{and} \quad 3y = y'.
\]

So \( x = \frac{x'}{2} \quad \text{and} \quad y = \frac{y'}{3} \).

We know that \( x^2 + y^2 = 1 \). Substituting for \( x \) and \( y \) in that equation, an equation for the image is

\[
\left(\frac{x'}{2}\right)^2 + \left(\frac{y'}{3}\right)^2 = 1.
\]

Because equations are usually written with \( x \) and \( y \), the primes are dropped. So an equation of the image is

\[
\frac{x^2}{4} + \frac{y^2}{9} = 1.
\]

This is an equation for an ellipse with a minor axis of length 4 and a major axis of length 6.
Check Substitute some points known to be on the image. Do their coordinates satisfy this equation?

Try (2, 0). \[ \frac{2^2}{4} + \frac{0^2}{9} = 1 + 0 = 1 \] It checks.

Try (-1.6, 1.8). \[ \frac{(-1.6)^2}{4} + \frac{(1.8)^2}{9} = \frac{2.56}{4} + \frac{3.24}{9} = 0.64 + 0.36 = 1 \] It checks.

The argument in Example 1 can be repeated with \( a \) in place of 2 and \( b \) in place of 3. It shows that any ellipse in standard form can be thought of as a stretched circle.

**Theorem**

The image of the unit circle with equation \( x^2 + y^2 = 1 \) under \( S_{a,b} \) is the ellipse with equation \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \).

Because the ellipse is related in so many ways to the circle it should not surprise you that the area of an ellipse is related to the area of a circle. In general for any figure, the scale change \( S_{a,b} \) multiplies the area of the preimage by \( ab \). Since the area of a unit circle, which has radius 1, is \( \pi(1)^2 = \pi \), the area of the ellipse that is its image under \( S_{a,b} \) has area \( \pi \cdot (ab) = \pi ab \).

**Example 2** Find the area of the ellipse in Example 1.

**Solution** The length of the major axis is 6 and the length of the minor axis is 4. So \( a = 3 \) and \( b = 2 \) and the area of the ellipse is \( \pi \cdot 3 \cdot 2 = 6\pi \).
Questions

1. An ellipse in which the major and minor axes are equal in length is called a(n) ___.

In 2 and 3, true or false.

2. Every circle is an ellipse.

3. All ellipses are circles.

In 4 and 5, consider the circle $x^2 + y^2 = 1$ and the scale change $S_{a,b}$.

4. What is an equation for the image of the circle under $S_{a,b}$?

5. a. What is the area of the circle?
   b. What is the area of its image?

In 6 and 7, consider the ellipse drawn at the right.

6. a. What scale change maps the unit circle to this ellipse?
   b. Find an equation for this ellipse.

7. Find its area.

8. a. Below are equations of the ellipses shown at the start of this lesson. Find their foci.

\[
\begin{align*}
(i) \quad & \frac{x^2}{64} + \frac{y^2}{1} = 1 \\
(ii) \quad & \frac{x^2}{64} + \frac{y^2}{16} = 1 \\
(iii) \quad & \frac{x^2}{64} + \frac{y^2}{49} = 1
\end{align*}
\]

b. As the distance between the foci decreases, what happens to the shape of an ellipse?

c. Use the relationship $a^2 - c^2 = b^2$ to find the distance between the foci for the circle $\frac{x^2}{64} + \frac{y^2}{64} = 1$.

d. Are your answers to parts b and c consistent?

9. a. True or false Under a scale change, a figure is similar to its image.
   b. Justify your answer to part a by using an example from this lesson.

10. Prove that the image of the unit circle under the scale change $S_{a,b}$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Hint: Follow the idea of Example 1.)

11. a. Sketch a circle that has area $16\pi$.
   b. Sketch three noncongruent ellipses whose areas are also $16\pi$. 

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12. In Australia, a type of football is played on elliptical regions called Aussie Rules fields. One such field has a major axis of length 185 m and minor axis of length 155 m. A 1-meter track surrounding the field is to be covered with turf. The outer boundary of the track is then not an ellipse but very close to being one. Assume it is an ellipse to estimate the area of the track.

In 13–17, match each equation with the best description. A letter may be used more than once. Do not graph. (*Lessons 12-1, 12-2, 12-4*)

13. \( x^2 + y^2 = 25 \)  
   (a) circle

14. \( \frac{x^2}{25} + \frac{y^2}{81} = 1 \)  
   (b) ellipse

15. \( 4x^2 + y^2 = 100 \)  
   (c) interior of circle

16. \( x^2 + y^2 < 25 \)  
   (d) interior of ellipse

17. \( \frac{x^2}{81} + \frac{y^2}{25} > 1 \)  
   (e) exterior of circle

   (f) exterior of ellipse

18. The ellipse at the right has \( x \)-intercepts of 5 and -5 and foci at \((0, 12)\) and \((0, -12)\). Find:
   a. \( F_1V_1 \)
   b. the focal constant
   c. an equation for this ellipse.
   (*Lesson 12-4*)

19. In Chicago, elliptical flower beds have been planted over the Monroe Street parking garage. A landscape architect in a botanical garden decides to copy this idea. The elliptical gardens will have a major axis with length 18 ft and a minor axis with length 14 ft. Assuming the center is at \((0, 0)\) and foci are on the \(x\)-axis, find an equation for the ellipse. (*Lesson 12-4*)
20. Below is the top view of a castle surrounded by a circular moat 15 feet wide. The distance from the center of the castle to the outside of the moat is 500 feet. If the center of the castle is considered the origin, write a system of inequalities to describe the set of points on the surface of the moat. (Lesson 12-2)

21. Multiple choice In a hyperbola the focal constant is less than the distance between the foci.
   (a) always
   (b) sometimes
   (c) never (Lesson 12-3)

In 22 and 23, consider the line \( l \) with equation \( y = \frac{1}{2} x + 4 \) and the point \( P = (3, -1) \).

22. Find an equation for the line through \( P \) parallel to \( l \). (Lesson 3-5)

23. Find an equation for the line through \( P \) perpendicular to \( l \). (Lesson 4-8)

24. A vacuum pump is designed so that each stroke leaves only \( 97\% \) of the gas in the chamber.
   a. What percent of the gas remains after 2 strokes?
   b. Write an equation to model the percent \( P \) of the gas left after \( s \) strokes.
   c. How many strokes are necessary so that no more than \( 5\% \) of the gas remains? (Lessons 8-3, 8-10)

25. a. Modify the BASIC program in Question 11 of Lesson 12-2 so that it prints points on an ellipse in standard form.
   b. Run your program for the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).
You have seen hyperbolas generated in two ways. First, in Chapter 2 you studied hyperbolas that arise from situations modeled by inverse variation of the form \( y = \frac{k}{x} \). For instance, the formula \( r = \frac{d}{t} \) can be used by a person traveling on a highway with mileage markers to check a speedometer by driving at a constant speed for one mile and timing how long it takes.

\[
\text{rate in \( \frac{\text{miles}}{\text{hour}} \)} = \frac{\text{distance in miles}}{\text{time in hours}}
\]

If the distance between markers is 1 mile, the above equation becomes

\[
\text{rate in \( \frac{\text{miles}}{\text{hour}} \)} = \frac{1}{\text{time in hours}}
\]

There are 3600 seconds in an hour, so when time is measured in seconds this relationship is equivalent to

\[
\text{rate in \( \frac{\text{miles}}{\text{hour}} \)} = 3600 \cdot \frac{1}{\text{time in seconds}} = \frac{3600}{\text{time in seconds}}
\]

Here are some pairs of values (rounded to the nearest tenth) that satisfy the equation \( r = \frac{3600}{t} \).

<table>
<thead>
<tr>
<th>time (sec)</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (mph)</td>
<td>90</td>
<td>80</td>
<td>72</td>
<td>65.5</td>
<td>60</td>
<td>55.4</td>
<td>51.4</td>
<td>48</td>
<td>45</td>
<td>42.4</td>
<td>40</td>
<td>36</td>
</tr>
</tbody>
</table>
Since $r$ is inversely proportional to $t$, the graph of $tr = 3600$ for $t > 0$ and $r > 0$ is one branch of a hyperbola.

You learned a second situation in Lesson 12-3. A hyperbola can be defined geometrically as the set of points $P$ in a plane such that $|PF_1 - PF_2| = d$, where $F_1$ and $F_2$ are two fixed points and $d$ is a constant with $0 < d < F_1F_2$. For instance, below is a sketch of the set of points where $F_1F_2 = 12$ and $|PF_1 - PF_2| = 10$.

To show that the algebraic view of hyperbolas studied in Chapter 2 is consistent with the geometric view of hyperbolas, we begin with simple numbers.

Consider the hyperbola with foci $F_1 = (6, 6)$ and $F_2 = (-6, -6)$ and focal constant 12. We now prove that this hyperbola has an equation of the form $y = \frac{k}{x}$. Specifically, its equation is $y = \frac{18}{x}$. 
Theorem:
The hyperbola with foci (6, 6) and (-6, -6) and focal constant 12 has equation \( y = \frac{18}{x} \).

Proof

1. Let \( P = (x, y) \) be a point on the hyperbola. One branch of the curve is given by

\[ PF_1 - PF_2 = d. \]

2. Here \( F_1 = (6, 6), F_2 = (-6, -6) \), and \( d = 12 \). From the distance formula,

\[ \sqrt{(x - 6)^2 + (y - 6)^2} - \sqrt{(x + 6)^2 + (y + 6)^2} = 12. \]

3. Adding one of the square roots to both sides,

\[ \sqrt{(x - 6)^2 + (y - 6)^2} = 12 + \sqrt{(x + 6)^2 + (y + 6)^2}. \]

4. Squaring both sides (the right side is like a binomial),

\[ (x - 6)^2 + (y - 6)^2 = 144 + 24\sqrt{(x + 6)^2 + (y + 6)^2} + (x+6)^2 + (y+6)^2. \]

5. Expanding the binomials,

\[ x^2 - 12x + 36 + y^2 - 12y + 36 = 144 + 24\sqrt{(x+6)^2 + (y+6)^2} + x^2 + 12x + 36 + y^2 + 12y + 36. \]

6. Adding \(-x^2 - y^2 - 72 - 12x - 12y - 144\) to each side,

\[ -24x - 24y - 144 = 24\sqrt{(x + 6)^2 + (y + 6)^2}. \]

7. Dividing each side by -24 and then squaring,

\[ (x + y + 6)^2 = (x + 6)^2 + (y + 6)^2. \]

8. Expanding the trinomial and the two binomials,

\[ x^2 + y^2 + 12x + 12y + 2xy + 36 = x^2 + 12x + 36 + y^2 + 12y + 36. \]

9. (The next steps are easy — can you see what was done?)

\[ 2xy = 36 \]

10. \[ y = \frac{18}{x} \]
By a long process just like that used in the preceding proof, the general theorem stated below can be proved.

**Theorem:**

The graph of \( xy = k \) is a hyperbola. (When \( k > 0 \), this is the hyperbola with foci \( (\sqrt{2k}, \sqrt{2k}) \) and \( (-\sqrt{2k}, -\sqrt{2k}) \) and focal constant \( 2\sqrt{2k} \).)

The foci are labeled \( (\sqrt{2k}, \sqrt{2k}) \) and \( (-\sqrt{2k}, -\sqrt{2k}) \) so no radicals appear in the final equation \( xy = k \). In the following example you are given the foci and asked to find \( k \).

**Example**  Find an equation for the hyperbola with foci at \( (4, 4) \) and \( (-4, -4) \) and focal constant 8.

**Solution**  In the theorem above \( \sqrt{2k} = 4 \). Then \( 2k = 16 \) and \( k = 8 \). Thus, an equation for this hyperbola is \( y = \frac{8}{x} \) or \( xy = 8 \).

**Check**  Sketch a graph. Verify that for points \( P \) on the graph, \( |PF_1 - PF_2| = 8 \). We use the point \( (2, 4) \).

\[
|\sqrt{(2 - 4)^2 + (4 - 4)^2} - \sqrt{(2 + 4)^2 + (4 + 4)^2}| = |\sqrt{4} - \sqrt{36 + 64}| = 12 - 101 = |8| = 8
\]

It checks.

By reversing the process used above you can conclude that the graph of \( y = \frac{3600}{x} \) is a hyperbola. Since \( k = 3600 \), the foci are at \( (\sqrt{7200}, \sqrt{7200}) \) and \( (-\sqrt{7200}, -\sqrt{7200}) \) and the focal constant is \( 2\sqrt{7200} \).
Recall that the $x$- and $y$-axes are asymptotes of all equations of the form $y = \frac{k}{x}$, where $k \neq 0$. A hyperbola with perpendicular asymptotes is called a **rectangular hyperbola**. Thus, graphs of equations of the form $y = \frac{k}{x}$ are rectangular hyperbolas.

In the next lesson you will learn to find equations for other hyperbolas, both rectangular and nonrectangular.

**Questions**

1. If it takes 75 seconds to drive a mile, what is the average speed in miles per hour?

2. If it takes $t$ seconds to drive a mile, what is the average speed in miles per hour?

3. Consider the hyperbola with equation $xy = 18$. Name its
   a. foci
   b. asymptotes
   c. focal constant.

4. Consider the hyperbola with equation $xy = k$. Name its
   a. foci
   b. asymptotes
   c. focal constant.

5. Verify that the point (8, 1) is on the hyperbola of the Example.

6. a. Find an equation for the hyperbola with foci at (10, 10) and (-10, -10) and focal constant 20.
   b. Verify that the point (-2, -25) is on this hyperbola.

7. A car travels the 2.5 miles around the Indianapolis Speedway in $t$ seconds at an average rate of $r$ mph. Racing fans with stopwatches can calculate how fast a car is traveling if they know the value of the constant $rt$. What is that value?

8. The product of two real numbers is 100.
   a. Graph all possible pairs of numbers.
   b. Identify the foci and focal constant for this hyperbola.

In 9 and 10, sketch a graph.

9. $xy > 8$

10. $xy \leq 8$

In 11 and 12, give an equation for the conic or region.

11. the rectangular hyperbola with vertices (8, 8) and (-8, -8)

12. the interior of the rectangular hyperbola with vertices (1.5, 1.5) and (-1.5, -1.5)
13. An ellipse has foci \( F_1 \) and \( F_2 \) on the \( x \)-axis, and \( F_1 F_2 = 4 \). Also \( PF_1 + PF_2 = 7 \). Find
   a. the length of the major axis
   b. an equation in standard form for the ellipse
   c. the area of the ellipse. \((\text{Lessons 12-4, 12-5})\)

14. An exhibition tent is in the form of half a cylinder with each cross-section a semiellipse (half an ellipse) having base 20 ft and height 8 ft. How close to either end can a person 5 ft tall stand straight up? \((\text{Lessons 12-2, 12-4})\)

In 15 and 16, sketch a graph. \((\text{Lessons 12-1, 12-4})\)
15. \( \{(x, y): x^2 + y^2 = 144\} \)  
16. \( \{(x, y): 9x^2 + y^2 = 144\} \)

17. Find two nonreal numbers whose product is 100. \((\text{Lessons 6-8, 6-9})\)

18. What number must be put into the blank to make the expression \( y^2 - 13y + \square \) a perfect square? \((\text{Lessons 6-5, 11-3})\)

19. Solve \( x^5 - 81x = 0. \) \((\text{Lesson 11-4})\)

20. In the rectangular coordinate plane, can the graph of a function be an ellipse? Why or why not? \((\text{Lessons 7-1, 12-3, 12-4})\)

21. A tank has a slow leak. Suppose the water level starts at 100 inches and falls \( \frac{1}{2} \) inch per day. \((\text{Lesson 3-1})\)
   a. Write an equation relating the number \( N \) of days the tank has been leaking and the water level \( L \).
   b. After how many days will the tank be empty?

22. The words ellipsis and hyperbole have meanings in grammar. What are these meanings?
The hyperbolas studied in Lesson 12-6 are special. Each has foci on the line $y = x$ and has the $x$- and $y$-axes for asymptotes. The resulting equations of the form $xy = k$ do not look like those of any of the other conic sections you have studied. We can generate an equation for a hyperbola which resembles an equation of an ellipse by choosing foci $(c, 0)$ and $(-c, 0)$ and a general focal constant.

**Theorem: (Equation for a Hyperbola):**

The hyperbola with foci $(c, 0)$ and $(-c, 0)$ and focal constant $2a$ has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

**Proof**

The proof is almost identical to the proof of the equation for an ellipse in Lesson 12-4. Let $P = (x, y)$ be any point on the hyperbola. By the definition of a hyperbola,

$$|PF_1 - PF_2| = 2a.$$ 

This equation is equivalent to

$$PF_1 - PF_2 = \pm 2a.$$ 

That is, with $P = (x, y)$, $F_1 = (-c, 0)$, and $F_2 = (c, 0)$,

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = \pm 2a.$$ 

Doing manipulations similar to those in steps 2-8 of that proof, the same equation in step 9 results.

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Then in step 10, for hyperbolas, $c > a > 0$, so $c^2 > a^2$. Thus we can let $b^2 = c^2 - a^2$. So $-b^2 = a^2 - c^2$. This accounts for the minus sign in the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
Example 1  Consider the hyperbola with distance between foci $F_1F_2 = 12$ and focal constant $|PF_1 - PF_2| = 10$. (It was drawn with conic graph paper in Example 2, Lesson 12-3.) Suppose a rectangular coordinate system is placed so that the $x$-axis coincides with $F_1F_2$ and the $y$-axis bisects it. Find an equation for this hyperbola.

![Graph of the hyperbola](image)

Solution  $F_1F_2 = 12$, so $2c = 12$, and $c = 6$. The focal constant $2a = 10$, so $a = 5$. Thus $b^2 = 6^2 - 5^2 = 11$. An equation for this hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{11} = 1.$$  

We say that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is standard form for the equation of a hyperbola. To graph $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, notice that $(a, 0)$ and $(-a, 0)$ satisfy the equation. Since the foci are on the $x$-axis, the hyperbola is symmetric about that axis. Thus, $(a, 0)$ and $(-a, 0)$ are the vertices. When $x = 0$, no real value of $y$ works, so the hyperbola does not intersect the $y$-axis. A rough drawing using this information is given below.

![Vertices and asymptotes](image)

But this drawing is a little too rough. More points or asymptotes are needed to do a better job. To find general equations for the asymptotes, it is helpful to examine the simplest hyperbola of this kind, that is, the hyperbola with equation $x^2 - y^2 = 1$.  

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Some points on the graph of $x^2 - y^2 = 1$ are given below at the left. Because of the hyperbola's symmetry, these points in the first quadrant have reflection images on the hyperbola in the other three quadrants.

$$(1, 0)$$
$$(2, \sqrt{3}) \approx (2, 1.73)$$
$$(3, \sqrt{8}) \approx (3, 2.83)$$
$$(4, \sqrt{15}) \approx (4, 3.87)$$
$$(5, \sqrt{24}) \approx (5, 4.90)$$

As $x$ gets larger, the points get closer to the line with equation $y = x$. For instance, the point $(100, \sqrt{9999}) \approx (100, 99.995)$ is on the hyperbola. The line whose equation is $y = x$ appears to be an asymptote.

Every hyperbola is symmetric to the line through its foci, in this case the $x$-axis. So the union of both asymptotes is also symmetric to that line. Thus we reflect $y = x$ over the $x$-axis to get the other asymptote $y = -x$. This can be verified algebraically. When $x^2 - y^2 = 1$,

$$y^2 = x^2 - 1.$$  
So
$$y = \pm \sqrt{x^2 - 1}.$$  
As $x$ gets larger, $\sqrt{x^2 - 1}$ becomes closer to $\sqrt{x^2}$, which is $|x|$. So $y$ gets closer to $x$ or $-x$.

The scale change $S_{a,b}$ transforms $x^2 - y^2 = 1$ onto $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The asymptotes $y = \pm x$ are mapped onto the lines $\frac{y}{b} = \pm \frac{x}{a}$. These lines become the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

**Theorem:**

The asymptotes of the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{y}{b} = \pm \frac{x}{a}$.  

LEsson 12-7 More Hyperbolas
Example 2  Sketch a graph of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Solution  $a^2 = 9$, so $a = 3$. Thus the vertices are $(3, 0)$ and $(-3, 0)$. The asymptotes are $\frac{y}{4} = \pm \frac{x}{3}$. Carefully graph the vertices and asymptotes. Then sketch the hyperbola.

\begin{center}
\begin{tikzpicture}
\draw[->, very thick] (-5,0) -- (5,0) node[right] {$x$};
\draw[->, very thick] (0,-5) -- (0,5) node[above] {$y$};
\filldraw[black] (-3,0) circle (2pt) node[below] {(-3,0)};
\filldraw[black] (3,0) circle (2pt) node[below] {(3,0)};
\filldraw[black] (0,4/3) circle (2pt) node[below] {$\frac{y}{4} = \frac{x}{3}$};
\filldraw[black] (0, -4/3) circle (2pt) node[below] {$\frac{y}{4} = -\frac{x}{3}$};
\end{tikzpicture}
\end{center}

Check  Find coordinates of a point on the hyperbola. If $x = 5$, then $\frac{25}{9} - \frac{y^2}{16} = 1$, from which $y = \pm \frac{16}{3}$. The points $(5, \frac{16}{3})$ and $(5, -\frac{16}{3})$ do seem to be on the hyperbola. It checks.

Questions

1. A hyperbola with foci $(c, 0)$ and $(-c, 0)$ and focal constant $2a$ has an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

2. Why is the hyperbola with equation $x^2 - y^2 = 1$ so useful?

In 3 and 4, give (a) the vertices and (b) the asymptotes of each hyperbola.

3. $1 = x^2 - y^2$

4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

5. Consider the hyperbola with equation $\frac{x^2}{25} - \frac{y^2}{11} = 1$ from Example 1.
   a. Name its vertices.
   b. State equations for its asymptotes.

In 6 and 7, consider the hyperbola with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

6. Graph this hyperbola.

7. Find two points on the curve with $x$-coordinate equal to 6.

8. True or false  A single equation for both asymptotes of $x^2 - y^2 = 1$ is $y = |x|$. 
9. Solve \( x^2 - y^2 = 1 \) for \( y \). Use your solution to graph \( x^2 - y^2 = 1 \) using an automatic grapher.

10. By making a table of values and analyzing properties, graph the set of points satisfying \( y^2 - x^2 = 1 \). (Hint: The graph is like a graph in this lesson with \( x \) and \( y \) switched.)

11. The point \((-7, 4)\) is on a hyperbola with foci \((5, 0)\) and \((-5, 0)\).

\[
\begin{align*}
&\text{a. Find the focal constant.} \\
&\text{b. Give an equation for that hyperbola in standard form. (Hint: Find } b^2 = c^2 - a^2.\) \\
&\text{c. Graph the hyperbola using either a rectangular grid or conic graph paper.}
\end{align*}
\]

In 12 and 13, (a) sketch a graph; (b) state the eccentricity of the hyperbola. (The eccentricity of a hyperbola is defined to be the ratio of the distance between its foci and its focal constant; that is, \( \frac{2c}{2a} \) or \( \frac{c}{a} \).)

12. \( \frac{x^2}{25} - \frac{y^2}{9} = 1 \)

13. \( \frac{x^2}{25} - \frac{y^2}{4} = 1 \)

14. Refer to the graphs in Questions 12 and 13. Make a conjecture about the shape of the hyperbola in relation to its eccentricity.

15. What hyperbola in this lesson is a rectangular hyperbola?

In 16–18, multiple choice. Choose the best term from the following:
(a) circle (b) ellipse (c) parabola (d) hyperbola
(Lessons 6-3, 12-1, 12-3, 12-4, 12-6)

16. What is the set of points satisfying the equation \( |\sqrt{(x - 3)^2 + (y - 3)^2} - \sqrt{(x + 3)^2 + (y + 3)^2}| = 6 \)?

17. What is the set of points equidistant from a given focus and directrix?

18. What is the set of points satisfying the equation \( 4x^2 + 5y^2 = 100 \)?
In 19 and 20, consider that the compulsory circles made by a figure skater must have radius 1.5 times the height of the skater.

Assuming a coordinate system shown in the figure at the right, write a sentence to describe, for a skater 5 ft 4 in. tall, the points

19. on the upper circle;
20. in the interior of the lower circle. *(Lessons 12-1, 12-2)*

21. Find the vertex of the parabola with equation \( y = 3x^2 - 18x + 7 \). *(Lesson 6-5)*

22. If $2000 is compounded continuously at 6.25%, in how many years will it quadruple? *(Lesson 9-9)*

23. How close does a hyperbola get to its asymptotes? In particular, how large does \( x \) have to be in order for a point \((x, y)\) on \( x^2 - y^2 = 1 \) to be within .001 of the line \( y = x \)?

24. a. Refer to the diagram below. To do the following you will need 3 tacks, a ruler, a piece of string of length shorter than that of the ruler, a pencil, and a piece of board.

(i) Tack the ruler to a piece of board so that it pivots at point \( A \).
(ii) Tack one end of the piece of string to the other end of the ruler (point \( B \)).
(iii) Take the other end of the string and tack it to the board at point \( C \). (The distance between tacks \( A \) and \( C \) must be larger than the difference between the length of the ruler and the length of the string.)
(iv) Holding the string taut against the ruler with a pencil, rotate the ruler about point \( A \).

b. Explain why the resulting curve must be one branch of a hyperbola.
You have now seen equations for all the different types of conics. Here are some of these equations in standard form.

- **y = ax^2 + bx + c**  
  parabola

- \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]  
  ellipse

- \[(x - h)^2 + (y - k)^2 = r^2\]  
  circle

- \[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } xy = k\]  
  hyperbola

Although the equations for the hyperbola and ellipse look similar, the others look different. However, all these equations only contain terms with \(x^2\), \(xy\), \(y^2\), \(x\), or \(y\) and constants. All the conic sections are special types of quadratic relations, and their equations can be written in the general form

\[Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,\]

where \(A\), \(B\), \(C\), \(D\), \(E\), and \(F\) are real numbers, and at least one of \(A\), \(B\), or \(C\) is nonzero.

**Example 1**  Show that the circle with equation \((x - 3)^2 + y^2 = 14\) is a quadratic relation.

**Solution**  To do this, the equation for this circle must be put into the general form of a quadratic relation. So first expand the binomial.

\[x^2 - 6x + 9 + y^2 = 14\]

Now add -14 to both sides. Then use the Commutative Property of Addition to reorder the terms so that they are in the order \(x^2\), \(xy\), \(y^2\), \(x\), \(y\), and constants.

\[x^2 + 0xy + y^2 - 6x + 0y - 14 = 0\]

This is in the desired form with \(A = 1\), \(B = 0\), \(C = 1\), \(D = -6\), \(E = 0\), and \(F = -14\).

By expanding the standard-form equation of a circle, \((x - h)^2 + (y - k)^2 = r^2\), you can show that it has no \(xy\) term. So \(B = 0\) and \(A = C\). Given an equation for a circle in general form, you can complete the square to determine the center and radius of the circle.
Example 2

Find the center and radius of the circle with equation
\[ x^2 + 10x + y^2 - 8y - 20 = 0. \]

Solution

Complete the square on \( x^2 + 10x \) and on \( y^2 - 8y \), and add the same numbers to both sides of the equation.

\[ x^2 + 10x + 25 + y^2 - 8y + 16 - 20 = 25 + 16 \]

Factor the perfect square trinomials.

\[ (x + 5)^2 + (y - 4)^2 - 20 = 41 \]

Add 20 to each side.

\[ (x + 5)^2 + (y - 4)^2 = 61 \]

The center is \((-5, 4)\) and the radius is \(\sqrt{61}\).

The values of \(A\), \(B\), and \(C\) in a quadratic relation tell quite a bit about that relation. For instance, when \(B = 0\), then the equation has no \(xy\) term. As a result, the relation is symmetric to either a horizontal line or a vertical line. The only relation with an \(xy\) term that you have studied is \(xy = k\), and that relation is symmetric to the lines \(y = x\) and \(y = -x\).

There is an amazing theorem that tells whether a quadratic relation is a hyperbola, parabola, or ellipse. The proof is beyond the scope of this book.

**Discriminant Theorem for Conics:**

If \(A, B, C, D, E,\) and \(F\) are real numbers and at least one of \(A, B,\) or \(C\) is nonzero, then the graph of

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

is an ellipse if \(B^2 - 4AC\) is negative,

a parabola if \(B^2 - 4AC = 0\),

and a hyperbola if \(B^2 - 4AC\) is positive.

Example 3

Identify the conic section with equation
\[ 3x^2 - 5xy + 4y^2 - 2x + 9y - 6 = 0. \]

Solution

Here \(A = 3\), \(B = -5\), and \(C = 4\). The other coefficients can be ignored. Since \(B^2 - 4AC = (-5)^2 - 4 \cdot 3 \cdot 4 = 25 - 48 < 0\), the conic section is an ellipse.
Check  If your automatic grapher allows you to plot quadratic relations in general form, enter the coefficients and draw the graph. The output below from our grapher shows an ellipse with major and minor axes not parallel to the coordinate axes.

You may wonder how there ever could be such a theorem. Recall that the conic sections are intersections of a plane with a double cone. The value of $B^2 - 4AC$ determines the angle of the plane that intersects the cone. Let $k$ be the measure of the acute angle between the axis and an edge of the cone. Let $\theta$ be the measure of the smallest angle between the axis and the plane.

There are special cases of the Discriminant Theorem in which the graph does not resemble an ellipse, a parabola, or a hyperbola. Geometrically, these occur when the intersecting plane contains the vertex $V$ of the cone. Then the ellipse *degenerates* to a single point $(V)$, the parabola to a single line (an edge through $V$), and the hyperbola to two lines (intersecting edges through $V$).
For instance, the graph of the equation \( x^2 + y^2 = 0 \) is the single point \((0, 0)\). Using the Discriminant Theorem with \( A = 1, B = 0, C = 1, \) and \( D = E = F = 0, B^2 - 4AC = -4 \), which is negative. The point is a degenerate ellipse.

**Questions**

In 1–4, (a) tell whether or not the sentence is an equation for a quadratic relation. (b) If so, put the equation in general form. If not, tell why not.

1. \( x^2 + 4xy^2 = 6 \)
2. \( \frac{1}{2} y - 13x^2 = \sqrt{5} x \)
3. \( x^2 + 2xy + 3y^2 + 4x + 5y = 6 \)
4. \( xy - 8 = 2xy \)

In 5–8, tell whether the graph is a hyperbola, parabola, or ellipse.

5. \( 3x^2 + 9x + 3y^2 + 12y = 0 \)
6. \( 25x^2 - 10xy + y^2 + 3x + 6y + 11 = 0 \)
7. \( x^2 - xy = 2 \)
8. \( 0 = x^2 + 5xy + 7y^2 - 32 \)

In 9–12, consider the equation \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \). What conic results from the given situation?

9. \( A = C \) and \( B = 0 \)
10. \( B = 0 \)
11. \( B^2 - 4AC = 0 \)
12. \( B^2 - 4AC > 0 \)

In 13 and 14, find the center and radius of the circle.

13. \( x^2 + 4x + y^2 + 4y + 2 = 0 \)
14. \( x^2 - 8x + y^2 + 10y - 6 = 0 \)

15. A degenerate parabola is a(n) \_\_\_.

In 16 and 17, show that the equation describes a quadratic relation by putting it in general form. Give the values of \( A, B, C, D, E, \) and \( F \).

16. \( \frac{x^2}{4} - \frac{y^2}{9} = 1 \)
17. \( y = 3(x + 1)^2 - 8 \)

18. If \( 3x^2 + 4xy + y^2 = 0 \), then \((3x + y)(x + y) = 0\), so \( 3x + y = 0 \) or \( x + y = 0 \). Thus the original equation is equivalent to the union of two lines.
   a. Graph all points satisfying \( 3x^2 + 4xy + y^2 = 0 \).
   b. The graph is a degenerate form of what conic?

19. Consider the equation \( 64x^2 + 12y^2 = 768 \).
   a. Identify the conic.
   b. Solve the equation for \( y \).
   c. Use the result of part b to graph the conic on an automatic grapher.
20. Consider the hyperbola with equation \( \frac{x^2}{25} - \frac{y^2}{36} = 1 \).
   a. What are its foci?    b. Name its vertices.
   c. State equations for its asymptotes. (Lesson 12-7)

21. In the hyperbola with equation \( xy = 148 \), what is the focal constant? (Lesson 12-6)

22. The picture at the left shows the entrances to an amphitheater in Yugoslavia. It is typical of early Roman construction in which a semicircular arch is built over a square opening. Suppose the gate is 10 ft wide.
   a. How high is it?
   b. Can a truck 6 ft wide and 12 ft high fit through the gate? (Lesson 12-2, Previous course)

23. Consider the following system:

   \[
   \begin{align*}
   y &= 4x \\
   2x - 3y &= -15
   \end{align*}
   \]

   (Lessons 5-2, 5-3, 5-4)
   a. Name three methods you can use to solve this system.
   b. Solve the system using any method.

24. Refer to the graph below which shows the number of calories of heat needed to raise the temperature so that ice will turn to water and eventually to steam. (Lesson 3-8)

   a. At what temperature in °C does ice melt?
   b. True or false Water boils at a constant temperature.
   c. Find the slope of the line between the points (100, 0) and (200, 100).
   d. What does your answer to part c mean?
25. If your automatic grapher cannot plot conics in the form \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \) directly, you can plot the general conic by first solving for \( y \). Rewrite the equation as

\[
Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0.
\]

Applying the quadratic formula to solve for \( y \) gives

\[
y = \frac{-(Bx + E) \pm \sqrt{(Bx + E)^2 - 4(C)(Ax^2 + Dx + F)}}{2C}
\]

Separate this into two equations, one using the + sign, the other the − sign, and input the constants \( A, B, C, D, E, \) and \( F \).

a. Use these equations to plot the curve in Example 3.

b. Plot some other general second degree equations with \( xy \) terms. Do the results agree with the Discriminant Theorem?

26. The polynomial \( ax^4 + bx^3 + cx^2 + dx + e \) can be written as

\[
(((ax + b)x + c)x + d)x + e.
\]

This latter expression can be easier to evaluate, particularly by computers, since it does not involve exponents and it can be described by an iterative, or repeating, algorithm. For any particular value for \( x \), multiply the first coefficient by \( x \), then add the next coefficient. Multiply that sum by \( x \), then add the next coefficient. Again, multiply that sum by \( x \), then add the next coefficient. And so on, until the last coefficient has been added. The process can be done without a computer, and is then called synthetic substitution.

First write down the coefficients \( a, b, c, d, \) and \( e \). Then follow the arrows.

For instance, to find \( P(5) \) for \( P(x) = 2x^4 - 9x^3 + 4x - 7 \), you would write the following:

\[
\begin{array}{cccccc}
2 & -9 & 0 & 4 & -7 \\
\downarrow & 10 & 5 & 25 & 145 \\
2 & 1 & 5 & 29 & 138 \\
\end{array}
\]

and find that \( P(5) = 138 \).

a. Verify that \( P(5) = 138 \) by substituting in the formula for \( P(x) \).

b. Use synthetic substitution to evaluate \( Q(7) \) when \( Q(x) = 3x^4 + 2x^3 - 20x^2 - 3x + 12 \).

c. Use synthetic substitution to evaluate \( Q(x) \) in part b when \( x = -3 \).
A quadratic system is a system that involves at least one quadratic sentence. As with linear systems, you may solve quadratic systems by

1. graphing,
2. substitution,
3. or linear combinations.

No new properties are needed. In this lesson, the systems consist of a quadratic and a linear equation.

From geometry you know that a line can intersect a circle in 2, 1, or 0 points.

Similarly, a system of one linear and one quadratic equation may have 2, 1, or 0 solutions.

Example 1 By graphing, approximate solutions to the following system:

\[
\begin{align*}
  y - 3x &= 1 \\
  xy &= 10
\end{align*}
\]

Solution Graph both curves on the same set of axes. From the graph you can see that there are two solutions, one in the first quadrant, the other in the third. Zoom or rescale to estimate the coordinates. One point appears to be (-2, -5); the other about (1.7, 6).

Check Substitute into the first equation.

-5 - 3(-2) = 1 and (-2)(-5) = 10.

So (-2, -5) is an exact solution.

6 - 3(1.7) = 0.9 = 1 and (1.7)(6) = 10.2 = 10.

So (1.7, 6) is an approximate solution.
With an automatic grapher, the solutions can be found to as much accuracy as you wish. Still, it is sometimes nice to get exact solutions. These can be found by substitution.

**Example 2** Find exact solutions to the system \( \begin{align*}
y - 3x &= 1 \\
x - y &= 10.
\end{align*} \)

**Solution** Solve the first sentence for \( y \).

\[ y = 3x + 1. \]

Substitute the expression \( 3x + 1 \) for \( y \) in the second sentence.

\[ x(3x + 1) = 10 \]

This is a quadratic equation that you can solve by the Quadratic Formula or by factoring.

\[ 3x^2 + x = 10 \]

\[ 3x^2 + x - 10 = 0 \]

\[ \begin{align*}
x &= \frac{-1 + \sqrt{1 - 4 \cdot 3(-10)}}{2 \cdot 3} \\
&= \frac{-1 + \sqrt{121}}{6} \\
x &= \frac{-1 - 11}{6} \text{ or } x = \frac{-1 + 11}{6} \\
x &= -2 \text{ or } x = \frac{5}{3}
\end{align*} \]

Now remember that \( y = 3x + 1. \)

When \( x = -2, y = 3(-2) + 1 = -5. \)

When \( x = \frac{5}{3}, y = 3\left(\frac{5}{3}\right) + 1 = 6. \)

**Check** The solutions \((-2, -5)\) and \(\left(\frac{5}{3}, 6\right)\) agree with the graph in Example 1.

In the above example, the substitution of the linear quantity into the quadratic relation resulted in a quadratic equation in one variable. Because quadratic equations may have 2, 1, or 0 solutions, a *quadratic-linear system* may also have 2, 1, or 0 solutions.

**Example 3** At the right are graphs of the equations \(6x^2 + y^2 = 100\) and \(y = -12x + 50\). It appears that they intersect in only one point. Is this so? Justify your answer.
Solution  Solve the system \[ \begin{align*} 6x^2 + y^2 &= 100 \\
y &= -12x + 50. \end{align*} \]
The second sentence is already solved for \( y \). Substitute for \( y \) in the first sentence.

\[ 6x^2 + (-12x + 50)^2 = 100 \]

Expand and rewrite in the general form of a quadratic equation.

\[ 6x^2 + 144x^2 - 1200x + 2500 = 100 \]
\[ 150x^2 - 1200x + 2400 = 0 \]

Divide each side by 150 to simplify.

\[ x^2 - 8x + 16 = 0 \]

The left side is a perfect square.

\[ (x - 4)^2 = 0 \]

So \( x = 4 \) is the only solution. When \( x = 4 \) in the first sentence,

\[ 6(4)^2 + y^2 = 100 \]
\[ y^2 = 4 \]
\[ y = \pm 2. \]

Thus, there are two possible solutions: \( (4, 2) \) and \( (4, -2) \). The point \( (4, 2) \) satisfies the equation \( y = -12x + 50 \), but the point \( (4, -2) \) does not. Therefore, there is only one solution to this system.

Quadratic systems, just as linear systems, can be inconsistent. The signal for inconsistency is that the solutions to the quadratic system are not real.

**Example 4**  Find the points of intersection of the line \( y = x - 1 \) and the parabola \( y = x^2 \).

**Solution 1**  Graphs of the line and parabola, shown at the right, show there is no solution.

**Solution 2**  Solve the system \( \begin{align*} y &= x - 1 \\
y &= x^2 \end{align*} \) algebraically. Since \( x - 1 \) and \( x^2 \) both equal \( y \), they equal each other.

\[ x - 1 = x^2 \]

Thus \( x^2 - x + 1 = 0 \).

Using the Quadratic Formula, \( x = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2} = \frac{1 \pm \sqrt{-3}}{2} \).

The nonreal solutions indicate there are no points of intersection.
Questions

1. Which of the three strategies for solving systems is not found in this lesson?

2. How many solutions may a system of one linear and one quadratic equation have?

In 3 and 4, solve by (a) graphing and (b) substitution.

3. \[
\begin{align*}
xy &= 18 \\
y &= 3x + 12
\end{align*}
\]

4. \[
\begin{align*}
x^2 + y^2 &= 25 \\
y &= \frac{3}{4}x
\end{align*}
\]

5. Find the points of intersection of the line \( y = x + 2 \) and the parabola \( y = x^2 \).

6. Find the points of intersection of the line \( y = x - 1 \) and the parabola \( y = 2x^2 \).

7. a. What name is given to a system which has no solutions?
   b. Give an example of such a system.

8. Refer to the equation \( 3x^2 + x - 10 = 0 \) in the solution of Example 2.
   a. Solve this equation by factoring.
   b. Compare your answers to those in the lesson.

9. A graph of the system
   \[
   \begin{align*}
y &= x^2 - 2x - 15 \\
x + y &= -3
\end{align*}
   \]
   is shown at the right.
   a. How many solutions are there?
   b. Approximate the solutions.
   c. Check that your estimates are close.

In 10 and 11, consider the figure at the right which suggests that the parabola \( y = x^2 - 8x + 18 \) and the line \( y = 2x - 7 \) intersect near the point (5, 3).

10. Check by substitution that this point is on both curves.

11. Solve the system algebraically and verify that this is the only solution.

In 12 and 13, use the following system:

12. Estimate the solutions to this system by graphing.

13. Solve the system algebraically.
14. Phillip has 150 m of fencing material and wants to form a rectangle whose area is 1300 square meters.
   a. Let \(x\) = the width of the field and \(y\) = its length. Write a system of equations that models this situation.
   b. Use graphing to estimate the dimensions of this region.
   c. Solve this system.

15. The sum of two real numbers is to be 10 and their product 30. Use graphs to show that this is impossible.

16. What conic section is described by the equation
   \[3x^2 + 4y^2 - 6x + 16y - 19 = 0\] (Lesson 12-8)

17. Find an equation for a hyperbola with foci at (2, 2) and (-2, -2) and focal constant 4. (Lesson 12-6)

18. Give an equation for a hyperbola that
   a. is a function.
   b. is not a function. (Lessons 7-1, 12-6, 12-7)

19. Halley’s comet has an elliptical orbit with the sun at one focus. Its closest distance to the sun is about
   \(9 \cdot 10^7\) km, while its farthest distance is about \(5.34 \cdot 10^9\) km.
   a. Find the length of the major axis of Halley’s comet’s orbit.
   b. What is the length of its minor axis?
   c. How long does it take Halley’s comet to complete each orbit? (Lesson 12-4)

20. A supersonic jet traveling parallel to the ground has a shock wave in the shape of a cone. The sonic boom is felt on all the points located on the intersection of the cone and the ground. What kind of conic section is that intersection? (Lesson 12-1)

21. Factor completely: \(x^4 - 8xy^3\) (Lesson 11-3)

22. A lemonade stand reports the following monthly profit \(P\) (in hundreds of dollars) in relation to the average monthly temperature \(T\) (in degrees Celsius).

<table>
<thead>
<tr>
<th>(T)</th>
<th>-10</th>
<th>0</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>-125</td>
<td>-50</td>
<td>25</td>
<td>100</td>
</tr>
</tbody>
</table>

   a. Does a linear function fit these data?
   b. If so, write \(P\) as a function of \(T\), and tell what quantity the slope represents. If not, tell why not. (Lessons 2-4, 3-1, 11-7)

23. Draw an example of a system involving a hyperbola and an oblique line that has exactly one solution.
Systems of hyperbolas are used to locate ships at sea. See page 728.

Quadratic-quadratic systems involve the intersection of curves represented by quadratic relations: circles, ellipses, hyperbolas, and parabolas. They are a bit more complicated than linear systems; there may be 0, 1, 2, 3, 4, or infinitely many solutions. The first two examples illustrate systems with 4 solutions each.

To find exact solutions, the first goal is always the same: work to get an equation in one variable. In Example 1 we use substitution to solve the system.

Example 1
At the right is pictured the following system:
\[
\begin{align*}
  x^2 + y^2 &= 25 & \text{circle with center (0, 0), radius 5} \\
  y &= x^2 - 13 & \text{parabola with vertex (0, -13), congruent to } y = x^2
\end{align*}
\]

Find the four solutions shown in the graph.

Solution
Substitute \(x^2 - 13\) for \(y\) in the first equation. The result is what is desired—an equation in one variable.

\[
\begin{align*}
  x^2 + (x^2 - 13)^2 &= 25 \\
  x^2 + x^4 - 26x^2 + 169 &= 25 \\
  x^4 - 25x^2 + 144 &= 0
\end{align*}
\]

It looks difficult! A fourth-degree equation! But let \(m = x^2\), and the resulting equation is quadratic.

\[
m^2 - 25m + 144 = 0
\]

Using the quadratic formula,

\[
m = \frac{25 \pm \sqrt{625 - 576}}{2} = \frac{25 \pm \sqrt{49}}{2} = \frac{25 \pm 7}{2}
\]

So \(m = 16\) or \(m = 9\).

Thus \(x^2 = 16\) or \(x^2 = 9\).

Therefore, \(x = 4, -4, 3,\) or \(-3\).
For each value of $x$, there is a corresponding value of $y$. Substitute in the equation $y = x^2 - 13$ to find that value. When $x = 4$ or $-4$, $y = 3$. When $x = 3$ or $-3$, $y = -4$.

The four solutions are $(4, 3), (-4, 3), (3, -4), \text{ and } (-3, -4)$.

In the next example, because both relations are symmetric to the $x$- and $y$-axes, so is the set of solutions. Note how the Linear Combination method studied in Lesson 5-3 is applied.

---

**Example 2** Find all points of intersection of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the hyperbola $x^2 - y^2 = 4$.

![Graph showing the ellipse and hyperbola](image)

**Solution** Multiply the first equation by $16 \cdot 9 = 144$ to remove fractions.

\[
\begin{align*}
9x^2 + 16y^2 &= 144 \\
x^2 - y^2 &= 4
\end{align*}
\]

Now the system can be solved using the linear combination method. Multiply the second equation by $-9$ and add the equations.

\[
\begin{align*}
9x^2 + 16y^2 &= 144 \\
-9x^2 + 9y^2 &= -36 \\
25y^2 &= 108
\end{align*}
\]

Solve for $y$. $y^2 = \frac{108}{25}$, $y = \pm \sqrt{\frac{108}{25}} = \pm \frac{6\sqrt{3}}{5}$

Use $x^2 - y^2 = 4$ to find $x^2$. Since $y^2 = \frac{108}{25}$, $x^2 - \frac{108}{25} = 4$, and so $x^2 = \frac{208}{25}$.

Thus for each value of $y$, $x = \pm \sqrt{\frac{208}{25}} = \pm \frac{4\sqrt{13}}{5}$.

The points of intersection are $\left(\frac{4\sqrt{13}}{5}, \frac{6\sqrt{3}}{5}\right)$, $\left(-\frac{4\sqrt{13}}{5}, \frac{6\sqrt{3}}{5}\right)$, $\left(\frac{4\sqrt{13}}{5}, -\frac{6\sqrt{3}}{5}\right)$, and $\left(-\frac{4\sqrt{13}}{5}, -\frac{6\sqrt{3}}{5}\right)$, or approximately $(2.88, 2.08)$, $(-2.88, 2.08)$, $(2.88, -2.08)$, and $(-2.88, -2.08)$.

**Check** The graph shows these solutions to be quite reasonable.
Recall from Lesson 12-1 that with a seismograph, a tracking station can determine its distance from the center of an earthquake. Two such tracking stations can determine that the center is at one of two locations.

Example 3

One station determines that the center of the quake is 30 miles away. A second station 40 miles east and 10 miles north of the first finds that it is 20 miles from the center. Where is the center of the earthquake?

Solution

First draw a rough picture. It seems there may be two points. We call them \( P \) and \( Q \). Locate a coordinate system at the center of the first station. Now find the equations of the circles.

Circle (1) has center \((0, 0)\) and radius 30:

\[
x^2 + y^2 = 900
\]

Circle (2) has center \((40, 10)\) and radius 20:

\[
(x - 40)^2 + (y - 10)^2 = 400
\]

The system has been determined. Now it must be solved. Expand the squares of the binomials in the equation for circle (2):

Now subtract equation (1).

\[
x^2 - 80x + 1600 + y^2 - 20y + 100 = 400 - 900
\]

\[
-x^2 + y^2 - 80x + 1600 - 20y + 100 = -500
\]

A linear equation results.

Solve for one of the variables in the linear equation. We solve for \( y \).

\[
2200 - 80x = 20y
\]

\[
110 - 4x = y
\]

Now substitute for \( y \) in the equation of one of the circles. We use \( x^2 + y^2 = 900 \), because it is simpler.

\[
x^2 + (110 - 4x)^2 = 900
\]

This is finally an equation in one variable.

\[
x^2 + (12100 - 880x + 16x^2) = 900
\]

\[
17x^2 - 880x + 11200 = 0
\]

Using the quadratic formula,

\[
x = \frac{880 \pm \sqrt{880^2 - 4 \cdot 17 \cdot 11200}}{34} = \frac{880 \pm \sqrt{12800}}{34}
\]

\[
\approx \frac{880 \pm 113.1}{34}
\]

So \( x \approx 29.2 \) or \( x \approx 22.6 \).
Corresponding estimates of $y$ can be found by substituting into $y = 110 - 4x$.

When $x = 29.2$, $y = -6.8$.
When $x = 22.6$, $y = 19.6$.

The points are near (29.2, -6.8) and (22.6, 19.6).

**Check 1** Examine the graph. The points seem correct. That is, from the given information the center of the quake is either about 29.2 miles east and 6.8 miles south of the first station, or about 22.6 miles east and 19.6 miles north of it.

**Check 2** Substitute into the original equations. Try each point in each equation. Does $(29.2)^2 + (-6.8)^2 = 900$? We get about 899, which is close enough. The three other substitutions are left to you.

Other situations can lead to systems of quadratic equations which require multiple substitutions to solve.

---

**Example 4**

One month, Wanda’s Western Wear took in $12,000 from boot sales. The next month, although Wanda sold 40 fewer pairs of boots, the store took in $12,800 from boot sales because they had raised the price by $20. Find the price of a pair of boots in each month.

**Solution** Let $n =$ the number of pairs of boots sold in the first month. $c =$ the cost of a pair of boots in the first month.

The equations for total sales in the first and second months respectively are:

(1) \[ nc = 12000 \]
(2) \[ (n - 40)(c + 20) = 12800 \]

From (1), you know \( c = \frac{12000}{n} \).

From (2), \( nc + 20n - 40c - 13600 = 0 \). The two forms of equation (1) allow you to make two substitutions into the expanded form of equation (2), namely 12000 for $nc$ and $\frac{12000}{n}$ for $c$, to get:

\[ 12000 + 20n - 40\left(\frac{12000}{n}\right) - 13600 = 0 \]

Simplify. \[ 20n - 1600 - \frac{480000}{n} = 0 \]

Multiply by $n$. \[ 20n^2 - 1600n - 480000 = 0 \]

Divide by 20. \[ n^2 - 80n - 24000 = 0 \]

Solve by factoring. \[ (n - 200)(n + 120) = 0 \]

\[ n = 200 \text{ or } n = -120 \]

**LESSON 12-10 Quadratic-Quadratic Systems 727**
The number of pairs of boots can only be positive, so use the positive answer and substitute in equation (1) to find the price.

\[ 200c = 12000 \]
\[ c = 60 \]

The boots cost $60 the first month, and \( c + 20 = 80 \) the second month.

**Check**
First month:
\[ (200)(60) = 12000 \]
Yes, it checks.

Second month:
\[ (200 - 40)(60 + 20) = (160)(80) \]
\[ = 12800 \]
Yes, it checks.

Intersections of hyperbolas are the basis for the LORAN system. In this system, three LOng RAnge Navigational stations simultaneously send electronic signals to a ship at sea. The ship receives these signals at slightly different times. If \( A \) and \( B \) are two stations, then by measuring the time differential and by knowing the speed of the radio waves, the ship \( P \) can be located on a hyperbola with foci at \( A \) and \( B \). A similar process locates the ship on a hyperbola with foci at stations \( B \) and \( C \). The intersection of the two hyperbolas gives the ship’s location.
Questions

1. a. How many solutions may a system of two quadratic equations in \(x\) and \(y\) have?
   
b. Which of these possibilities is illustrated in Examples 1-4?

In 2–4, refer to Example 1.

2. What technique was used in obtaining the equation
   \[ x^2 + (x^2 - 13)^2 = 25? \]

3. What substitution changed \(x^4 - 25x^2 + 144 = 0\) into a quadratic equation?

4. Explain how the step \(m = 16\) leads to \(x = 4\) or \(x = -4\).

In 5 and 6, refer to Example 2.

5. What in the original equations signifies that the solutions will be symmetric to the axes?

6. The four solutions are vertices of what figure?

7. Refer to Example 3. True or false The solution to this system used both a linear combination and substitution.

8. Refer to Example 2 of Lesson 12-1. Find the coordinates of the two possible epicenters \(A\) and \(B\) of the earthquake.

In 9 and 10, refer to Example 4.

9. What were the two substitutions that transformed the second equation into an equation with only one variable?

10. If in the second month Wanda had instead raised prices by $30 and earned $10,800 from 80 fewer sales than the previous month, what would be the second equation?

In 11 and 12, refer to the relations \(x^2 + y^2 = 9\) and \(x^2 + 4y^2 = 16\) graphed at the right.

11. Estimate the solutions from the graph.

12. Find the exact solutions algebraically.

13. Solve the following system:
   \[
   \begin{align*}
   y &= x^2 - 4x + 3 \\
   y &= x^2 - 9
   \end{align*}
   \]
14. Refer to Example 3. A third station 50 miles west and 20 miles south of the first station determines that the center of the quake is about 80 miles away.
   a. Find an equation to describe this information.
   b. Using the points found in Example 3, determine which point was really the location of the epicenter.

In 15 and 16, consider this situation. The product of two numbers is 1073. If one number is increased by 3 and the other is decreased by 7, the new product is 960.

15. Multiple choice Which of the following systems represents this situation?
   (a) \(\begin{align*}
   xy &= 960 \\
   (x + 3)(y - 7) &= 1073
   \end{align*}\)
   (b) \(\begin{align*}
   xy &= 1073 \\
   (x - 3)(y + 7) &= 960
   \end{align*}\)
   (c) \(\begin{align*}
   xy &= 1073 \\
   (x + 3)(y - 7) &= 960
   \end{align*}\)
   (d) \(\begin{align*}
   xy &= 1073 \\
   (x - 3)(y - 7) &= 960
   \end{align*}\)

16. Find the numbers.

17. Without doing any calculations, solve the following system:
   \(\begin{align*}
   (x - 3)^2 + y^2 &= 4 \\
   (x + 3)^2 + y^2 &= 4
   \end{align*}\)
   Explain how you found your answer.

18. One circle has a center at the origin, the other at (4, 0). Each has a radius of 2. Where do they intersect? Check by graphing.

19. Draw two parabolas which intersect in:
   a. exactly one point   
   b. exactly two points   
   c. exactly three points

In 20 and 21, how many possible solutions could there be to the following systems?

20. a circle and a parabola   
21. two circles

22. By graphing the system \(\begin{align*}
   xy &= 10 \\
   y &= 4x + 5
   \end{align*}\)
   a. determine the number of real solutions, and
   b. estimate them. (Lesson 12-9)

In 23 and 24, for the ellipse pictured at the right, find

23. the area

24. an equation in standard form. (Lessons 12-4, 12-5)
25. A skydiver jumping from a plane falls about 16 ft the first second, 48 ft the next second, and 80 ft the third second, if air resistance is ignored. How many feet will the diver fall in the thirtieth second? (Lesson 3-6)

26. Consider the geometric sequence 8, -12, 18, -27, ... . Find
a. the next term
b. an explicit formula for the nth term. (Lesson 8-3)

27. Give an equation for a quadratic relation that intersects the unit circle \(x^2 + y^2 = 1\): (a) in no points; (b) in exactly one point; (c) in exactly two points; (d) in exactly three points; (e) in exactly four points.

28. Consider the ellipse with equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), and the circle whose diameter has endpoints at the foci \((c, 0)\) and \((-c, 0)\). The area of the ellipse is \(\pi ab\); the area of the circle is \(\pi c^2\). Are these two areas ever equal? If so what relation exists between \(a\), \(b\), and \(c\)?
In this chapter you studied quadratic relations in two variables, their graphs, and geometric properties of these figures. A quadratic equation in two variables is of the form

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \]

where not all of \( A, B, \) and \( C \) are zero.

### Conic

<table>
<thead>
<tr>
<th>Conic</th>
<th>Equation in Standard form</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>circle</td>
<td>((x - h)^2 + (y - k)^2 = r^2)</td>
<td><img src="https://via.placeholder.com/150" alt="Circle" /></td>
</tr>
<tr>
<td>ellipse</td>
<td>(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1)</td>
<td><img src="https://via.placeholder.com/150" alt="Ellipse" /></td>
</tr>
<tr>
<td>hyperbola</td>
<td>(xy = k)</td>
<td><img src="https://via.placeholder.com/150" alt="Hyperbola" /></td>
</tr>
</tbody>
</table>

The Discriminant Theorem states that

- if \( B^2 - 4AC < 0 \), the curve is an ellipse,
- if \( B^2 - 4AC = 0 \), the curve is a parabola,
- and if \( B^2 - 4AC > 0 \), the curve is a hyperbola.

Additionally, you can identify the curve by knowing the standard-form equations as given below.
The graph of every quadratic equation in two variables is a conic section. That is, it can be formed by the intersection of a plane and a double cone. If the plane does not contain the vertex of the cone, the intersection may be an ellipse (of which the circle is a special case), a hyperbola, or a parabola. If the plane contains the vertex of the cone, degenerate conic sections—a point, line, or two lines—are formed.

These curves may also be generated geometrically in two dimensions. In a plane a circle is the set of points at a given distance (its radius) from a fixed point (its center); an ellipse is the set of points such that the sum of its distances to two fixed points (its foci) is constant; and a hyperbola is the set of points such that the difference of its distances from two fixed points (its foci) is constant.

Conic sections appear naturally as orbits of planets and comets, in paths of objects thrown in the air, as energy waves radiating from the epicenter of an earthquake, and in many manufactured objects such as tunnels, windows, and satellite receiver dishes.

Systems of equations with quadratic sentences are solved much the same as linear systems, that is, by graphing, by substitution, or by using linear combinations. A system of one linear and one quadratic equation may have 0, 1, or 2 solutions; a system of two quadratics may have 0, 1, 2, 3, 4, or infinitely many solutions.

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with a *.

### Lesson 12-1
- Quadratic equation in two variables
- Quadratic relation in two variables
- Conic section, conic
- Circle, radius, center
- Center-Radius Equation for a Circle Theorem
- Lattice point

### Lesson 12-2
- Interior, exterior of a circle
- Interior and Exterior of a Circle Theorem

### Lesson 12-3
- Foci, focal constant of an ellipse or hyperbola
- Ellipse
- Vertices of an ellipse or hyperbola
- Hyperbola
- Asymptotes of a hyperbola
- Eccentricity

### Lesson 12-4
- Standard position for an ellipse
- Standard form of equation for an ellipse
- Equation for an Ellipse Theorem
- Major axis, minor axis, center of an ellipse

### Lesson 12-5
- Area of an ellipse

### Lesson 12-6
- Rectangular hyperbola

### Lesson 12-7
- Standard form for an equation of a hyperbola

### Lesson 12-8
- General form of a quadratic relation
- Discriminant Theorem for Conics
- Degenerate form of a conic

### Lesson 12-9
- Quadratic system
- Quadratic-linear system

### Lesson 12-10
- Quadratic-quadratic system

### Vocabulary

**Lesson 12-1**
- *quadratic equation in two variables
- *quadratic relation in two variables
- conic section, conic
- *circle, radius, center
- Center-Radius Equation for a Circle Theorem
- *lattice point

**Lesson 12-2**
- *interior, exterior of a circle
- Interior and Exterior of a Circle Theorem

**Lesson 12-3**
- foci, focal constant of an ellipse or hyperbola
- *ellipse
- vertices of an ellipse or hyperbola
- *hyperbola
- asymptotes of a hyperbola
- eccentricity

**Lesson 12-4**
- standard position for an ellipse
- standard form of equation for an ellipse
- Equation for an Ellipse Theorem
- *major axis, minor axis, center of an ellipse

**Lesson 12-5**
- area of an ellipse

**Lesson 12-6**
- *rectangular hyperbola

**Lesson 12-7**
- *standard form for an equation of a hyperbola

**Lesson 12-8**
- *general form of a quadratic relation
- Discriminant Theorem for Conics
- degenerate form of a conic

**Lesson 12-9**
- quadratic system
- quadratic-linear system

**Lesson 12-10**
- quadratic-quadratic system

---

**CHAPTER 12 Summary and Vocabulary 733**
Take this test as you would take a test in class. Use graph paper and a ruler. Then check your work with the solutions in the Selected Answer section in the back of the book.

In 1 and 2, consider the equation
\[ x^2 + 9x + y^2 - 26y - 163 = 0. \]

1. Determine which conic section the equation represents.
2. Rewrite the equation in standard form for that conic.

In 3-5, consider the image of \( x^2 + y^2 = 1 \) under the scale change \( S_{3,4} \).
3. State an equation for the image.

4. Multiple choice  The image is a(n)
   (a) circle  (b) ellipse  (c) parabola  (d) hyperbola.
5. Find the coordinates of the vertices of the image.

In 6 and 7, refer to the ellipse drawn below.

6. Determine an equation for this ellipse.
7. Find its area.

In 8 and 9, consider the following system:
\[ \begin{align*}
  y &= x - 2 \\
  y &= 4x - x^2 \\
\end{align*} \]
8. Estimate the solutions by graphing the system.
9. Find the exact solutions algebraically.

10. Pluto has an elliptical orbit with the Sun at one focus. Its closest distance to the Sun is about 2.8 billion miles, while its farthest distance is about 4.6 billion miles. What is the length of the major axis of Pluto’s orbit?

11. Use conic graph paper with centers 12 units apart to sketch the set of points \( P \) such that \( |PF_1 - PF_2| = 5 \).

12. Graph the set of points \((x, y)\) such that \( 9 < x^2 + y^2 < 25 \).

In 13 and 14, one earthquake station determines that the center of a quake is 40 miles away. A second station 25 miles west and 60 miles north of the first station finds that it is 30 miles from the center.
13. Write an equation to describe each situation.
14. Where is the epicenter of the quake (to the nearest tenth of a mile)?
15. The graph of \( xy = 2 \) is a hyperbola. What are the asymptotes of this hyperbola?
16. Graph the set of points \((x, y)\) satisfying \( \frac{x^2}{9} - \frac{y^2}{4} = 1 \).
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

**Objective A:** Convert from the general form of a quadratic equation in two variables to standard form for a particular curve, and vice versa. (Lesson 12-8)

In 1 and 2, rewrite in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

1. $(x - 3)^2 + (y + 7)^2 = 100$
2. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

In 3–6, rewrite the equation in standard form.

3. ellipse: $25x^2 + 75y^2 = 150$
4. parabola: $y + 3x = x^2 - 5$
5. hyperbola: $2x^2 - 4y^2 = 8$
6. circle: $x^2 + y^2 + 6x - 9y - 12 = 0$

**Objective B:** Write equations or inequalities for quadratic relations given sufficient conditions. (Lessons 12-1, 12-2, 12-4, 12-6, 12-7)

In 7 and 8, find an equation for the circle satisfying the conditions.

7. center at origin, radius 6
8. center is (-7, 5), radius 12

9. a. Solve the equation $x^2 + y^2 = 20$ for $y$.

b. Explain how the graph of the equation $x^2 + y^2 = 20$ is related to the graph of your response to part a.

10. Determine a quadratic relation describing the interior of the ellipse with equation $x^2 + 3y^2 = 75$.

In 11 and 12, write an equation for the ellipse satisfying the given conditions.

11. foci are (0, 5) and (0, -5); focal constant is 26
12. foci are (9, 0) and (-9, 0); minor axis has length 6

In 13 and 14, find an equation for the hyperbola satisfying the given conditions.

13. foci at (7, 0) and (-7, 0); focal constant 8
14. vertices are (1, 1) and (-1, -1); asymptotes are the x- and y-axes

**Objective C:** Find the area of an ellipse. (Lesson 12-5)

In 15 and 16, find the area of the ellipse satisfying the given conditions.

15. Its equation is $\frac{x^2}{121} + \frac{y^2}{9} = 1$.
16. The endpoints of its major and minor axes are (0, 10), (0, -10), (5, 0), and (-5, 0).

17. Which has a larger area: a circle of radius 5 or an ellipse with major and minor axes of lengths 12 and 8, respectively? Justify your answer.

18. Find the area of the shaded region below, which is between an ellipse with major axis of length 10 and minor axis of length 8, and a circle with diameter 8.
Objective D. Solve systems of one linear and one quadratic equation or two quadratic equations by substitution or linear combination. (Lessons 12-9, 12-10)

In 19–26, solve.

19. \( \begin{cases} y = x^2 + 5 \\ y = -x^2 + 5x + 8 \end{cases} \)

20. \( \begin{cases} 2x + y = 23 \\ y = 2x^2 - 7x + 5 \end{cases} \)

21. \( \begin{cases} y = x^2 + 3x - 4 \\ y = 2x^2 + 5x - 3 \end{cases} \)

In 26 and 27, identify the center and radius of the circle with the given equation.

26. \((x + 8)^2 + y^2 = 196\)

27. \(x^2 + y^2 = 5\)

In 28 and 29, consider the ellipse with equation \(\frac{x^2}{169} + \frac{y^2}{400} = 1\).

28. Name its vertices.

29. State the length of its minor axis.

Objective E: Identify characteristics of circles, ellipses, and hyperbolas. (Lessons 12-1, 12-4, 12-6, 12-7)

In 26 and 27, identify the center and radius of the circle with the given equation.

26. \((x + 8)^2 + y^2 = 196\)

27. \(x^2 + y^2 = 5\)

In 28 and 29, consider the ellipse with equation \(\frac{x^2}{169} + \frac{y^2}{400} = 1\).

28. Name its vertices.

29. State the length of its minor axis.

Objective F: Classify curves as circles, ellipses, parabolas, or hyperbolas using algebraic or geometric properties. (Lessons 12-1, 12-3, 12-8)

In 33 and 34, consider two fixed points \(F_1\) and \(F_2\) and a focal constant \(d\). Identify the set of points \(P\) satisfying the given conditions.

33. \(F_1P + F_2P = d\), where \(d > F_1F_2\)

34. \(|F_1P - F_2P| = d\), where \(d < F_1F_2\)

In 35 and 36, consider the equation \(Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\). What conic results from the given conditions?

35. \(A = C\) and \(B = 0\)

36. \(B^2 - 4AC = 0\)

In 37 and 38, tell whether the graph is a hyperbola, a parabola, or an ellipse.

37. \(5x^2 + 10x - y^2 + 3y = 0\)

38. \(x^2 - 3x + y^2 + 4y = xy\)

39. The figure below shows a double cone intersected by four planes \(A\), \(B\), \(C\), and \(D\). Identify the curve produced by each intersection.

---

PROPERTIES deal with the principles behind the mathematics.
Objective G: Describe relations between conics. (Lessons 12-3, 12-5)
In 40–43, true or false.
40. Every circle is an ellipse.
41. The image of the unit circle under a scale change is an ellipse.
42. A hyperbola can be considered as the union of two parabolas.
43. All quadratic relations in two variables can be determined from the intersection of a plane and a double cone.

USES deal with applications of mathematics in real situations.

Objective H: Use circles, ellipses, and hyperbolas to solve real world problems. (Lessons 12-1, 12-2, 12-4, 12-5)
44. A truck 10 ft high and 5 ft wide approaches a semicircular tunnel with a radius of 12 ft. Will the truck fit through the tunnel? Justify your answer.
45. The elliptically shaped pool below is to be surrounded by tile so that the outer boundary of the tile is also an ellipse. The tiler needs to know the area of the shaded region to determine how much tile to buy. The major axis of the pool is 15 m and the minor axis of the pool is 8 m. The major axis AB is 18 m and the minor axis DC is 11 m. What is the area of the shaded region?

46. The orbit of the Earth around the Sun is elliptical with the Sun as one focus. The closest and farthest distances of the Earth from the Sun are 91.4 and 94.5 million miles, respectively.
a. How far is $F_2$, the second focus, from the Sun?
b. What is the length of the minor axis of the Earth’s orbit?

In 47–50, a computer programmer needs to write instructions to draw a figure, such as the one above at the right, with concentric circles with radii of 10, 30, and 50 pixels. The center of the circles is at the point (200, 100).
47. What sentence does the checkerboard region satisfy?
48. What sentence does the light-shaded ring satisfy?
49. What sentence lets the programmer describe points $(x, y)$ in the exterior of the circle with equation $(x + 8)^2 + y^2 = 5$?
50. a. For the circle $x^2 + y^2 = 1$, what sentence is the image of the circle under the scale change $S: (x, y) \rightarrow (6x, 9y)$?
b. What kind of curve is the image in part a?
Objective I: Use systems of quadratic equations to solve real problems. (Lessons 12-9, 12-10)

51. A rectangular Oriental rug has an area of 216 square feet and a perimeter of 60 feet. Find the dimensions of the rug.
In 52 and 53, suppose a seismograph shows the epicenter of an earthquake to be about 50 miles away from station 1. Another station, which is 60 miles east and 40 miles south of station 1, finds that the quake is also 50 miles away.
52. Find the possible locations for the epicenter.
53. Station 3, 70 miles west and 20 miles north of station 1, finds that the same quake is about 106 miles away. Where is the actual epicenter of the earthquake?

54. The demand function for Peewee’s Sports Company is \( xp = 250 \), where \( x \) is the number of baseballs in hundreds, and \( p \) is the unit price of a baseball. The supply function for the Giant Baseball Manufacturer is \( p = 2x^2 \). Find the equilibrium point, that is, the point where supply and demand are the same.

55. Eileen’s Eye Extravaganza took in $5600 in sunglass sales for last year. This year Eileen lowered the price by two dollars, sold seventy more pairs of sunglasses, and made $5880.
   a. How much is she selling her sunglasses for now?
   b. How many pairs did she sell this year?

Objective J: Graph quadratic relations given sentences for them, and vice-versa. (Lessons 12-1, 12-2, 12-3, 12-4, 12-6, 12-7)
In 56 and 57, use conic graph paper with centers 10 units apart to draw the set of points \( P \) satisfying the given condition.

In 58–61, sketch a graph.

58. \( \frac{x^2}{16} + \frac{y^2}{81} = 1 \)
59. \( \frac{x^2}{16} - \frac{y^2}{81} = 1 \)
60. \( xy = 12 \)
61. \( x^2 + y^2 \leq 9 \)

In 62 and 63, state an equation for the curve.
62. a circle tangent to the coordinate axes at (0, -1) and (1, 0)
63. an ellipse with x-intercepts \( \pm 7 \) and y-intercepts \( \pm 4 \)

In 64 and 65, multiple choice. Select the equation that best describes each graph.

(a) \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)  
(b) \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)  
(c) \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \)  
(d) \( xy = a; \ a > 0 \)

64.

65.

 Objective K: Solve systems of quadratic equations geometrically. (Lessons 12-9, 12-10)

In 66 and 67, solve by graphing.

66. \[
\begin{align*}
  y &= x^2 - 10 \\
  y &= 11 - x 
\end{align*}
\]

67. \[
\begin{align*}
  x^2 + y^2 &= 81 \\
  x^2 + (y + 18)^2 &= 81 
\end{align*}
\]

In 68 and 69, draw an example showing how the situation can occur.

68. a circle and a hyperbola that intersect in 4 points

69. two parabolas that do not intersect

70. Refer to the graphs below of the curves \( \frac{x^2}{40} + \frac{y^2}{10} = 1 \) and \( x + y = 1 \).

a. Estimate the points of intersection from this sketch.

b. Use an automatic grahper to estimate the solutions to the nearest tenth.
CHAPTER 13

Series, Combinations, and Statistics
Addition is as fundamental in advanced mathematics as it is in arithmetic. In this chapter, you will study sums of terms of various sequences, learn a special notation for sums, and see many applications of Pascal’s Triangle, a triangular array which is formed by adding.

\[
\begin{array}{cccccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array}
\]

The applications of these ideas are diverse: to counting committees, to powers of binomials, to probability, to statistics, and even to IQ scores, scores on standardized tests, and bell-shaped distribution curves.

**13-1**: Arithmetic Series  
**13-2**: Geometric Series  
**13-3**: The Σ and ! Symbols  
**13-4**: Infinite Geometric Series  
**13-5**: Pascal’s Triangle  
**13-6**: The Binomial Theorem  
**13-7**: Subsets and Combinations  
**13-8**: Probabilities and Combinations  
**13-9**: Descriptive Statistics  
**13-10**: Binomial and Normal Distributions  
**13-11**: Polls and Sampling

*Standardized tests are familiar to students entering or graduating from academic, technical, or professional training.*
There is a story often told about Karl Friedrich Gauss. (At age 20, he proved The Fundamental Theorem of Algebra; see page 639.) When he was in third grade, his class misbehaved and the teacher gave the following problem as punishment.

"Add the integers from 1 to 100."

It is said that Gauss solved the problem in almost no time at all. His method was something like the following. Let $S$ be the desired sum.

\[
S = 1 + 2 + 3 + \ldots + 98 + 99 + 100
\]

Using the Commutative and Associative Properties, the sum can be rewritten in reverse order.

\[
S = 100 + 99 + 98 + \ldots + 3 + 2 + 1
\]

Now add corresponding terms. The sums are the same!

\[
2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101
\]

\[
\text{100 terms}
\]

\[
2S = 100 \cdot 101
\]

\[
S = 5050
\]

The story is that Gauss wrote only the number 5050 on his slate, having done all the figuring in his head. The teacher (who had hoped the problem would keep the students working for a long time) was quite disturbed. However, the teacher did recognize that Gauss was extraordinary and gave him some advanced books to read. Gauss’ method of solution is the basis for the proof of the next theorem.

**Theorem:**

The sum of the integers from 1 to $n$ is $\frac{1}{2}n(n + 1)$.

**Proof:**

1. Let $S = 1 + 2 + \ldots + (n - 1) + n$
2. Reversing the order of the terms.
   \[
   S = n + (n - 1) + \ldots + 2 + 1
   \]
3. Add corresponding terms.
   \[
   2S = (1 + n) + (2 + n - 1) + \ldots + (n - 1 + 2) + (n + 1)
   \]
   \[
   = (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1)
   \]
   \[
   n \text{ terms}
   \]
4. \[
2S = n(n + 1)
\]
5. \[
2S = n(n + 1)
\]
6. So, \[
S = \frac{1}{2}n(n + 1).
\]
Using the formula, the sum of the integers from 1 to 100 is \( \frac{1}{2}(100)(101) = 5050 \), the answer Gauss gave. Similarly, the sum of the integers from 1 to 4 is \( \frac{1}{2}(4)(5) = 10 \), a result easy to check.

Recall that an arithmetic or linear sequence is one in which the difference between consecutive terms is constant. An arithmetic sequence has the form \( a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + (n - 1)d \). The integers from 1 to 100 form a finite arithmetic sequence with \( a_1 = 1, n = 100, \text{ and } d = 1 \). Reasoning similar to that of Gauss can be used to find the sum of the consecutive terms of any finite arithmetic sequence.

### Example 1
Find the sum of the first 30 terms of the arithmetic sequence

4, 11, 18, 25, ...

**Solution**  First, calculate the 30th term. The common difference is 7. The 30th term is \( 4 + 29 \cdot 7 \), or 207.

Thus, \( S = 4 + 11 + \ldots + 200 + 207 \).

Also, \( S = 207 + 200 + \ldots + 11 + 4 \).

So, \( 2S = \underbrace{211 + 211 + \ldots + 211 + 211}_{30 \text{ terms}} \).

Thus, \( S = \frac{1}{2}(30)(211) = 3165 \).

In general, an indicated sum of terms is called a **series**. If the terms form an arithmetic sequence with first term \( a_1 \) and common difference \( d \), the indicated sum of the terms is called an **arithmetic series**. The sum of the first \( n \) terms, represented \( S_n \), is

\[
S_n = a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n.
\]

You can find a formula for \( S_n \) by noticing that the series can be written in two ways:

(i) Start with the first term \( a_1 \) and successively add the common difference \( d \).

(ii) Start with the last term \( a_n \) and successively subtract the common difference \( d \).

\[
S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n - 1)d] \\
S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + [a_n - (n - 1)d]
\]
When you add corresponding pairs of terms of these two formulas, as Gauss did, each of the \( n \) pairs adds to the same amount, \( a_1 + a_n \): 

\[
S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)
\]

\( n \) terms

So 

\[
2S_n = n(a_1 + a_n)
\]

and, multiplying both sides by \( \frac{1}{2} \), 

\[
S_n = \frac{n}{2}(a_1 + a_n).
\]

This formula is convenient if the first and \( n \)th terms are known. If the \( n \)th term is not known, an alternative formula can be found using the formula for the \( n \)th term of an arithmetic sequence, 

\[
a_n = a_1 + (n - 1)d.
\]

Substituting for \( a_n \) in the right side of the expression for \( S_n \),

\[
S_n = \frac{n}{2}[a_1 + a_1 + (n - 1)d].
\]

That is, 

\[
S_n = \frac{n}{2}[2a_1 + (n - 1)d].
\]

This argument proves the following theorem.

**Theorem:**

Let \( S_n = a_1 + a_2 + \ldots + a_n \) be an arithmetic series with constant difference \( d \). Then the value \( S_n \) of that series is 

\[
S_n = \frac{n}{2}(a_1 + a_n)
\]

or 

\[
S_n = \frac{n}{2}[2a_1 + (n - 1)d].
\]

**Example 2** An auditorium has 15 rows, with 20 seats in the front row and 2 more seats in each row thereafter.

a. How many seats are there in the last row?

b. How many seats are there in all?
Solution

a. The sequence of seats is 20, 22, 24, … . The 15th row has 20 + 14 \cdot 2 seats, or 48 seats.
b. Use the formula

\[ S_n = \frac{n}{2} (a_1 + a_n). \]

In this case, \( n = 15 \), \( a_1 = 20 \), and \( a_n = a_{15} = 48 \).

\[ S_{15} = \frac{15}{2} (20 + 48) = \frac{15}{2} \cdot 68 = 510 \]

There are 510 seats in the auditorium.

Check Use the formula \( S_n = \frac{n}{2} (2a_1 + (n - 1)d) \).

Then \( S_{15} = \frac{15}{2} (2 \cdot 20 + (15 - 1) \cdot 2) \)
\[ = \frac{15}{2} (40 + 28) = 510. \text{ It checks.} \]

Questions

1. If Gauss was 8 years old when in third grade, what year was that?

2. What properties ensure that

\[ 1 + 2 + \ldots + (n - 1) + n = n + (n - 1) + \ldots + 2 + 1? \]

3. Consider 20 + 18 + 16 + 14 and 20, 18, 16, 14.
   a. Which is an arithmetic sequence?
   b. Which is an arithmetic series?

4. What is the \( n \)th term of the arithmetic sequence with first term \( a_1 \) and constant difference \( d \)?

5. Find the sum of the integers from 1 to 1000.

6. a. Write out all the terms in the arithmetic series

   \[ 5 + 9 + 13 + \ldots + 37. \]
   b. How many terms are there?
   c. What is the sum of all the terms?

7. Suppose a theater has 26 seats in the first row and that each row has 4 more seats than the previous row. If there are 30 rows in the theater,
   a. how many seats are in the last row?
   b. how many seats are there in all?

8. Find the sum of the odd integers from 25 to 75.

9. Finish this sentence: The sum of the \( n \) terms of an arithmetic sequence equals the average of the first and last terms multiplied by \( ? \)
10. Find the sum of all the positive even integers with 3 digits.

11. Let $S_n$ be the sum of the first $n$ terms of the sequence defined by $a_n = 11n - 3$. Find:
   a. $S_2$
   b. $S_3$
   c. $S_{25}$

12. An organization has new officers for the year and is ordering new stationery. In January, a mailing is sent to the 325 current members. If the membership increases each month by 5 members, how many envelopes will be needed for one year’s monthly mailings?

In 13 and 14, two salaries are compared.

13. Suppose a firefighter earns $24,000 the first year on the job. The second year and each year thereafter, the firefighter earns $1200 more than the previous year.
   a. Write the first and last terms of the series whose sum gives the total amount earned in 8 years.
   b. Find the total amount earned in 8 years.

14. After 6 months on the job, and every 6 months thereafter, another firefighter gets a raise of $600. If that firefighter earns

   12,000 the 1st half-year,
   12,600 the 2nd half-year,
   13,200 the 3rd half-year,

find the total amount earned by this firefighter in 8 years.

15. The following BASIC program generates recursively the terms of an arithmetic sequence and the sum of the terms of that sequence.

   ```basic
   10 REM PROGRAM TO PRINT TERMS OF ARITHMETIC
   15 LET N = 1
   20 LET TERM = 10
   25 LET SUM = 0
   30 LET SUM = SUM + TERM
   35 PRINT "N", "TERM", "SUM"
   40 PRINT N, TERM, SUM
   45 FOR N = 2 TO 15
   50   TERM = TERM + 3
   55   SUM = SUM + TERM
   60   PRINT N, TERM, SUM
   65 NEXT N
   70 END
   ``

   a. Run this program and list the last line of output.
   b. What explicit formulas could have been used to calculate the last term and sum directly?
   c. Modify the program so it generates the sequence and series determined by Question 13.
16. In the triangle at the right, find:
   a. \( m \angle Q \) to the nearest degree
   b. \( \sin Q \) to the nearest thousandth. \((Lesson 10-2)\)

17. To estimate the height of a tall building you walk 100 meters from it and look up. Taking your protractor from your notebook, you estimate that you need to turn your head up \( 60^\circ \) to see the top of the building. How much above eye level is the top of the building? \((Lesson 10-2)\)

In 18 and 19, factor as much as possible. \((Lesson 11-3)\)
18. \( 9x^4 - 9y^2 \)  19. \( x - ax \)

20. \textit{Multiple choice}  What is the 20th term of the geometric sequence that begins 3, 6, \ldots ? \((Lesson 8-3)\)
   (a) \( 3 \cdot 20 \)  (b) \( 3 \cdot 2 \cdot 20 \)
   (c) \( 3 \cdot 2^{19} \)  (d) \( 3 \cdot 2^{20} \)

In 21–24, write as a power of 5. \((Lessons 8-1, 8-4, 8-6)\)
21. \( 5^{10} \cdot 5^4 \)  22. \( 5^6 \cdot 5 \)
23. \( 5^9 \div 5^3 \)  24. \( \sqrt{5} \)

25. The polynomial function \( y = 3(x - 4)^2 + k \) contains the point (2, -1).
   a. What is the value of \( k \)?
   b. Describe the graph of this function. \((Lesson 6-4)\)

26. The number 9 can be written as the sum of an arithmetic sequence \( 9 = 1 + 3 + 5 \). What other numbers from 1 to 100 can be written as the sum of an arithmetic sequence whose terms are positive integers? \((Assume the sequence must have at least three distinct terms.)\)
Legend has it that, when he first learned to play chess, the king of Persia was so impressed that he summoned the game's inventor to offer a reward. The inventor pointed to the chessboard, and said that a satisfactory reward would be one grain of wheat on the first square, two on the second, four on the third, eight on the fourth, and so on for all sixty-four squares. The king protested that this was surely not enough reward, but the inventor insisted. What do you think? Is this a large or small reward?

Recall that a geometric sequence is a sequence in which the ratio of consecutive terms is constant. The number of grains on the squares form a geometric sequence with first term 1 and constant ratio 2. The $n$th term of this sequence is $2^{n-1}$. The total number of grains of wheat on the chessboard is the sum of the first 64 terms of this sequence. Call this total $S_{64}$.

$$S_{64} = 1 + 2 + 4 + 8 + \ldots + 2^{63}$$

An indicated sum like this of successive terms of a geometric sequence is called a geometric series.

To evaluate $S_{64}$, first write the series in reverse order:

$$S_{64} = 2^{63} + 2^{62} + \ldots + 8 + 4 + 2 + 1$$

Notice that if each term of $S_{64}$ is doubled, many values identical to those in the first series are generated:

$$2S_{64} = 2^{64} + 2^{63} + 2^{62} + \ldots + 16 + 8 + 4 + 2$$

Subtracting the first equation from the second gives

$$2S_{64} - S_{64} = 2^{64} + (2^{63} - 2^{63}) + (2^{62} - 2^{62}) + \ldots + (8 - 8) + (4 - 4) + (2 - 2) - 1;$$

i.e.,

$$S_{64} = 2^{64} - 1.$$
This is about $1.84 \times 10^{19}$ grains of wheat. If you assume that each grain can be approximated by a tiny rectangular box, $4 \text{ mm} \times 1 \text{ mm} \times 1 \text{ mm}$, the total volume of wheat would be about 74 cubic kilometers, quite a bit more than all the wheat in the world. We do not know what happened to the inventor after the king found out he had been tricked.

The above procedure can be generalized to find the value $S_n$ of the geometric series with first term $g$, constant ratio $r$, and $n$ terms.

$$S_n = g + gr + gr^2 + \ldots + gr^{n-1}$$

Multiply by $r$.

$$rS_n = gr + gr^2 + \ldots + gr^{n-1} + gr^n$$

Subtract.

$$S_n - rS_n = g - gr^n$$

Factor.

$$(1 - r)S_n = g(1 - r^n)$$

Divide both sides by $1 - r$.

$$S_n = \frac{g(1 - r^n)}{1 - r}$$

The constant ratio $r$ cannot be 1 in this formula, but that is not a problem. If $r = 1$, the series is $g + g + g + \ldots + g$, with $n$ terms, and its sum is $ng$.

This argument proves the following theorem.

**Theorem:**

Let $g + gr + gr^2 + \ldots + gr^{n-1}$ be a geometric series with $r \neq 1$. Then the value $S_n$ of that series is

$$S_n = \frac{g(1 - r^n)}{1 - r}.$$
The formula for a geometric series works even when the ratio is negative.

Example 2  Find the sum of the first 100 terms of the geometric series
\[ 5 - 10 + 20 - 40 + 80 - \ldots \]

Solution  \[ S_n = \frac{g(1 - r^n)}{1 - r} \]. In this case \( g = 5 \), \( r = -2 \), and \( n = 100 \).

\[ S_{100} = \frac{5(1 - (-2)^{100})}{1 - (-2)} = \frac{5 - 5 \cdot 2^{100}}{3} \approx -2.11 \times 10^{30} \]

Check  The sum is negative. This is what you would expect after an even number of terms.

When the constant ratio \( r > 1 \), it is often more convenient to use the formula \( S_n = \frac{g(r^n - 1)}{r - 1} \), derived by multiplying the numerator and denominator of \( S_n = \frac{g(1 - r^n)}{1 - r} \) by -1. This is the case when evaluating polynomials that arise from compound interest situations. They can be evaluated using this formula more quickly than by adding each deposit's yield.

Example 3  If \$100 is deposited on January 1st of the years 2000, 2001, 2002, and so on to 2009, with an annual yield of 7\%, how much will there be on January 1st, 2010?

Solution  The scale factor in this situation is 1.07. On January 1st, 2010, there will be

\[ 100(1.07)^{10} + 100(1.07)^9 + \ldots + 100(1.07)^2 + 100(1.07). \]

Think of this as a geometric series with the term at the right, \( g = 100(1.07) \), as the first term, and ratio \( r = 1.07 \). There are 10 terms, so \( n = 10 \). Use the formula for the value of a geometric series.

\[ S_{10} = \frac{g(r^n - 1)}{r - 1} = \frac{100(1.07)(1.07^{10} - 1)}{1.07 - 1} = \frac{107(1.07^{10} - 1)}{0.07} \]

\[ \approx \$1478.36 \]

Check  From the middle of the year 2005, the value accrued would be \( 100(1.07)^5 \), which is about \$140.26. Multiplying that middle value by 10, an estimate is \$1402.60. The answer \$1478.36 seems reasonable.
Questions

1. According to the story about the king of Persia, how many grains of wheat were on the first two rows of the chess board?

2. In 2–5, give the value of the series.
   2. \( 3 + 12 + 48 + \ldots + 3 \cdot 4^9 \)
   3. \( 50 + 10 + 2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} \)
   4. \( 50 - 10 + 2 - \frac{2}{5} + \frac{2}{25} - \frac{2}{125} \)
   5. \( 1 + b + b^2 + \ldots + b^{16} \)

6. a. In the formula for the value of a geometric series, what value can \( r \) not have?
   b. Why can it not have this value?

7. If \$200\) is deposited on January 1st of five consecutive years and earns an annual yield of \(8\%\), how much will there be on January 1st of the sixth year?

8. A superball is dropped from a height of 2 meters and bounces to \(90\%\) of its height on each bounce. When it hits the ground for the eighth time, how far has it traveled?

9. a. Write the first 8 terms of the geometric series from the sequence
   \[
   \left\{ \begin{array}{l}
g_1 = 6 \\
g_n = -\frac{3}{2}g_{n-1} \text{ for } n > 1. \\
\end{array} \right.
   
   b. Find the sum of these terms.

10. On the first day of each month Mollie pays \$100 on a car loan. Suppose she had no loan and could earn \(1\%\) per month on this money. How much would she have at the end of the year?

In 11–13, as shown at the right, midpoints of a 12 by 16 rectangle have been connected to form a rhombus. Then midpoints of the rhombus are connected to form a rectangle, and so on.

11. a. List the perimeters of the first 10 rectangles formed (including the largest rectangle).
   b. What is the sum of the perimeters of the first 10 rectangles formed?

12. What is the sum of the perimeters of the first 10 rhombi formed?

13. If the figure contains 10 rectangles and 10 rhombi, into how many regions has the original rectangle been divided?
14. If the chessboard inventor had wanted 1 grain on the first square, 2 on the second, 3 on the third, and so on in arithmetic sequence, how many grains would have been the reward? (Lesson 13-1)

15. Find the sum of all integers between 100 and 1000 that are divisible by 3. (Lesson 13-1)

16. Give the coordinates of the points of intersection of the line

\[ y = 2x + 5 \]

and the parabola \( x = y^2 \). (Lesson 12-9)

17. Give an equation for the line parallel to \( 3x + 2y = 10 \) and containing \( (8, 4) \). (Lesson 13-1)

18. a. Identify the quadrilateral graphed at the right.
b. Prove or disprove: The diagonals of this quadrilateral have the same length. (Previous course)

19. Use the formula \( h = h_0 + v_0t - 4.9t^2 \) to estimate the amount of time a skydiver is in free fall if the skydiver leaves a plane at an altitude of 2500 meters and opens the parachute at 500 meters. (Lesson 6-2)

20. An ellipse is inscribed in a rectangle 20 cm long and 12 cm wide. The area of the ellipse is what percent of the area of the rectangle? (Lesson 12-5, Previous course)

21. Refer to Lesson 13-1. Modify the program in Question 15 so it generates terms in the geometric series in the story about chess and the king of Persia. Write a short paragraph about some interesting aspect of your computer output.

22. Your ancestors consist of 2 parents, 4 grandparents, 8 great-grandparents, 16 great-great-grandparents, and so on. Pick some estimate for the number of years in a generation.
a. Use that estimate to help calculate the total number of ancestors you have had in the past 2000 years.
b. Must there have been some duplicates (people from whom you descended in two different ways)? Explain your answer.
Mathematicians use symbols to shorten writing. Consider the arithmetic sequence 8, 11, 14, ... . Any term of this sequence can be calculated from the formula \( a_n = 3n + 5 \). For example, the 1000th term is 3005.

The sum of the first 1000 terms of this sequence is

\[ 8 + 11 + 14 + \ldots + 3005. \]

A shorthand notation for such sums uses the Greek capital letter \( \Sigma \) (sigma).

"The sum of the numbers of the form \( 3n + 5 \) for integer values of \( n \) from \( n = 1 \) to \( n = 1000 \)" is written

\[ \sum_{n=1}^{1000} (3n + 5). \]

This is called \( \Sigma \)-notation, which is read as sigma notation or summation notation. In \( \Sigma \)-notation:

\[ 8 + 11 + 14 + \ldots + 3005 = \sum_{n=1}^{1000} (3n + 5). \]

In summation notation, the variable \( n \) under the \( \Sigma \) sign is the index variable or index. In this book, index variables have only integer values. It is common to use letters \( i, j, k, \) or \( n \) as index variables. (In summation notation, \( i \) is not the complex number \( \sqrt{-1} \) unless it is so specified.) To evaluate an expression written in sigma notation, substitute into the expression following the sigma sign each integer from the lower value of the index variable to the upper value, and add the results.

**Example 1**

a. Read and b. write \( \sum_{i=5}^{11} 2^i \) without using \( \Sigma \).

**Solution**

a. "The sum of the numbers of the form 2 to the \( i \)th power for integer values of \( i \) from 5 to 11."

\[ \sum_{i=5}^{11} 2^i = 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} \]

or \( 32 + 64 + 128 + 256 + 512 + 1024 + 2048 \)

The expression in Example 1 can be evaluated using the formula for a geometric series with \( g = 2^5, r = 2, \) and \( n = 7 \).

\[ \sum_{i=5}^{11} 2^i = \frac{2^5(2^7 - 1)}{2 - 1} = 32 \cdot 127 = 4064 \]
Each of the theorems of the last two lessons can be restated using \( \Sigma \)-notation. Notice that \( i \) is used as the index variable to avoid confusion with the variable \( n \). Compare these restatements with the original statements.

**Sum of integers from 1 to \( n \):**

\[
\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)
\]

**In an arithmetic sequence** \( a_1, a_2, a_3, \ldots, a_n \) with constant difference \( d \):

\[
\sum_{i=1}^{n} a_i = \frac{1}{2}n(a_1 + a_n) = \frac{n}{2}[2a_1 + (n - 1)d]
\]

**In a geometric sequence** \( g_1, g_2, g_3, \ldots, g_n \) with constant ratio \( r \):

\[
\sum_{i=1}^{n} g_i = g_1 \frac{(1 - r^n)}{1 - r}
\]

---

**Example 2** Evaluate \( \sum_{n=1}^{1000} (3n + 5) \).

**Solution** This is the arithmetic series mentioned at the start of this lesson. Its first term is \( 3 \cdot 1 + 5 \), or 8, and the constant difference is 3. There are 1000 terms. Using the formula

\[
S_n = \frac{n}{2}(2a_1 + (n - 1)d),
\]

\[
S_{1000} = \frac{1000}{2}(2 \cdot 8 + (1000 - 1)3)
\]

\[
= 500(16 + 2997)
\]

\[
= 1,506,500
\]

A number in base 10, our familiar decimal system, is a sum of powers of 10.

\[834.57 = 8 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 + 5 \cdot 10^{-1} + 7 \cdot 10^{-2}\]

If all the digits are alike, then the number can be represented in \( \Sigma \)-notation.

\[9999.9 = 9 \cdot 10^3 + 9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0 + 9 \cdot 10^{-1}\]

Each term has the form \( 9 \cdot 10^i \) where \( i \) goes from 3 to -1. When using \( \Sigma \)-notation, the smallest and largest values of the index variable are written below and above the sigma, respectively.
Using $\Sigma$-notation, 9999.9 can be written as
\[
\sum_{i=-1}^{3} (9 \cdot 10^i).
\]

Notice that the index variable may have negative values.

Another symbol with immediate application is the *factorial* symbol, an exclamation point. The symbol $n!$ is read "$n$ factorial".

**Definition:**

\[
n! = \text{product of the integers from } n \text{ to } 1.
\]

The **factorial function** is defined by the equation $f(n) = n!$. For now, we take the domain of the factorial function to be the set of positive integers. In Lesson 13-5, the domain is extended to include 0. Small values of the factorial function can be calculated by hand or in your head.

\[
\begin{align*}
f(1) &= 1! = 1 \\
f(2) &= 2! = 2 \cdot 1 = 2 \\
f(3) &= 3! = 3 \cdot 2 \cdot 1 = 6 \\
f(4) &= 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
f(5) &= 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
f(6) &= 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720
\end{align*}
\]

Larger values require a calculator or computer. Many scientific calculators have a **factorial key** $\text{!}$. For instance, to calculate $20!$, key in $20 \text{!}$. The display indicates that $20! \approx 2.4329 \cdot 10^{18}$, or about 2,432,900,000,000,000,000.

The factorial function gives the number of possible arrangements of $n$ different objects in a row. These different arrangements are called **permutations**. For instance, with three objects, $A$, $B$, and $C$, there are six possible permutations: $ABC$, $ACB$, $BAC$, $BCA$, $CAB$, and $CBA$, and $3! = 6$.

**Example 3** Find the number of possible orders in which four runners, Alan, Bob, Carl, and David, might finish a race.

**Solution** The number of possible orders is the number of permutations of the four runners. List the possible orders. We use only the runners' initials.

\[
\begin{align*}
ABCD & \quad BACD & \quad CABD & \quad DABC \\
ABDC & \quad BADC & \quad CADB & \quad DACB \\
ACBD & \quad BCAD & \quad CBAD & \quad DBAC \\
ACDB & \quad BCDA & \quad CBDA & \quad DBCA \\
ADBC & \quad BDAC & \quad CDAB & \quad DCAB \\
ADCB & \quad BDCA & \quad CDBA & \quad DCBA
\end{align*}
\]

There are 24 permutations. This equals $4!$. 

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If you wished to list the possible ways in which five people could finish a race, you could begin with the list in Example 3. Call the 5th racer \( E \). In each permutation in the list, you can insert the \( E \) at the beginning, in three middle spots, or at the end. For instance, inserting \( E \) into \( ABCD \) yields \( EABCD, AEBCD, ABED, ABCED, \) or \( ABCDE \). This means that the number of permutations of 5 objects is 5 times the number of permutations of 4 objects. So the number of permutations of 5 objects is \( 5 \cdot 4! \), which equals \( 5! \). Similarly the number of permutations of 6 objects is \( 6 \cdot 5! \), which equals \( 6! \). Extending this argument proves the following theorem.

**Theorem:**

There are \( n! \) permutations of \( n \) distinct objects.

Numbers of permutations grow quickly. With 20 objects, there are 20! permutations, the large number estimated on the previous page.

### Questions

1. The symbol \( \Sigma \) is the Greek letter ___.

2. In \( \Sigma \)-notation, the variable under the \( \Sigma \) sign is the ___ variable.

In 3–6, multiple choice.

3. \[ \sum_{i=1}^{3} i^2 = \]
   (a) \( 3^2 \) (b) \( 1 + 4 + 9 \) (c) \( 1 + 2 + \ldots + 9 \) (d) none of these

4. \[ \sum_{k=1}^{4} 3k = \]
   (a) 16 (b) 30 (c) 82 (d) 94

5. \[ \sum_{n=1}^{5} (2n + 1) = \]
   (a) 3 + 5 + 7 + 9 + 11 (b) 3 + 11 (c) 1 + 5 + 11 (d) 2 + 4 + 6 + 8 + 10 + 1

6. \[ 3 + 6 + 9 + 12 + 15 + 18 + 21 = \]
   (a) \( \sum_{i=3}^{7} i \) (b) \( \sum_{i=1}^{7} 3i \) (c) \( \sum_{i=3}^{21} (3i) \) (d) none of these

In 7–10, give the value of the sum.

7. \[ \sum_{i=1}^{36} i \]

8. \[ \sum_{i=1}^{100} (2i - 1) \]

9. \[ \sum_{k=1}^{6} (2 \cdot 3^k) \]

10. \[ \sum_{n=-2}^{3} (4 \cdot 10^n) \]
11. In \( \sum_{i=100}^{200} (4i) \), how many terms are added?

12. The symbol \( n! \) is read ___.

13. Evaluate (a) 4!  (b) 6!  (c) 21!

14. a. Write out all permutations of the 4 letters \( P, E, R, M \).
   b. How many permutations are there?

Applying the Mathematics

In 15–17, write the series using \( \Sigma \)-notation.

15. \( 2 + 4 + 6 + 8 + 10 + 12 + 14 \)

16. \( 9 + 18 + 36 + 72 + 144 + 288 + 576 + 1152 \)

17. the sum of the squares of the integers from 1 to 100

18. a. Translate this statement into an algebraic formula using \( \Sigma \)-notation:
   The sum of the cubes of the integers from 1 to \( n \) is the square of the sum of the integers from 1 to \( n \).
   b. Verify part a when \( n = 4 \).

19. Write the average (mean) of the \( n \) numbers \( a_1, a_2, \ldots, a_n \) using \( \Sigma \)-notation.

20. Consider the sequence with the recursive definition
   \[
   \begin{align*}
   a_1 &= 1 \\
   a_n &= n \cdot a_{n-1} \text{ for } n > 1.
   \end{align*}
   \]
   a. Give the first 7 terms of the sequence.
   b. What is an appropriate name for this sequence?

21. Simplify (a) \( \frac{15!}{14!} \) and (b) \( \frac{(n + 1)!}{n!} \).

22. Show, by listing, that the number of different permutations of the letters of the word \( DEEDED \) is \( \frac{6!}{3!3!} \).

In 23 and 24, refer to the array of dots at the right.

23. If the array continued until there were 100 dots in the bottom row, how many dots would there be in all? (Lesson 13-1)

24. If the total number of dots is 496, how many rows are there?

25. Graph \( \{(x, y): x^2 + y^2 = 1\} \) and \( \{(x, y): x^2 - y^2 = 1\} \) on the same axes. (Lessons 12-1, 12-7)
26. Consider the following investment. Ima Saver deposits $50 on the first day of every month and earns 6% compounded monthly.
   a. How much interest will the first $50 deposit earn after 12 months?
   b. How much will there be in Ima’s account just before she makes the 12th deposit? (Assume the account starts with $0, and that there are no withdrawals.) *(Lessons 8-2, 11-1, 13-2)*

27. A snail is crawling straight up a wall. The 1st hour it climbs 16 inches; the 2nd hour it climbs 12 inches; each succeeding hour it climbs only $\frac{3}{4}$ the distance it climbed the previous hour. Assume this pattern holds indefinitely.
   a. How far does the snail climb during the 7th hour?
   b. What is the total distance climbed in 7 hours? *(Lessons 8-3, 13-2)*

28. If 3 blobs and 4 globs weigh 170 kg and 7 blobs and 6 globs weigh 330 kg, what will 4 blobs and 2 globs weigh? *(Lesson 5-3)*

29. How much louder is a sound of 100 decibels than one of 80 decibels? *(Lesson 9-3)*

30. Arrange from smallest to largest without using a calculator.
   *(Lessons 9-4, 9-6)*
   
   $2 \log 3, 3 \log 2, \log 3 + \log 2$

31. Consider the series of reciprocals of integers:

   $$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

   a. How many terms of the series are needed before the sum exceeds 2?
   b. How many terms of the series are needed before the sum exceeds 3?
   c. How many terms of the series are needed before the sum exceeds 10?
   d. Do you think the sum ever gets larger than 100? Why or why not?
Suppose a ball is dropped from a height of 6 feet and on each bounce rebounds to $\frac{2}{3}$ of its previous height. Here is a diagram of the distance it travels.

The lengths of the downward paths form a geometric sequence with first term 6 and constant ratio $\frac{2}{3}$. The lengths of the upward paths form a geometric sequence with first term 4 and constant ratio $\frac{2}{3}$. In Lesson 13-2, you learned how to calculate the total lengths for finite sequences. But in theory, this ball will bounce forever. (In reality, friction causes the ball to stop.)

Distance down = $6 + 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \ldots$

Distance up = $4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \ldots$

These sums have infinitely many terms. They are infinite geometric series.

Do these "infinite sums" represent particular numbers? To answer this question, form a sequence $S_n$ of partial sums. Let $S_n$ be the sum of the first $n$ terms of the sequence of distances down.

$S_1 = 6$
$S_2 = 6 + 4 = 10$
$S_3 = 6 + 4 + \frac{8}{3} = 12\frac{2}{3}$
$S_4 = 6 + 4 + \frac{8}{3} + \frac{16}{9} = 14\frac{1}{3}$
$S_5 = 6 + 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} = 15\frac{17}{27}$
$S_6 = 6 + 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \frac{64}{81} = 16\frac{34}{81}$

As $n$ gets larger, if $S_n$ gets closer and closer to some number $L$, then we say "as $n$ increases without bound, $S_n$ approaches $L$ as a limit." We write

$S_n \rightarrow L$

or

$\lim_{n \rightarrow \infty} S_n = L$.  

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The expression ‘‘\( n \to \infty \)’’ is read ‘‘\( n \) approaches infinity’’ or ‘‘\( n \) gets larger and larger without bound’’. The last line is read ‘‘the limit as \( n \) goes toward infinity of \( S_n \) is \( L \).’’

If there is such a number (or limit), that number \( L \) is called the **value** or **sum of the infinite series**. You can see that the partial sums of the above sequence keep on increasing. But as \( n \) increases, the difference between \( S_{n+1} \) and \( S_n \) gets smaller. Do you think there is a limit? The answer is given later in this lesson.

Many infinite geometric series have limits. Consider the infinite geometric sequence with first term \( \frac{6}{10} \) and constant ratio \( \frac{1}{10} \).

\[
\frac{6}{10}, \frac{6}{100}, \frac{6}{1000}, \frac{6}{10,000}, \ldots
\]

Here is the sequence of partial sums.

\[
\begin{align*}
S_1 &= \frac{6}{10} = 0.6 \\
S_2 &= \frac{6}{10} + \frac{6}{100} = 0.66 \\
S_3 &= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} = 0.666 \\
S_4 &= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10,000} = 0.6666 \\
&\vdots
\end{align*}
\]

As \( n \) gets larger, \( S_n \) approaches the infinite repeating decimal 0.\( \overline{6} \) as a limit. Since \( 0.\overline{6} = \frac{2}{3} \),

\[
S_n \to \frac{2}{3}.
\]

Even though you have seen many infinite decimals before, you may not have thought of them as infinite sums.

\[
\sum_{n=1}^{\infty} 0.6 \cdot \left(\frac{1}{10}\right)^{n-1} = \frac{2}{3}
\]

We say that the sum of numbers of the form \( 6 \cdot \left(\frac{1}{10}\right)^n \), as \( n \) goes from 1 to infinity, is \( \frac{2}{3} \).

Some infinite geometric series definitely have no limit. Consider the sequence of integer powers of 2: 1, 2, 4, 8, 16, 32, \ldots. The infinite geometric series \( 1 + 2 + 4 + 8 + 16 + 32 + \ldots \) has the following partial sums.

\[
\begin{align*}
S_1 &= 1 = 1 \\
S_2 &= 1 + 2 = 3 \\
S_3 &= 1 + 2 + 4 = 7 \\
S_4 &= 1 + 2 + 4 + 8 = 15 \\
&\vdots
\end{align*}
\]
Clearly the sums increase and ultimately get larger than any given number. There is no limit. Some people write
\[ S_n \to \infty \]
or
\[ \lim_{n \to \infty} S_n = \infty \]
read "S_n goes to infinity," or "S_n becomes infinitely large."

The geometric series \( \sum \frac{6}{10^k} + \frac{6}{1000} + \ldots \) has a limit. In 1 + 2 + 4 + ... , there is no limit. When does a geometric series have a limit? Remember that the sum of the first n terms of the infinite geometric series with first term g and ratio r is
\[ S_n = g \cdot \frac{1 - r^n}{1 - r}. \]

When \( |r| < 1 \), \( r^n \) gets nearer and nearer to zero as n gets larger. That is, if r is between -1 and 1, then as n gets larger,
\[ r^n \to 0 \]
So,
\[ (1 - r^n) \to 1. \]

Multiplying,
\[ \frac{g}{1 - r}(1 - r^n) \to \frac{g}{1 - r}. \]

This informally demonstrates the next theorem. A formal proof would require more advanced mathematics.

**Theorem:**
If \( |r| < 1 \), the infinite geometric series with first term g and ratio r has the value
\[ S = \frac{g}{1 - r}. \]

Using Σ-notation, this theorem can be stated more succinctly. If
\[ |r| < 1, \sum_{n=1}^{\infty} gr^{n-1} = \frac{g}{1 - r}. \]

For instance, in the sequence 0.6, 0.06, 0.006, ..., where the ratio is \( \frac{1}{10} \) and the first term g is \( \frac{6}{10} \), the sum of all terms is
\[ \frac{g}{1 - r} = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{2}{3}. \]

This agrees with the result on the previous page. By this method, a simple fraction for any infinite repeating decimal can be found.
Example 1  Find a simple fraction equal to $8.521212121 \ldots$

**Solution**  

$$8.5\overline{21} = 8.5 + 0.021212121 \ldots$$

$$= 8.5 + 0.021 + 0.00021 + 0.0000021 + \ldots$$

The terms beginning with 0.021 form an infinite geometric sequence with first term 0.021 and constant ratio 0.01. Using the theorem,

$$S = 8.5 + \frac{0.021}{1 - 0.01}$$

$$= 8.5 + \frac{0.021}{0.99} = \frac{85}{10} + \frac{21}{990}$$

Thus,

$$S = \frac{1406}{165}.$$  

Now let us return to the bouncing ball. Because the ratio $\frac{2}{3}$ is between -1 and 1, the series has a limit.

Example 2  Find the total distance traveled by the ball discussed at the beginning of this lesson.

**Solution**

Distance falling  

$$= 6 + 4 + \frac{8}{3} + \ldots = \frac{6}{1 - \frac{2}{3}} = 18 \text{ feet}$$

Distance rising  

$$= 4 + \frac{8}{3} + \frac{16}{9} + \ldots = \frac{4}{1 - \frac{2}{3}} = 12 \text{ feet}$$

Total distance  

$$= 18 + 12 = 30 \text{ feet}.$$  

Questions

In 1–4, (a) does the infinite geometric series have a value? (b) If so, what is this value? If not, why not?

1. $1 + 2 + 4 + 8 + \ldots$

2. $10 + 5 + 2.5 + 1.25 + \ldots$

3. $1 - 2 + 4 - 8 + \ldots$

4. $g + gr + gr^2 + gr^3 + \ldots$

5. Write in words: $\lim_{n \to \infty} S_n = L$.

6. If $S_1 = 0.6$, $S_2 = .54$, $S_3 = .546$, $S_4 = .5454$, and so on, what is $\lim_{n \to \infty} S_n$?

In 7–9, an infinite repeating decimal is given.

a. Write the decimal as an infinite geometric series.

b. Find a fraction equal to the value of the series.

7. $0.4444 \ldots$

8. $9.252525 \ldots$

9. $2.46\overline{8}$
10. A ball is dropped from a height of 2 meters and on each bounce rebounds to \( \frac{1}{4} \) of its previous height.
   a. What is the total length of the downward paths?
   b. What is the total length of the upward paths?
   c. How far does the ball travel before it stops bouncing?

11. The Greek philosopher Zeno was bothered by the idea of the infinite. He did not feel that time could be split into infinitely many parts. (This kind of split is done when one assumes that a dropped ball bounces infinitely often before stopping.) So he invented situations to make his point. One of these is the race between Achilles and the tortoise. (Achilles was a legendary fast Greek runner; a tortoise is a land turtle.)

Achilles, being faster than the tortoise, gives the tortoise a 10-meter head start. Achilles runs 10 times as fast. By the time Achilles runs 10 meters, the tortoise has gone 1 meter. When Achilles runs that 1 meter, the tortoise has gone 0.1 meter farther. When Achilles runs that 0.1 meter, the tortoise has gone 0.01 meter farther. This continues forever.

Zeno argued that Achilles never catches up to the tortoise. What do you think? Justify your answer.

In 12 and 13, a snowflake curve is the limit of a sequence of polygons formed in the following manner. \( F_1 \) is an equilateral triangle. \( F_n \) is formed by drawing an equilateral triangle outward on the middle third of each side of \( F_{n-1} \) and then deleting that middle third.

12. Suppose the perimeter of \( F_1 \) is 3.
   a. Find the perimeter of \( F_2 \).
   b. Find the perimeter of \( F_3 \).
   c. What is the perimeter \( P_n \) of \( F_n \)?
   d. Does the perimeter of \( F_3 \) have a limit as \( n \) gets larger and larger? If so, what is that limit? If not, why not?

13. Suppose the area of \( F_1 \) is \( \frac{\sqrt{3}}{4} \).
   a. Find the area of \( F_2 \).
   b. Find the area of \( F_3 \).
   c. What is the area \( A_n \) of \( F_n \)?
   d. Does the area of \( F_n \) have a limit as \( n \) gets larger and larger? If so, what is that limit? If not, why not?
14. The ball in Question 10 is dropped from three times the height.
   a. Without calculating, how many times farther do you think the ball
      would travel before it stops bouncing?
   b. Calculate how far the ball would travel, using the formula.

15. a. Use the formula for an infinite geometric series to find the value
    of the infinite repeating decimal 0.9999999 ….
    b. Use part a to give two different decimals equal to $\frac{1}{2}$.

16. a. Write out the first five terms of the series $\sum_{n=1}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^{n-1}$.
    b. What is the value of this series?

17. a. Expand $\sum_{n=1}^{10} (5n + 1)$. That is, write down the terms of the sum.
    b. Find its value. (Lessons 13-1, 13-3)

18. Find the value of $100 + 100x + 100x^2 + 100x^4 + 100x^8$ when
    $x = 1.06$ using: (a) direct substitution and (b) a formula for geometric
    series. (Lesson 13-2)

19. Find the points of intersection of the
    circle $x^2 + y^2 = 25$ and the ellipse
    $\frac{x^2}{64} + \frac{y^2}{16} = 1$ graphed at the right.
    (Lesson 12-10)

20. A line is perpendicular to $y = 3x$.
    a. What is its slope?
    b. What is its y-intercept? (Lesson 4-8)

21. Multiple choice   Which does not equal the others?
   (Lessons 10-3, 10-4, 10-5, 10-8)
   (a) $\sin 30\degree$  (b) $\sin 150\degree$  (c) $\sin 330\degree$  (d) $\sin 390\degree$

22. Let $f(n) = n!$ Evaluate. (Lesson 13-3)
   a. $f(3)$  b. $\frac{f(5)}{f(4)}$  c. $\frac{f(n+1)}{f(n)}$

23. Which is larger, $100!$ or $100^{100}$? (Lesson 13-3)

24. Five candidates, Jerry, Kerry, Larry, Mary, and Perry, are to give
    speeches at an election assembly. In how many different orders could
    the candidates speak? (Lesson 13-3)

In 25–27, find an equal expression of the form $ax^n$. (Lessons 8-1, 8-4)

25. $(2x)^3$  
26. $x^5(x^2)^4$  
27. $9x^{-2} \cdot (9x)^{-2}$

28. Zeno (see Question 11) is also known for a paradox called the "arrow
    paradox." Using other books, find out what this paradox is.
Very often an idea from one part of mathematics has applications to another part of mathematics. The triangular array below is such an idea. (The top row of the array is called row 0 because this is convenient in applications of the array.)

The triangular array below was known in ancient India. It was rediscovered and discussed in 1544 by Michael Stifel, a German mathematician. But the array is known as Pascal’s triangle, named after Blaise Pascal (1623–1662), the French mathematician and philosopher who discovered many properties relating the elements (numbers) in the array. Pascal himself called it the Triangle Arithmetique, literally the “arithmetical triangle.”

```
Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
```

The dots indicate that the array goes on without end.
Pascal’s triangle is formed in a very simple way. You can think of Pascal’s triangle as a two-dimensional sequence. Each element is determined by a row and its position in that row. The only element in the top row (row 0) is 1. The first and last elements of every other row are also 1. If \( x \) and \( y \) are located next to each other on a row, the element just below and directly between them is \( x + y \), as illustrated below.

\[
\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
\leftarrow \text{row 4}
\]

\[
\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(1 + 4) & (4 + 6) & (6 + 4) & (4 + 1) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
\leftarrow \text{row 5}
\]

This is also shown by the arrows between rows 4 and 5. All other elements in the array follow the same patterns.

There is a standard symbol used to represent elements in Pascal’s triangle.

**Definition:**

The \((r + 1)\text{st}\) element in row \(n\) of Pascal’s triangle is denoted by \(\binom{n}{r}\).

For instance, the 1st element in the 7th row is \(\binom{7}{0}\). The entire 7th row consists of the following elements.

\[
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}
\]

That is, \(\binom{7}{0} = 1\), \(\binom{7}{1} = 7\), \(\binom{7}{2} = 21\), and so forth. We found these values by referring to row 7 as it was given on the previous page. The top row has one element, \(\binom{0}{0} = 1\).

The method of construction of Pascal’s triangle lends itself to a recursive definition of the triangle using the above symbol. Since the triangle is a sequence in two directions—down and across—the recursive rule involves two variables.
Definition:

Pascal's triangle is the sequence satisfying

\[
\binom{n}{0} = \binom{n}{n} = 1
\]

and

\[
\binom{n + 1}{r + 1} = \binom{n}{r} + \binom{n}{r + 1},
\]

where \( n \) and \( r \) are any integers with \( 0 \leq r \leq n \).

Part (1) of the definition gives the "sides" of the triangle. Part (2) is a symbolic way of stating that adding two adjacent elements in one row gives the element below them in the next row.

In order to determine the elements in the 14th row of the triangle from the definition, you would have to construct the first 13 rows. The next theorem was first proved by Isaac Newton and shows how to calculate \( \binom{n}{r} \) without constructing the triangle. The theorem is a surprising application of factorials and requires that 0! be defined to equal 1.

Theorem:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

Before proving this theorem, here are some instances of it.

Example 1 Calculate \( \binom{4}{2} \).

Solution Here \( n = 4 \) and \( r = 2 \). So \( n - r = 2 \) also.

\[
\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1(2 \cdot 1)} = 6
\]

Check This agrees with \( \binom{4}{2} \) being the 3rd element in the 4th row of Pascal's triangle.
Example 2 Calculate \( \binom{11}{3} \).

**Solution**

\[
\binom{11}{3} = \frac{11!}{3!(11-3)!}
\]

\[
= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 165
\]

**Check** Use a calculator. \([11! \div (3! \times 8!)] = \) yields 165.

Notice how easily the fraction of Example 2 can be simplified because of the common factors.

In Example 3, the equality \(0! = 1\) must be used.

Example 3 Calculate \( \binom{7}{0} \).

**Solution**

\[
\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{7!}{1 \cdot 7!} = 1
\]

**Check** This agrees with \( \binom{7}{0} \) being the 1st element in the 7th row.

For a proof of the theorem, \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \), it is enough to show that the factorial formula \( \frac{n!}{r!(n-r)!} \) satisfies the relationships involving \( \binom{n}{r} \) which define Pascal’s triangle. Does the formula for \( \binom{n}{0} \) equal the formula for \( \binom{n}{n} \) and equal 1?

\[
\binom{n}{0} = 1 \quad \text{and} \quad \frac{n!}{0!(n-0)!} = \frac{n!}{0! \cdot n!} = \frac{n!}{1 \cdot n!} = 1, \quad \text{so} \quad \binom{n}{0} = \frac{n!}{0!(n-0)!}.
\]

\[
\binom{n}{n} = 1 \quad \text{and} \quad \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1, \quad \text{so} \quad \binom{n}{n} = \frac{n!}{n!(n-n)!}.
\]

Thus the formula works for the “sides” of Pascal’s triangle.

To prove that the formula for \( \binom{n+1}{r+1} \) is the sum of the formulas for \( \binom{n}{r} \) and \( \binom{n}{r+1} \) requires substantial algebraic manipulation; it is omitted here.

Important applications of Pascal’s triangle are given in the remaining lessons of this chapter.
Questions

1. Write down rows 0 through 7 of Pascal’s triangle from memory.

2. What are the rules by which Pascal’s triangle is defined?

3. Write down rows 8 through 10 of Pascal’s triangle. (It is a good idea to keep rows 0 through 10 handy for reference.)

4. a. The symbol \( \binom{n}{r} \) denotes what element in which row of Pascal’s triangle?

   b. In terms of factorials, \( \binom{n}{r} = \_\_ \).

In 5–12, calculate.

5. \( \binom{8}{2} \)  
6. \( \binom{3}{2} \)  
7. \( \binom{10}{5} \)  
8. \( 0! \)

9. \( \binom{6}{6} \)  
10. \( \binom{15}{0} \)  
11. \( \binom{15}{14} \)  
12. \( \binom{20}{2} \)

13. If \( 10 \cdot 9! = x! \), then \( x = \_\_ \).

14. If \( (n - r)(n - r - 1)! = y! \), then \( y = \_\_ \).

In 15 and 16, true or false.

15. \( \binom{99}{17} \) is an integer.

16. \( \frac{n!}{(n - 2)!} \) is always an integer when \( n \geq 2 \).

17. If \( \binom{10}{5} + \binom{10}{6} = \binom{x}{y} \), then \( x = \_\_ \) and \( y = \_\_ \).

18. If \( \binom{9}{2} + \binom{a}{b} = \binom{10}{2} \), then \( a = \_\_ \) and \( b = \_\_ \).

In 19–21, tell where in Pascal’s triangle the following sequence can be found.

19. the positive integers

20. the triangular numbers: 1, 3, 6, 10, 15, ...

21. the sequence of partial sums of triangular numbers: 1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10, ...

22. a. Write the repeating decimal 0.297297297297 ... as an infinite geometric series.

   b. Find the fraction in lowest terms equal to this decimal. (Lesson 13-4)
23. Consider the squares below whose sides form a geometric sequence.

\[
\begin{array}{ccccccccc}
& & & & & & & & 4 \\
& & & & & & 3 & & \\
& & & & & 9 & & & \\
& & & & & & & & 4 \\
& & & & & & & & D \\
& & & & & & & & D \\
& & & & & & & & D \\
& & & & & & & & D \\
& & & & & & & & D \\
& & & & & & & & D \\
\end{array}
\]

a. What is the sum of the areas of the nine squares pictured?
b. What is the sum of the areas of this infinite geometric sequence of squares? (Lessons 13-2, 13-4)

24. Very young children often play with rods whose lengths form the arithmetic sequence \( x, 2x, 3x, \ldots, 10x \).

a. What is the sum of the lengths of these rods? (Lesson 13-1)
b. If you wanted to make a set of such rods for a young relative out of a piece of wood 6 feet long, how long could you make the shortest rod? (Lesson 1-7)

25. Evaluate \( \sum_{n=1}^{4} (n^2 + 3n) \). (Lesson 13-3)

26. The sum of the first \( n \) terms of the arithmetic sequence \( 1, 5, 9, 13, \ldots \) is 2415. What is \( n \)? (Lesson 13-1)

27. In the quadratic equation \( ax^2 + bx + c = 0 \), when \( a, b, \) and \( c \) are integers, state what you can conclude from the following.

a. \( b^2 - 4ac = 0 \)  
b. \( b^2 - 4ac = -8 \)  
c. \( b^2 - 4ac = 16 \)  
(Lesson 6-7)

28. Multiply the complex numbers \( 2i \) and \( 2 - 2i \). (Lesson 6-9)

29. In how many possible orders might six girls in a diving event finish? (Lesson 13-3)

30. There are six elements surrounding each element not on a side of Pascal’s triangle. For instance, around 15 in row 6 are the elements 5, 10, 20, 35, 21, and 6. In 1969, an amazing property about the product of these elements was discovered by Verner Hoggatt and W. Hansell of San Jose State University.

a. Find this product for all the elements not on the sides of the first five rows of the triangle.
b. What is true of these products?
c. Verify your answer to part b by calculating the product for other elements in the triangle.
You have seen powers of binomials in many places in this book. Here are a few examples, with the binomials identified in bold type.

\[ A = P(1 + r)^t \quad \text{Compound Interest Formula (Lesson 8-2)} \]
\[ y - k = a(x - h)^2 \quad \text{vertex form of the equation of a parabola (Lesson 6-4)} \]
\[ e = \left(1 + \frac{1}{n}\right)^n \quad \text{definition of } e \text{ (Lesson 9-7)} \]

The geometric sequence with first term 1 and the binomial \( a + b \) as its constant ratio generates all the positive integer powers of binomials.

\[ 1, a + b, (a + b)^2, (a + b)^3, (a + b)^4, \ldots \]

You know the binomial \( (a + b)^2 = a^2 + 2ab + b^2 \). It is natural to try to expand the other powers of binomials. The results are known as binomial expansions.

\[
\begin{align*}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
&\vdots
\end{align*}
\]

Looking at the expansions, it may seem that there is no pattern. But if the \( a \)s and \( b \)s are ignored, and only the coefficients are written, then Pascal’s triangle appears!

<table>
<thead>
<tr>
<th>Powers of ((a + b))</th>
<th>Coefficients of the expansion</th>
<th>Row of Pascal’s triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a + b)^0)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>((a + b)^1)</td>
<td>1 1</td>
<td>1</td>
</tr>
<tr>
<td>((a + b)^2)</td>
<td>1 2 1</td>
<td>2</td>
</tr>
<tr>
<td>((a + b)^3)</td>
<td>1 3 3 1</td>
<td>3</td>
</tr>
<tr>
<td>((a + b)^4)</td>
<td>1 4 6 4 1</td>
<td>4</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots \vdots</td>
<td>\vdots \vdots</td>
</tr>
</tbody>
</table>
As a consequence, the expansions can be written using the \( \binom{n}{r} \) symbolism.

\[
(a + b)^0 = \binom{0}{0} \\
(a + b)^1 = \binom{1}{0}a + \binom{1}{1}b \\
(a + b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 \\
(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \\
(a + b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4 \\
\vdots
\]

Notice how easy Pascal’s Triangle makes the expansion of \((a + b)^n\):

1. All the powers of \(a\) from \(a^n\) to \(a^0\) occur in order.
2. In each term, the exponents of \(a\) and \(b\) add to \(n\).
3. If the power of \(b\) is \(r\), then the coefficient of the term
   \[\binom{n}{r}\] is

This information is summarized in a famous theorem, which was known to Omar Khayyam, the famous Persian poet, mathematician, and astronomer who died around the year 1123. Of course, he did not have our modern notation.

### Binomial Theorem:

\[
(a + b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r.
\]

A formal proof of the Binomial Theorem requires a knowledge of mathematical induction, a powerful proof technique not discussed in this book, and first used by Pascal when he discussed the array.

**Example 1** Expand \((a + b)^7\).

**Solution** First, fill in powers of \(a\) and \(b\).

\[
(a + b)^7 = \underline{a^7} + \underline{a^6b} + \underline{a^5b^2} + \underline{a^4b^3} + \underline{a^3b^4} + \\
\underline{a^2b^5} + \underline{ab^6} + \underline{b^7} \\
\]
Second, put in the coefficients.

\[ (a + b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7 \]

Finally, evaluate the coefficients, either by referring to row 7 of Pascal’s Triangle or by using the formula \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

\[ (a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \]

The Binomial Theorem can be used to expand powers of any binomial by substituting for \( a \) and \( b \).

---

**Example 2** Expand \((5x - 2y)^3\).

**Solution** Use \((a + b)^3\) with \(5x\) as \(a\) and \(-2y\) as \(b\).

\[ (a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \]

Substituting,

\[ (5x - 2y)^3 = 1(5x)^3 + 3(5x)^2(-2y) + 3(5x)(-2y)^2 + 1(-2y)^3 \]
\[ = 125x^3 - 150x^2y + 60xy^2 - 8y^3 \]

**Check** Substitute specific values for \(x\) and \(y\). Let \(x = 2\) and \(y = 3\). Then the given expression \((5x - 2y)^3 = (10 - 6)^3 = 64\).

The value of the expanded form is:

\[ 125x^3 - 150x^2y + 60xy^2 - 8y^3 = 125\cdot8 - 150\cdot4\cdot3 + 60\cdot2\cdot9 - 8\cdot27 \]
\[ = 1000 - 1800 + 1080 - 216 \]
\[ = 64. \text{ It checks.} \]

---

**Example 3** Expand \((x^2 + 1)^4\).

**Solution** Think of \(x^2\) as \(a\) and 1 as \(b\) and follow the form of \((a + b)^4\).

\[ (x^2 + 1)^4 = \binom{4}{0}(x^2)^4 + \binom{4}{1}(x^2)^3 \cdot 1 + \binom{4}{2}(x^2)^2 \cdot 1^2 + \binom{4}{3}(x^2)^1 \cdot 1^3 + \binom{4}{4} \cdot 1^4 \]
\[ = x^8 + 4x^6 + 6x^4 + 4x^2 + 1 \]

**Check** Let \(x = 2\). Then \((x^2 + 1)^4 = (4 + 1)^4 = 5^4 = 625\). You should verify that the value of the polynomial when \(x = 2\) is also 625. Also note as a check that the exponents of each variable in each expansion form an arithmetic sequence.
Due to their use in the binomial theorem, the numbers in Pascal’s triangle are known as **binomial coefficients**. The binomial theorem has a surprising number of applications in estimations, counting problems, probability, and statistics.

**Questions**

In 1–4, expand each binomial power.

1. \((a + b)^2\)  
2. \((a + b)^3\)  
3. \((a + b)^4\)  
4. \((a + b)^5\)

5. State the Binomial Theorem.

In 6–11, expand each binomial power.

6. \((x + 1)^5\)  
7. \((a - b)^3\)  
8. \((2 - m)^4\)  
9. \((x + y)^6\)

10. \((8x + y)^3\)  
11. \((a + 2b)^4\)

12. Multiply the binomial expansion for \((a + b)^4\) by \(a + b\) to check the expansion for \((a + b)^5\).

In 13 and 14, convert to an expression in the form \((a + b)^n\).

13. \(\sum_{r=0}^{n} \binom{n}{r} x^{n-r}3^r\)  
14. \(\sum_{i=0}^{n} \binom{n}{i} y^{n-i}(2a)^i\)

15. a. Multiply and simplify:
   \((a^2 + 2ab + b^2)(a^2 + 2ab + b^2)\)
   
b. Your answer to part a should be a power of \(a + b\). Which one? Why?

In 16 and 17, use the Binomial Theorem to approximate some powers quickly. Here is an example.

\[(1.002)^3 = (1 + .002)^3\]
\[= 1^3 + 3 \cdot 1^2 \cdot (.002) + 3 \cdot 1 \cdot (.002)^2 + (.002)^3\]
\[= 1 + .006 + .000012 + .000000008\]
\[= 1.006012008\]

Since the last two terms in the expansion are so small, they might be ignored in an estimate. \((1.002)^3 \approx 1.006\) to the nearest thousandth.

16. Estimate \((1.004)^3\) to the nearest thousandth. Check your answer with a calculator.

17. Estimate \((1.001)^{10}\) correct to fifteen decimal places.

18. Write row 9 of Pascal’s Triangle. *(Lesson 13-5)*
19. Consider the infinite geometric sequence 
64, 48, 36, 27, ...

a. Find the sum of the first six terms of the sequence.
b. Find the sum of all terms of the sequence.
c. Multiple choice Which represents the sum in part b? 
   (Lessons 13-2, 13-3, 13-4)

\[
\begin{align*}
(i) & \sum_{i=1}^{\infty} 64 \cdot \left(\frac{3}{4}\right)^i \\
(ii) & \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i \\
(iii) & \sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{i-1} \\
(iv) & \sum_{i=1}^{\infty} 64 \left(\frac{3}{4}\right)^{i-1}
\end{align*}
\]

In 20–22, suppose you are offered two jobs. The first pays $10,000 the first year, with an annual increase of $1,500 each year thereafter. The second pays $6,000 the first year, with a 25% increase (compounded annually) each year thereafter.

20. List your projected annual salary for the first three years for each of the jobs.

21. Find the first year in which the salary for the second job exceeds that of the first job.

22. If you plan to stay in one of these jobs for 10 years, which will give the largest total salary? Justify your answer. (Lessons 3-6, 8-3, 13-1, 13-2)

23. If the sum of the first \( n \) terms of the geometric sequence 
\(-2, 4, -8, 16, \ldots\) is \(-86\), what is \( n \)?

24. If the measurements of a single brick are \(3\frac{1}{2}'' \times 7\frac{3}{4}'' \times 2\frac{1}{4}''\), what is its volume, to the nearest cubic inch? (Previous course)

25. A pile of bricks is 10 bricks high, 6 bricks deep, and 15 bricks wide. If a single brick weighs between 4 and \(4\frac{1}{4}\) lb, what is the largest possible weight of the pile? (Previous course)

26. Multiple choice Which polynomial equals \(\frac{(n + 2)!}{n!}\)? (Lesson 13-3)
   (a) \(n + 1\)  (b) \(n + 2\)  (c) \((n + 1)(n + 2)\)  (d) none of these

27. a. How many permutations of the letters of the word MOUSE are possible?
   b. List the first and last in dictionary order. (Lesson 13-3)

28. Calculate \(11^n\) for \(n = 0, 1, 2, 3, 4, 5, \) and \(6\). Explain how you can obtain all these powers from the binomial coefficients.
A subcommittee of 3 people is to be chosen from the 10-person committee pictured below. In how many ways can this be done?

To answer this question, represent the full committee by the set \( \{A, B, C, D, E, F, G, H, I, J\} \). Each set of three members from this committee is a subset of the original set. For instance, two possible subcommittees are Alan, Barbara, and Carlos, and Frank, Delphine, and Joe. These subcommittees can be represented as the subsets \( \{A, B, C\} \) and \( \{F, D, J\} \). Order in sets makes no difference; \( \{F, D, J\} \) and \( \{D, F, J\} \) are the same subset. Thus the question can be viewed as a problem in counting subsets: How many subsets of 3 elements are possible from a set of 10 elements?

Think of forming the subsets one element at a time. There are 10 possibilities for the first element. Once the first element has been selected, there are 9 possibilities for the second element. Once the first two elements have been chosen, there are 8 possibilities for the third element. So it seems there are \( 10 \cdot 9 \cdot 8 \) possibilities. This assumes that the order in which the elements are chosen makes a difference. But \( \{F, D, J\} \) and \( \{D, F, J\} \) are the same subset, and in fact there are \( 3! \) or 6 different orders which give rise to the same subset \( \{F, D, J\} \). This is true of all 3-element subsets, so the answer \( 10 \cdot 9 \cdot 8 \) is \( 3! \) times what we need. The number of subsets is thus

\[
\frac{10 \cdot 9 \cdot 8}{3!}.
\]

Now multiply the fraction by \( \frac{7!}{7!} \). This multiplier equals 1, so it does not change the fraction's value. But it does change the way the fraction looks.

\[
\frac{10 \cdot 9 \cdot 8}{3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = \frac{10!}{3! \cdot 7!}
\]
The answer is a binomial coefficient! Evaluating directly, or looking for the 4th element in the 10th row of Pascal’s triangle, the answer is seen to be 120.

The following theorem and its proof generalize the above argument.

**Theorem:**

The number of subsets of \( r \) elements which can be formed from a set of \( n \) elements is \( \frac{n!}{r!(n-r)!} \), the binomial coefficient \( \binom{n}{r} \).

**Proof**

There are \( n \) choices for the first element in a subset. Once that element has been picked, there are \( n - 1 \) choices for the second element. This continues until all \( r \) elements have been picked. There are \( (n - r + 1) \) choices for the \( r \)th element. So, if different orders are considered different, there are

\[
\frac{n(n-1)(n-2) \ldots (n-r+1)}{r!}
\]

ways to pick them. But each subset is one of \( r! \) subsets with the same elements. So the number of different subsets is

\[
\frac{n(n-1)(n-2) \ldots (n-r+1)}{r!}
\]

Multiplying both numerator and denominator by \( (n-r)! \), the theorem results.

Here are several applications.

**Example 1** Five points are labeled in a plane, with no three collinear. How many triangles have these points as vertices?

**Solution 1** Draw a picture. Label the points \( A, B, C, D, \) and \( E \). Form triangles with the points as vertices. The possible triangles are (in alphabetical order) \( ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, \) and \( CDE \). So 10 triangles can be formed.
Solution 2  Any choice of 3 points from the 5 points determines a triangle. So use the theorem with \( n = 5 \) and \( r = 3 \). The number of possible triangles is \( \binom{5}{3} = \frac{5!}{3!(5 - 3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10 \).

Solution 3  Use the idea of the proof of the theorem. The first vertex of the triangle can be chosen in 5 ways. The second vertex can then be chosen in 4 ways. And the third vertex then can be chosen in 3 ways. So, if order made a difference, there would be \( 5 \cdot 4 \cdot 3 \), or 60 different triangles. But order doesn’t make a difference and each triangle is determined \( 3! \) or 6 times. So divide 60 by 6, yielding 10.

Any choice of \( r \) objects from \( n \) objects is called a combination. The theorem of this lesson can be restated: The number of combinations of \( r \) objects from \( n \) objects is \( \binom{n}{r} \). Example 1 shows that the number of combinations of 3 objects from 5 objects is 10; in some books, you may see the symbol \( _nC_r \). Like \( \binom{n}{r} \), it stands for the number of combinations of \( r \) objects from \( n \) objects; \( _5C_3 = \binom{5}{3} = 10 \). The subcommittee situation at the beginning of this lesson shows that \( _{10}C_3 = \binom{10}{3} = 120 \).

Example 2  There are 30 items on a menu in a Vietnamese restaurant. A group of friends plans to order 5 items. In how many ways can this be done?

Solution  Choosing 5 items from 30 items on a menu is equivalent to choosing subsets from a whole.

\[
_{30}C_5 = \binom{30}{5} = \frac{30!}{5!25!}. \text{ To evaluate } \frac{30!}{5!25!} \text{ with a calculator, press } 30 \left[ \begin{array}{c} n! \end{array} \right] \pm \left( \begin{array}{c} 25 \end{array} \right) \left[ \begin{array}{c} n! \end{array} \right] \times 5 \left[ \begin{array}{c} n! \end{array} \right] = 142,506. \text{ Without a calculator, work as follows: } \frac{30!}{5!25!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 25!}. \text{ Either way, you should get 142,506.}
\]

There are 142,506 ways to select 5 items from 30.
Combinations enable you to determine the number of subsets of a particular size. You may wonder how many subsets there are in all. The answer is simple and surprising. For instance, consider the subsets of \{A, B, C, D\}.

\{ \} 
\{A\}, \{B\}, \{C\}, \{D\}  
\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}  
\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}  
\{A, B, C, D\}

1 subset has 0 elements.  
4 subsets have 1 element.  
6 subsets have 2 elements.  
4 subsets have 3 elements.  
1 subset has 4 elements.

The numbers from Pascal’s Triangle appear again! The total number of subsets is 1 + 4 + 6 + 4 + 1, or 16. In general, the total number of subsets of a set with \(n\) elements is the sum of the elements in the \(n\)th row of Pascal’s Triangle.

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}
\]

Multiply by powers of 1 to make the sum look like the Binomial Theorem.

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}1^{n-i}1^{i} \\
= (1 + 1)^n \\
= 2^n
\]

**Theorem:**

A set with \(n\) elements has \(2^n\) subsets.

When \(n = 4\), \(2^n = 16\). This agrees with the number of subsets found above for a set with 4 elements.

**Questions**

1. **Multiple choice** Which is not a subset of \(\{T, E, A, M\}\)?
   (a) \(\{M, E, A, T\}\)  
   (b) \(\{\}\)  
   (c) \(\{A, M\}\)  
   (d) \(\{T, E, A, M, S\}\)

2. a. How many subsets of \(\{T, E, A, M\}\) have 3 elements?
   b. List them.
In 3–5, consider the set \{p, q, r\}.

3. List all the subsets of \{p, q, r\}.

4. How many subsets are there with the indicated number of elements?
   a. 0           b. 1           c. 2           d. 3

5. How many subsets does \{p, q, r\} have?

6. Any choice of \(r\) objects from \(n\) objects is called \(\binom{n}{r}\). 

7. The symbol \(\binom{n}{r}\) is another way of writing \(\frac{n!}{r!(n-r)!}\).

8. How many subcommittees of 4 people can be formed from a committee of 10?

9. Ten points are in a plane, with no three collinear. How many triangles have these points as vertices?

10. a. How many triangles have three given points as vertices?
     b. Does this agree with the formula for \(\binom{n}{r}\)?

11. In how many ways can 6 dishes be chosen from a menu with 25 options?

12. a. What is the sum of the entries in row 7 of Pascal’s Triangle?
     b. What does that have to do with this lesson?

13. A set with 8 elements has how many subsets?

14. Simplify:
   \(9C_0 + 9C_1 + 9C_2 + 9C_3 + 9C_4 + 9C_5 + 9C_6 + 9C_7 + 9C_8 + 9C_9\).

15. Copy and complete this pattern.

\[
\begin{array}{c|c}
1 & n \\
2 & n - 1 \\
3 & n - 2 \\
4 & \frac{?}{?} \\
7 & \frac{?}{?} \\
r & \frac{?}{?} \\
\end{array}
\]

16. Simplify: \(n \cdot (n - 1) \cdot \ldots \cdot (n - r + 1) \cdot ((n - r)!)\).

17. a. Suppose you pick one card from a 52-card playing deck. In how many ways can this be done?
     b. Suppose you pick two cards from a 52-card playing deck. In how many ways can this be done?
     c. If you pick 4 cards from the deck, what are the chances of selecting the 4 aces?

In 18–20, recall that the U.S. Congress consists of 100 senators and 435 representatives.

18. How many five-person Senatorial Committees are possible?
19. In how many ways can a "committee of the whole" be chosen in the House of Representatives?

20. What is the total number of committees of any size (except the empty set) which can be formed in the Congress?

In 21 and 22, expand. (*Lesson 13-6*)

21. \((a + b)^8\)  
22. \((2x - 3y)^3\)

In 23 and 24, use the sequence of rectangles below. The largest rectangle has base 8 and height 6. Each subsequent rectangle is a size change image of the next larger with a magnitude of \(\frac{1}{2}\).

23. What is the sum of the perimeters of the five largest rectangles?  (*Lesson 13-2*)

24. What is the sum of the perimeters of all the rectangles (assuming the rectangles could be shrunk forever)? (*Lesson 13-4*)

25. A runner, training for a marathon, runs 10 miles one day and then one more mile each day until 20 miles are run in a day. What is the total number of miles run? (*Lesson 13-1*)


\[
\begin{align*}
(a) & \quad \sum_{n=1}^{10} (n + 9) \\
(b) & \quad \sum_{n=10}^{20} n \\
(c) & \quad \sum_{n=1}^{11} (n + 9) \\
(d) & \quad \sum_{n=0}^{10} (10 + n)
\end{align*}
\]

27. *Multiple choice*  Which does not equal the number 2? (*Lesson 9-5*)

\[
\begin{align*}
(a) & \quad \log 100 \\
(b) & \quad \log_2 4 \\
(c) & \quad \log_3 6 \\
(d) & \quad \log_4 16
\end{align*}
\]

28. a. Graph the ellipse with equation \(\frac{x^2}{9} + \frac{y^2}{4} = 1\).  
   b. Where are its foci? (*Lesson 12-4*)

29. In this lesson, the sum of the elements in the \(n\)th row of Pascal’s Triangle was found to be \(2^n\). Find one or more of these expressions:  
   a. the sum of the squares of all the elements in the \(n\)th row of the triangle  
   b. the total in the \(n\)th row when alternate minus and plus signs are put between the elements of the row  
   c. the third element from the end of the \(n\)th row (i.e., \(1 + 3 + 6 + 10 + \ldots\)).
You have seen applications of Pascal’s Triangle to powers of binomials where the elements are called *binomial coefficients*, and to counting subsets where the elements are called *combinations*. Still another important application is to probability, and the elements are called *events*.

Suppose a hat contains 25 slips of paper numbered 1 through 25. As an experiment, you pick a slip blindfolded. The probability of getting the slip numbered 17 is \( \frac{1}{25} \). This means that if the experiment is repeated many times, over the long run you could expect the event ‘‘getting the 17’’ to occur about \( \frac{1}{25} \) of the time. The probability of getting an even-numbered slip is \( \frac{12}{25} \) because there are 12 even-numbered slips.

**Definition:**

If a situation has a total of \( t \) equally likely possibilities and \( e \) of these possibilities satisfy conditions for a particular event, then the

\[
\text{probability of the event} = \frac{e}{t}.
\]

Some probabilities are calculated using combinations.

**Example 1** You take a 5-question true-false test on a subject you know nothing about, so you guess. If a question is as likely to be true as false, and you need 75% correct to pass, what is your probability of passing?
Solution  The equally likely possibilities are the ways you could answer the questions, \( R \) (for right) and \( W \) (for wrong). For instance, \( RRRRW \) would mean that you had the first 4 questions right and the last one wrong. There are 32 possibilities, written here in alphabetical order.

\[
\begin{align*}
RRRR & \quad RWRR & \quad WRRR & \quad WWRR \\
RRRW & \quad RWRR & \quad WRRR & \quad WWRR \\
RRWR & \quad RWWR & \quad WRRW & \quad WWWR \\
RRWW & \quad RWWR & \quad WRWR & \quad WWWR \\
RRWR & \quad RWWW & \quad WRWW & \quad WWWR \\
RRWW & \quad RWWW & \quad WRWW & \quad WWWW
\end{align*}
\]

Counting gives the following information.

<table>
<thead>
<tr>
<th>Number correct</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 possibility has 0 Rs</td>
<td>( \frac{1}{32} )</td>
</tr>
<tr>
<td>5 possibilities have 1 R</td>
<td>( \frac{5}{32} )</td>
</tr>
<tr>
<td>10 possibilities have 2 Rs</td>
<td>( \frac{10}{32} )</td>
</tr>
<tr>
<td>10 possibilities have 3 Rs</td>
<td>( \frac{10}{32} )</td>
</tr>
<tr>
<td>5 possibilities have 4 Rs</td>
<td>( \frac{5}{32} )</td>
</tr>
<tr>
<td>1 possibility has 5 Rs</td>
<td>( \frac{1}{32} )</td>
</tr>
</tbody>
</table>

In order to pass you must get either 4 or 5 questions right. There are 6 possibilities with either 4 Rs or 5 Rs. Since there are 32 possibilities in all, the probability of passing is \( \frac{6}{32} \), which is about 0.19 or 19%.

In a True-False test, the same situation—choosing \( T \) or \( F \)—occurs again and again. Each such choosing is called a trial. The numbers in Pascal’s triangle always occur when calculating probabilities involving repeated trials.

**Theorem:**

If a situation consists of \( n \) trials with the same two equally likely outcomes for each trial, then the probability of one of these outcomes occurring exactly \( r \) times is

\[
\frac{n \choose r}{2^n}
\]
Proof

Think of the \( n \) trials as slots into which a \( T \) or \( F \) can be put.

\[
\begin{array}{cccccc}
T & T & T & T & T & T \\
F & F & F & F & \ldots & F \\
\hline
\end{array}
\]

\( n \) trials

Selecting the \( r \) slots where a \( T \) is placed is selecting a combination of \( r \) elements from \( n \). It can be done in \( \binom{n}{r} \) ways. The total number of possible selections equals the total number of subsets, or \( 2^n \).

From the definition of probability, the theorem follows.

A **fair** or **unbiased** coin is one with an equal probability of landing on either side. When each trial is the tossing of a fair coin, \( \binom{n}{r} \) is the probability of getting \( r \) heads in \( n \) tosses.

---

**Example 2** What is the probability of getting 2 heads in 4 tosses of a fair coin?

**Solution 1** If the coin is tossed 4 times, there are 16 possible ways of getting \( H \) (heads) and \( T \) (tails).

\[
\begin{array}{cccc}
HHHH & HTHH & THHH & TTHH \\
HHHT & HTHT & THHT & THTH \\
HHTH & HHTH & THTH & TTTT \\
HHTT & HTTT & THTH & THTH \\
\end{array}
\]

Of these, \( \binom{4}{2} \), or 6, have exactly 2 \( H \)s. They are circled. So \( \frac{6}{16} = \frac{3}{8} \) is the probability of obtaining 2 heads in 4 tosses of a fair coin.

**Solution 2** Use the theorem. The probability of getting exactly

\[
\frac{2 \text{ heads in 4 tosses of a fair coin}}{2^4} = \frac{\binom{4}{2}}{16} = \frac{6}{16} = \frac{3}{8}, \text{ or } 37.5%.
\]

Many people think the answer to Example 2 is \( \frac{1}{2} \). Until the beginnings of the theory of probability were put forth by Pascal and Pierre Fermat in the 17th century, these questions were quite difficult to answer.
Questions

In 1–4, slips of paper numbered 1 to 20 are tossed in a hat. One slip is picked out of the hat at random. Give the probability that its number is:

1. 15
2. even
3. odd
4. prime.

In 5 and 6, a student is given the following test.

Question 1: In 1950, which city had the larger population, Boston or Washington D.C.?
Question 2: Was Joan of Arc born before or after 1400?
Question 3: Which city is farther north, Havana, Cuba, or Mexico City?
Question 4: Who died first, Pascal or Fermat?

5. List all possible ways the test might be corrected. (Assume that all questions are answered.)

6. Assuming that the student guesses on each item, calculate each probability.
a. The student gets all 4 correct.
b. The student gets exactly 3 correct.
c. The student gets exactly 2 correct.
d. The student gets exactly 1 correct.
e. The student gets none correct.

In 7–9, a fair coin is tossed 6 times. Give the probability of each event.

7. exactly 3 heads
8. exactly 2 heads
9. 6 heads

10. A fair coin is tossed \( n \) times. State
a. the number of ways \( r \) tails may occur.
b. the total number of ways the coin may fall.
c. the probability of \( r \) tails.
11. Give the probabilities of getting 0, 1, 2, ..., 8 heads in 8 tosses of a fair coin.

12. On a 10 question true-false test, what is the probability you will get 8 or more correct
   a. if you guess on all 10 questions?
   b. if you know 4 items and guess on 6 others?

In 13 and 14, Gregor Mendel founded the field of genetics in the 19th century by studying the crossbreeding of peas and other plants. Mendel noted that some peas were smooth and others wrinkled. To explain the occurrence of this characteristic, Mendel assumed that each parent pea contributes one gene to its offspring and that determines the type of pea produced. The gene determining smoothness is represented as $S$, and the one for wrinkled is $W$.

<table>
<thead>
<tr>
<th>Genes</th>
<th>both $S$</th>
<th>1$S$, 1$W$</th>
<th>both $W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of pea</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

13. The crossbreeding of plants of pure stock in the first generation is shown below.

<table>
<thead>
<tr>
<th>gene from 2nd plant</th>
<th>W</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>gene from 1st plant</td>
<td>$S$</td>
<td>SW</td>
</tr>
<tr>
<td></td>
<td>$S$</td>
<td>SW</td>
</tr>
</tbody>
</table>

The entries in the body of the table show that all peas in the second generation have one $S$ gene and one $W$ gene. In a second-generation cross of pure smooth and pure wrinkled peas, state the probability that the offspring will appear
   a. smooth
   b. wrinkled

14. Suppose two second generation peas are crossed.
   a. Complete the table below showing the possible genetic makeup of the third generation.

<table>
<thead>
<tr>
<th>gene from 2nd plant</th>
<th>S</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>gene from 1st plant</td>
<td>S</td>
<td>W</td>
</tr>
</tbody>
</table>

   b. State the probability that plants in the third-generation will appear
      (i) smooth
      (ii) wrinkled.
15. Expand \((p + q)^9\). (Lesson 13-6)

16. How many bridge foursomes can be formed from eight people? (Lesson 13-7)

\[ \sum_{i=1}^{5} a_i \] if \( a_1 = 3 \), \( a_2 = 8 \), \( a_3 = 9 \), \( a_4 = -7 \), and \( a_5 = 2 \). (Lesson 13-3)

17. Evaluate \( \sum_{i=1}^{5} a_i \) if \( a_1 = 3 \), \( a_2 = 8 \), \( a_3 = 9 \), \( a_4 = -7 \), and \( a_5 = 2 \). (Lesson 13-3)

18. Four positive even integers form an increasing arithmetic sequence. The sum of the integers is 100.
   a. Give the mean of the numbers.
   b. True or false  The four numbers can be determined exactly. (Lesson 13-1)

19. A geometric series has 5 terms, the constant ratio is \( \frac{3}{5} \), and the sum is 1441. What is its first term? (Lesson 13-2)

20. If \( \log n = 5 \), what is the value of \( \log(n^2) \)? (Lesson 9-6)

In 21–23, as shown below, three tennis balls with diameter 2.5" fill a cylindrical can with the same diameter.

21. Find the surface area of one ball.

22. Find the internal surface area of the can. (Include the top and bottom.)

23. What percent of the volume of the can is occupied by the balls? (Previous course, Lesson 8-1)

24. Toss 4 coins and record how many heads appear. Repeat this at least 25 times. How closely do your results agree with what would be predicted by the theorem of this lesson?
Ten top students in a school take a college entrance exam and receive the scores shown at the right. These scores make up a data set. A data set is a set in which an element may be listed more than once.

Suppose you wished to describe these scores quickly. One way is to calculate a single number which in some way describes the entire set of scores. Three common numbers used for this purpose are the mean, the median, and the mode.

<table>
<thead>
<tr>
<th>College Entrance Scores for Ten Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>750</td>
</tr>
<tr>
<td>742</td>
</tr>
<tr>
<td>736</td>
</tr>
<tr>
<td>725</td>
</tr>
<tr>
<td>725</td>
</tr>
<tr>
<td>690</td>
</tr>
<tr>
<td>662</td>
</tr>
<tr>
<td>660</td>
</tr>
<tr>
<td>650</td>
</tr>
<tr>
<td>640</td>
</tr>
</tbody>
</table>

Data Set I

For the given college entrance test scores, the mean score is $\frac{6980}{10} = 698$. The median is considered to be the mean of the two middle scores, 690 and 725. So it is 707.5. The mode is the most common score, 725.

The mean, median, and mode are called measures of central tendency because they are intended to give a number which in some sense is at the "center" of the set. Geometrically, here is how these numbers and measures look on a number line.

The mean is most often used when the terms of the sequence are fairly closely grouped, as in finding bowling averages.

The median is used when there are a few low or high terms which could greatly affect the mean, as with personal incomes.
The *mode* is particularly useful when the terms are the results of rounding, as often occurs when recording the ages of people.

These three measures of central tendency are examples of *statistical measures*. A *statistical measure* is a single number which is used to describe an entire set of numbers.

Here is a second set of numbers which might be college entrance scores of ten students in a different school.

Compare this set to the previous set of scores. Is this set of scores better?

Actually, the mean, median, and mode are identical to those in the first set. But these scores are more widely dispersed, or spread out, than the scores given earlier. One measure of dispersion is *standard deviation*.

### Definition:

Let $S$ be a data set of $n$ numbers $\{S_1, S_2, \ldots, S_n\}$. Let $m$ be the mean of $S$. Then the *standard deviation* $\text{s.d.}$ of $S$ is

$$\text{s.d.} = \sqrt{\frac{\sum_{i=1}^{n} (S_i - m)^2}{n}}$$

### Example

Calculate the standard deviation $\text{s.d.}$ of the college entrance scores of set II.

**Solution**

Use the formula. The mean $m$ was previously calculated to be 698. The number $n$ of scores is 10.

$$\text{s.d.} = \sqrt{\frac{\sum_{i=1}^{n} (S_i - m)^2}{n}} = \sqrt{\frac{\sum_{i=1}^{10} (S_i - 698)^2}{10}}$$

---

College Entrance Scores for Ten Students

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
</tr>
<tr>
<td>792</td>
</tr>
<tr>
<td>786</td>
</tr>
<tr>
<td>725</td>
</tr>
<tr>
<td>725</td>
</tr>
<tr>
<td>690</td>
</tr>
<tr>
<td>662</td>
</tr>
<tr>
<td>610</td>
</tr>
<tr>
<td>600</td>
</tr>
<tr>
<td>590</td>
</tr>
</tbody>
</table>

Data Set II

- 725 = mode
- 707.5 = median
- 698 = mean
To calculate the sum under the radical, organize your work.

\[ S_i \quad S_i - 698 \quad (S_i - 698)^2 \]

\[
\begin{array}{ccc}
800 & 102 & 10404 \\
792 & 94 & 8836 \\
786 & 88 & 7744 \\
725 & 27 & 729 \\
725 & 27 & 729 \\
690 & -8 & 64 \\
662 & -36 & 1296 \\
610 & -88 & 7744 \\
600 & -98 & 9604 \\
590 & -108 & 11664 \\
\end{array}
\]

\[
\text{Sum} = 58814
\]

Thus \( s.d. = \sqrt{\frac{\sum_{i=1}^{10} (S_i - 698)^2}{10}} = \sqrt{\frac{58814}{10}} = \sqrt{5881.4} \approx 76.7 \)

The steps in finding the standard deviation of a data set are as follows:

- **Step 1**: Calculate the mean of \( S \).
- **Step 2**: Subtract the mean from each term of \( S \).
- **Step 3**: Square these differences.
- **Step 4**: Add up the squares.
- **Step 5**: Divide the sum by \( n \) (the number of terms).
- **Step 6**: Find the square root of this quotient.

When this is done for Data Set I,

\[
s.d. = \sqrt{\frac{16014}{10}} = \sqrt{1601.4} \approx 40.0.
\]

Note that the standard deviation for Data Set II is larger than the standard deviation for Data Set I. Also, Data Set II is more widely dispersed. In general, the larger the standard deviation, the more widely dispersed are the scores. Although hard to calculate by hand, standard deviations are easily calculated by computers and are very widely used. Some calculators have special keys to calculate standard deviations.

The four measures mentioned in this section—mean, median, mode, and standard deviation—are by no means the only statistical measures in common use. There are many others.
Statistics is a large and relatively new branch of mathematics. It has many applications in the social, biological, and physical sciences. In fact, statistical methods have even been used to determine authorship of unsigned writings and to analyze languages. Although you probably know that statistics can be, and have been, used to distort information and mislead people, the wide use of statistical methods indicates the confidence which people have in statistics which are properly used and interpreted.

Questions

1. How does a data set differ from ordinary sets?

2. Give the mean, median, and mode of \{1, 2, 2, 3, 3, 4, 4, 4, 4\}.

3. Here is a set of low temperatures for an Alaskan city for a week in January: \{-14, -14, -9, 2, 3, -4, 0\}.
   a. Give the mean, median, and mode of the data.
   b. Which of these numbers seems most representative of the set?

4. Name a statistic which is not a measure of central tendency.

5. A person bowls games of 182, 127, 161, and 155.
   a. Which measure of central tendency is usually used to describe bowling scores?
   b. Give that measure for this data set.

6. In the lesson, the standard deviation of Data Set I is reported to be about 40.0. Do the calculations to verify this value, organizing your work as in the example.

In 7 and 8, calculate the mean and standard deviation of the data set.

7. \{10, 20, 30, 40, 50\}  
8. \{88, 90, 90, 90, 92\}

9. Copy and complete  The larger the standard deviation of a data set, the \_\_\_ the numbers in the set are.

10. A store has two managers who each earn $35,000 a year, six employees who earn $20,000 a year, and three employees who earn $15,000 a year. Give the mean, median, and mode of the salaries paid by the store.

11. a. Why is median income often considered a better indicator of the wealth of a community than mean income?

   b. Why is the mode income not used at all?

12. The mean of 2 scores is \(x\) and of 3 scores is \(y\). Find the mean if all the scores are considered together.
13. Calculate the mean and standard deviation of the elements in row 6 of Pascal's Triangle.

14. The mean of three numbers in a geometric sequence is 28. The first number is 12. What might the other numbers be?

15. Give an example, different from the one in the lesson, of two different data sets that have the same mean but different standard deviations.

16. Suppose a fair coin is tossed 2 times. What are the probabilities of 0, 1, and 2 heads? (Lesson 13-8)

17. You and two friends in your math class hope to be selected from the 20 students in your class to represent the school. What is the probability that if 3 students are selected from your class at random, it will be you and your friends? (Lesson 13-8)

18. How many subsets of \( \{V, I, O, L, E, T\} \) contain 4 elements? (Lesson 13-7)

19. Expand \( (x - 2y)^4 \). (Lesson 13-6)

20. Suppose \( y \) varies as \( x^4 \). If \( x \) is multiplied by 3, what is the effect on \( y \)? (Lesson 2-3)

In 21 and 22, use the graph below.

![Graph of the U.S. House of Representatives members by party from 1948 to 1988](Image)

21. The graph has an approximate line of symmetry. What is an equation for that line of symmetry and why does the graph have this property?

22. In what year did the Republicans gain the most members in the House?

23. a. The graph in Questions 21 and 22 is called misleading by some people because the y-axis begins at 100, not 0. How would beginning at 0 affect the graph?

   b. Find an example of a misleading graph and tell why it seems to be misleading.

   c. Find an example of an interesting effective graph in a newspaper or other source, and tell what makes it interesting and effective.
If a fair coin is tossed 5 times, the probabilities of getting 0, 1, 2, 3, 4, or 5 heads can be graphed. These values are given in the table in Example 1 of Lesson 13-8.

Let \( P(n) = \text{the probability of } n \text{ heads in 5 tosses of a fair coin.} \) Then \( P(n) = \frac{1}{32}\binom{5}{n} \). We call \( P \) a probability function. A probability function or probability distribution is a function which maps a set of events onto their probabilities.

If a fair coin is tossed 10 times, the possible numbers of heads are 0, 1, 2, ..., 10, so there are 11 points in the graph of the corresponding probability function. \( P(x) \), the probability of \( x \) heads, is

\[
\frac{\binom{10}{x}}{1024},
\]

as given by the theorem of Lesson 13-8. The 11 probabilities are easy to calculate because the numerators in the fractions are the numbers in the 10th row of Pascal’s triangle. That is, they are binomial coefficients.

<table>
<thead>
<tr>
<th>Number of heads</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{1}{1024} )</td>
<td>( \frac{10}{1024} )</td>
<td>( \frac{45}{1024} )</td>
<td>( \frac{120}{1024} )</td>
<td>( \frac{210}{1024} )</td>
<td>( \frac{252}{1024} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of heads</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{210}{1024} )</td>
<td>( \frac{120}{1024} )</td>
<td>( \frac{45}{1024} )</td>
<td>( \frac{10}{1024} )</td>
<td>( \frac{1}{1024} )</td>
</tr>
</tbody>
</table>

For this reason, the function graphed at the top of the next page is called a binomial probability distribution, or simply, a binomial distribution.
Examine this 11-point graph closely. The individual probabilities are all less than \( \frac{1}{4} \). Notice how unlikely it is to get no heads or 10 heads in a row (the probability for each is less than \( \frac{1}{1000} \)). Even for 9 heads in 10 tosses the probability is less than \( \frac{1}{100} \).

There are not enough points in the first (6-point) graph to see any pattern emerging. But the points of the 11-point binomial distribution could be connected by a fairly smooth curve. As the number of tosses is increased, lower rows in Pascal’s triangle will give the probabilities, and the points more closely outline a curve shaped like a bell.

Below is the bell-shaped curve in the position where its equation is simplest—symmetric to the y-axis with y-intercept \( \frac{1}{\sqrt{2\pi}} \).

The function which determines this graph is called a **normal distribution**, and the curve is called a **normal curve**. Notice that its equation, shown on the graph, involves the constants \( e \approx 2.718 \) and \( \pi \approx 3.14 \). Every normal curve is the image of the above graph under a composite of translations or scale transformations.

Normal curves are models for many natural phenomena. The graph of the correspondence below would be very close to a normal curve.

height to the nearest inch \( \rightarrow \) number of men in the U.S. with that height

The curve would have its highest point around 5’ 10” or 5’ 11”.
Normal curves are often good mathematical models for the distribution of scores on an exam. The graph below shows an actual distribution of scores on a 40-question test given by one of the authors to 209 geometry students. (It was a hard test!) A possible corresponding normal curve is shown in blue dots.

For other tests, the scores are standardized or normalized. This means that a person’s score is not the number of correct answers, but some score chosen so that the distribution of scores is a normal curve.

SAT scores were standardized so that the original mean was 500 and the standard deviation was 100. Many IQ tests are normalized so that the mean IQ is 100 and the standard deviation is 15. One advantage of normalizing scores is that you need to know no other scores to know how a person’s score compares with the scores of others.

The next graph shows percentages of scores in certain intervals of a normal distribution with mean \( m \) and standard deviation \( s \). (An equation is also given.) Each percentage gives the probability of scoring in a particular interval. Actual values for endpoints of these intervals are given below the graph for particular applications.

The graph indicates that about 34.1% of IQ’s are between 100 and 115. Thus, about 68.2% of the IQ scores are within one standard deviation of the mean. Other information may be similarly read from the graph.
Questions

Covering the Reading

1. Let \( P(n) = \left( \frac{5}{n} \right) \).  
   a. Calculate \( P(3) \) and indicate what it could represent.  
   b. What kind of function is \( P? \)

2. a. What is the domain of the function \( P: x \rightarrow \left( \frac{10}{x} \right) \) graphed in this lesson?  
   b. What is the range of this function?

3. If a fair coin is tossed 10 times, what is the probability of getting exactly 5 heads?

4. Give the simplest equation for a normal curve.

5. Give one application of normal curves.

6. a. What does it mean for scores to be standardized?  
   b. What is one advantage of doing this?

7. Approximately what percent of people score above 700 on SAT tests?

8. Approximately what percent of people have IQs below 85?

9. Approximately what percent of scores on a normal curve are within one standard deviation of the mean?

Applying the Mathematics

10. Let \( P(n) = \) the probability of \( n \) heads in 6 tosses of a fair coin. Graph \( P. \)

11. If you tossed a fair coin 10 times, about what percent of the time would you expect to get from 4 heads to 6 heads?

12. Some tests are standardized so that the mean is the grade level at which the test is taken and the standard deviation is 1 grade level. So, for students who take a test at the beginning of 10th grade, the mean is 10.0 and the standard deviation is 1.0.  
   a. On such a test taken at the beginning of grade 10, what percent of students would be expected to score below 8.0 grade level?  
   b. If a test is taken in the middle of 8th grade (grade level 8.5), what percent of students score between 7.5 and 10.5?

13. ACT scores range from 1 to 35 with a mean near 19 and a standard deviation near 6. What percent of students have an ACT score above 25?

14. Let \( y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \) Estimate \( y \) to the nearest thousandth when \( x = 1.5. \)
15. Copy and complete In a normal distribution, 0.13% of the scores lie more than 3 standard deviations away from the mean (in each direction). This implies that 1 out of \( \frac{2}{10000} \) people has an IQ over \( 2 \).

16. Find the mean, median, mode, and standard deviation of these scores. (Lesson 13-9)

\[ 83, 85, 88, 92, 92 \]

17. Beginning with an equilateral triangle with sides of length 1 unit, new triangles are formed by connecting the midpoints of the sides of previous triangles. What is the sum of the perimeters of all triangles formed? (Lesson 13-4)

In 18 and 19, evaluate. (Lesson 13-5)

18. \( \binom{15}{12} \)

19. \( \binom{n}{0} \)

20. Expand \((1 - x^2)^3\). (Lesson 13-6)

21. True or false The probability of getting exactly 25 heads in 50 tosses of a fair coin is less than \( \frac{1}{10} \). (Lesson 13-8)

In 22 and 23, solve (a) exactly; (b) to the nearest hundredth. (Lessons 6-6, 9-9)

22. \( 3^x = 10 \)

23. \( 5x^2 + 3x = 10 \)

24. a. Graph \( \{(x, y): x^2 + (y - 5)^2 = 1\} \).

b. Describe in words the graph of \( \{(x, y): x^2 + (y - 5)^2 < 1\} \). (Lessons 12-1, 12-2)

25. Together with some other students, toss 25 coins and count the number of heads, but do this at least 200 times. Let \( P(h) = \) the number of times \( h \) heads appear.

a. How close is \( P(h) \) to a normal distribution?

b. What is the mean of the distribution (the mean number of heads)?

c. Estimate the standard deviation of the distribution.
Polls and Sampling

In a large school with 510 seniors, the administration wanted to know the percent of seniors who owned cars. The principal walked into a senior homeroom and polled the 25 seniors there. Four of the students said they owned cars. Since \( \frac{4}{25} = 16\% \), the principal concluded that about 16\% of the seniors in the school owned cars. The principal made an inference based on sampling.

In sampling the population is the set of all people or events or items that could be sampled. The sample is the subset of the population actually studied. Above, the population is the set of 510 seniors. The sample is the set of 25 seniors polled by the principal.

Example 1  A company is testing light bulbs to see how long they shine before burning out. What is the sample and what is the population?

Solution  The population is the set of all light bulbs that could be tested, perhaps all the light bulbs that have been or will be made by this company. The sample is the set of light bulbs actually tested.

The principal could have polled all seniors, but perhaps there was no time. Sampling is often used to save time. However, in Example 1, sampling is absolutely necessary because testing destroys the light bulbs. Hence the manufacturer cannot sample all light bulbs.

Sampling is also necessary when the population is infinite. For instance, suppose a coin is tossed 100 times to determine whether or not it is fair. The population is the infinite set of all tosses that could be made. The sample is the set of 100 tosses actually used.

A use of sampling familiar to you is in getting ratings of television programs. Ratings are percents of households tuned to the program. The higher the rating for a program, the more a television station
can charge for advertising, so the more money the station earns. Because there are so many people who watch television, polling everyone would be too costly. So ratings companies use a sample of households, usually from 1000 to 3000 in number. The population for TV ratings is the set of all households with televisions. If 23.1% of all households sampled are tuned to a particular show, then the rating is 23.1.

The reliability of a sample depends on its being representative of the population. The only sure way to make it representative is for each element of the population to have the same probability of being selected for the sample. We then call the sample a random sample. If seniors in the school described above are assigned to homerooms according to extracurricular interests, the principal’s sample may not have been a random sample. Coin tossing is closer to random.

TV stations often want to split the ratings sample to determine whether teenagers or senior citizens or other groups are watching. (Advertisers may be aiming their products at these groups.) Random sampling may not give them enough people in each of these smaller samples. So they stratify the sample, often by age. A stratified sample is a sample in which the population has first been split into subpopulations and then, from each subpopulation, a sample is selected. A stratified random sample occurs when the smaller samples are chosen randomly from the subpopulations.

How many ways can a sample be chosen? What is the probability that a sample will have particular characteristics? How large must a sample be in order to give accurate results? The answers involve Pascal’s triangle and the normal distribution.

To see this, examine the table on page 801. This table is part of a larger table of random numbers, so called because it was constructed so that each digit from 0 to 9 has the same probability of being selected, each pair of digits from 00 to 99 has the same probability of being there, each triple of digits from 000 to 999 has the same probability of being there, and so on.

You can use this table of random numbers to simulate what the principal might find if 20% of the seniors actually owned cars. Think of each senior as being represented by a digit. To simulate the 20%, a 0 or 1 will mean that the senior owns a car. A digit of 2 through 9 means the senior does not.

To use such a table, you must start randomly as well. With your eyes closed, point to a pair of digits on the page; use that pair as the row. Then point again to a pair of digits; use that pair as the column. For instance if you point to 32 and then to 07, start at the 32nd row, 7th column. If you point to a pair of digits whose number does not refer to a row or column, ignore that and point again.
Now suppose you begin at the digit in the 32nd row, 7th column. It is a 7. Examining the next 25 numbers is like going into a home-room and asking 25 seniors whether they own a car. Now choose a direction to go in—up, down, left or right—perhaps by rolling a die. We go right. The next 25 numbers are 6, 2, 2, 2, 3, 6, 0, 8, 6, 8, 4, 6, 3, 7, 9, 3, 1, 6, 1, 7, 6, 0, 3, 8, 6. Since 4 of the digits are either 0 or 1, in this sample 4 seniors own a car. If there are 510 digits (seniors) to choose from, there are \( \binom{510}{25} \) potential samples, a very large number (over \( 10^{42} \)). These samples would have from 0 to 25 seniors who own cars, but more of them will have 5 seniors than any other number of seniors. Slightly fewer samples will have 4 or 6 seniors. Again slightly fewer will have 3 or 7 seniors. A small percentage of the \( 10^{42} \) possible samples will have 0 seniors and a very, very tiny percentage near 25 seniors.

Let \( P(x) \) be the probability that a sample of 25 from 510 random digits contains \( x \) digits that are 0s or 1s. That is, \( P(x) = \) the probability that a sample contains \( x \) seniors with cars. A famous theorem from statistics, called the **Central Limit Theorem**, states that the function \( P \) is very closely approximated by a normal distribution whose mean is 20% (the mean of the population) and whose standard deviation is \( \sqrt{25 \cdot 20\% \cdot 80\%} \), which in this case is 2.

**Central Limit Theorem:**

Suppose random samples of size \( n \) are chosen from a population of events in which the probability of an event having a certain characteristic is \( p \). Let \( P(x) \) equal the number of elements in that sample with the characteristic. Then \( P \) is approximated by a normal distribution with mean \( np \) and standard deviation \( \sqrt{np(1 - p)} \).

That function is graphed here.
Recall the percents within given standard deviations for a normal distribution. If 20% of the seniors own cars and this principal polled 25 seniors at random, about 68% of the time the principal would find that from 3 to 7 seniors in the sample owned cars. That is what happened here. In these cases, the principal would infer that 12% to 28% of the seniors owned cars. That isn’t too far off even with a sample of 25. About 95% of the time the principal would infer that from 1 to 9 seniors in the sample (between 4% and 36% of the sample) owned cars. That’s a wider interval, but the principal could be 95% confident of the results.
Now let us turn to the TV polling example. Suppose that in reality
20% of households are tuned in to a particular show. Consider all
the random samples of 1600 people. These samples will have a
mean of 320 people (20%) tuned to the show. The standard
deivation of these samples is \( \sqrt{1600 \cdot 20\% \cdot 80\%} \), or 16. That
means that 68% of the time the samples will have between 304 and
336 people (between 19% and 21%) watching the show. The sample
percentage will be within 1% of the actual. Also, 95% of the time
the sample will have between 288 and 352 watching the show; that
is, between 18% and 22%. So 95% of the time, the sample is within
2% of the actual amount. This accuracy is probably good enough for
the networks.

\[ \text{distribution of sample means}
\]
\[ \text{with } n = 1600, \ p = 20\% \]

\[ \begin{align*}
18\% & \quad 19\% & \quad 20\% & \quad 21\% & \quad 22\%
\end{align*} \]

**Questions**

1. Give two reasons for sampling.

In 2 and 3, identify the population and the sample in the sampling
situation.

2. sampling to obtain TV ratings

3. polling potential voters to see which candidate is favored

4. What is the size of the samples often used in TV ratings?

5. What is the difference between a random sample and one that is not
random?

6. Why are stratified samples often used to obtain TV ratings?

7. What does a TV rating of 18.6 mean?

8. *Copy and complete* If 1600 people are polled randomly for TV rat-
ings, 68% of the time the rating will be within \( \_\% \) of the actual
percent of people watching the program.
9. *Multiple choice*  In this lesson, 20% of a senior class of 510 owned cars. The principal walked into a class and found that 4 of 25, or 16% of the seniors he polled, owned cars. What is the best reason that the percents are not equal?
   a. The sample was not random.
   b. Sample percents vary.
   c. Students may not have been telling the truth.

10. Means of samples of size \(n\), from a distribution in which the probability of a characteristic is \(p\), approximate a normal distribution with what mean and what standard deviation?

In 11 and 12, a fair coin is tossed 1000 times.

11. The mean number of heads in such samples is \(\_\) and the standard deviation is \(\_\).

12. This implies that 68% of the time, from \(\_\) to \(\_\) heads are expected.

In 13 and 14, consider that in BASIC a function named RND generates random numbers with decimal values between 0 and 1. RND always has the argument 1 so in programs you must use RND(1) to generate such a number.

13. a. Run this program and describe its output.

```basic
10 FOR N = 1 TO 10
20 PRINT RND(1)
30 NEXT N
40 END
```

b. Run the program again and write a sentence or two comparing its output to that in part a.

14. The following program simulates tossing a coin.

```basic
10 REM COIN TOSS SIMULATION
20 REM NMTOS = NUMBER OF TOSES
30 REM X = A RANDOM NUMBER
40 REM H = NUMBER OF HEADS, T = NUMBER OF TAILS
50 INPUT "HOW MANY TOSES"; NMTOS
60 FOR I = 1 TO NMTOS
70 LET X = RND(1)
80 IF X < .5 THEN H = H+1 ELSE T = T+1
90 NEXT I
100 PRINT H; "HEADS AND"; T; "TAILS"
110 END
```

a. Run the program for 50 tosses, and record the output.

b. Run the program for 500 tosses, and record the output.

c. Calculate the percent heads and percent tails for each run above. Which run more closely approximates the probability of getting a head on a toss of one coin?
15. You call a classmate to find out if he or she thinks there will be a test on Chapter 13 next Friday. What is the population and what is the sample?

16. Construct a data set whose mean is 10, whose median is 9, and whose mode is 8. (Lesson 13-9)

17. Give the standard deviation of the data set 
\{2, 4, 6, 8, 10, 12, 14, 16, 18\}. (Lesson 13-9)

18. Graph the binomial distribution for tossing a fair coin 3 times. (Lesson 13-10)

19. What is the probability of answering exactly 5 questions correctly on a 6-question test in which you have a 50% chance of getting each question correct? (Lesson 13-8)

20. True or false Justify your answer. \(3 \left( \sum_{n=1}^{4} n^2 \right) = \sum_{n=1}^{4} (3n^2)\). (Lesson 13-3)

21. If \(2^x = 45\), what is \(x\)? (Lesson 9-9)

22. Solve: \(r^{-1/2} = 81\). (Lesson 8-10)

23. Find equations for two parabolas congruent to \(y = x^2\) and having vertex \((6, 5)\). (Lesson 6-4)

24. Simplify \(\sqrt{4} \cdot \sqrt{9} + \sqrt{-4} \cdot \sqrt{9} + \sqrt{-4} \cdot \sqrt{-9} + \sqrt{4} \cdot \sqrt{-9}\). (Lessons 6-1, 6-8)

25. Give an equation for the right angle graphed here. (Lesson 7-5)

26. Sample at least 25 people on controversial issues of concern to you. From the size of your sample and the results you find, make reasonable inferences using the Central Limit Theorem.
A series is an indicated sum of terms of a sequence. The sum \( x_1 + x_2 + \ldots + x_n \) can be represented by \( \sum_{i=1}^{n} x_i \). Values of finite arithmetic (linear) sequences and of finite and infinite geometric (exponential) sequences may be calculated from the following formulas.

In an arithmetic sequence \( a_1, a_2, \ldots, a_n \) with common difference \( d \):

\[ S_n = \sum_{i=1}^{n} a_i = \frac{1}{2}n(a_1 + a_n) = \frac{n}{2}[2a_1 + (n - 1)d] \]

In a geometric sequence \( g_1, g_2, \ldots, g_n \) with common ratio \( r \):

\[ S_n = \sum_{i=1}^{n} g_i = g_1 \frac{1 - r^n}{1 - r} \]

If \( |r| < 1 \) then the infinite geometric series with first term \( g_1 \) and common ratio \( r \) has the value

\[ S = \sum_{n=1}^{\infty} g_1 r^{n-1} = \frac{g_1}{1 - r} \cdot \]

Pascal’s Triangle is a 2-dimensional sequence. The \((r + 1)\)st element in the \(n\)th row is denoted by \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \). The expression \( \binom{n}{r} \), also denoted \( n \choose r \), appears in several other important applications. It is the coefficient of \( a^{n-r}b^r \) in the binomial expansion of \((a + b)^n\). It is the number of subsets, or combinations, with \( r \) elements taken from a set with \( n \) elements. And if a situation consists of \( n \) trials with two equally likely outcomes (say heads/tails on the toss of a coin), then the probability of getting exactly one of these outcomes \( r \) times is \( \frac{n}{2^n} \).

A statistical measure is a number which is used to describe a data set. Measures of central tendency include the mean, median, and mode. The standard deviation of a data set is a measure of spread or dispersion.

Distributions of numbers such as test scores often resemble the graphs of probability values related to Pascal’s Triangle. As the number of the row of Pascal’s Triangle increases, the distribution takes on a shape more and more like a normal curve. Some tests are standardized so that their scores fit that shape. In a normal distribution, 68\% of the data are within one standard deviation of the mean, and 95\% within two standard deviations.

Sampling is a procedure by which one tries to describe a larger set (the population) by looking at a smaller set. Statistics calculated from samples are used as estimates of a population statistic. If the sample is random its mean can be compared to other possible means because the distribution of means is close to a normal distribution. This information can be used to obtain the accuracy of a sample of a particular size.

On the next page are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with a *. For all other terms you should be able to give a general description or a specific example.
Vocabulary

Lesson 13-1
*series
*arithmetic series

Lesson 13-2
*geometric series

Lesson 13-3
Σ, sigma
Σ-notation, summation notation
index variable
!, factorial notation

Lesson 13-4
*infinite geometric series
limit
partial sums
sum of a series
snowflake curve

Lesson 13-5
Pascal's Triangle

Lesson 13-6
Binomial Theorem
binomial expansion

Lesson 13-7
subset
*combination

Lesson 13-8
*probability (of an event)
trial
fair coin, unbiased coin

Lesson 13-9
data set
*mean
*median
*mode
measure of central tendency
standard deviation
statistical measure
measure of dispersion

Lesson 13-10
binomial distribution
probability distribution
probability function
normal curve
normal distribution
standardized scores, normalized scores

Lesson 13-11
*population
*sample
random sample
stratified sample
random numbers
simulation
Central Limit Theorem
Take this test as you would take a test in class. Then check your work with the solutions in the Selected Answers section in the back of the book.

1. Consider the sequence defined as follows.
   \[
   \begin{align*}
   g_1 &= 12 \\
   g_{n+1} &= \frac{1}{2}g_n \text{ for } n > 1.
   \end{align*}
   \]
   a. Write the first four terms of the sequence.
   b. Find the sum of the first 12 terms.

2. A concert hall has 30 rows. The first row has 12 seats. Each row has two more seats than the preceding row. How many seats are in the concert hall?

3. Evaluate and write as a decimal. \[ \sum_{i=2}^{3} 4(10)^i \]

4. Expand. \((x^2 - 3)^4\)

5. A pizza restaurant menu contains 15 possible ingredients for pizza. You order 3 of them. How many such combinations are possible?

In 6 and 7, evaluate.

6. \(C_{10}^8\)

7. \(\binom{40}{38}\)

8. Consider a fair coin tossed 10 times.
   a. State the total number of ways the coin may fall.
   b. What is the probability of getting exactly 5 heads and 5 tails?

In 9–11, consider Sheila’s scores on math quizzes this term: 80, 80, 88, 90, 93. Find the

9. mode

10. mean

11. score needed on the next quiz to bring her average up to 88.

12. If the sum of the first \(n\) integers is 300, what is the value of \(n\)?

13. Write using summation notation:
   \[1^3 + 2^3 + 3^3 + \ldots + 20^3\]

In 14 and 15, consider that on a recent administration of the ACT, composite scores had a mean of 18.8 and a standard deviation of 5.9. Assume that these scores are normally distributed.

14. About what percent of scores are within two standard deviations of the mean?

15. About what percent of scores are at or above 24.7?

16. Let \(P(n) = \binom{6}{n} \cdot \frac{2^n}{2^n}\).
   a. Make a table of values for this function for integers \(n\) from 0 to 6.
   b. Graph the function.
   c. Describe in words what \(P(n)\) represents in the context of tossing a coin.

17. a. For what value(s) of \(x\) does the following infinite geometric series have a value?
   \[1 - x + x^2 - x^3 + \ldots\]
   b. What is that sum?

18. a. Write rows zero through five of Pascal’s Triangle.
   b. If the top row is considered the 0th row, what is the sum of the numbers in the \(n\)th row?

19. A poll of 1000 registered voters shows that 60% favor a school referendum. What is the population and what is the sample?

20. True or false The second term in the binomial expansion of \((x - y)^7\) is \(\binom{7}{2} x^6(-y)^1\).

21. Multiple choice In how many ways can 8 different letters be ordered?
   (a) \(8!\)   (b) \(2^8\)   (c) \(8^2\)   (d) \(\binom{8}{1}\)

22. Why is sampling necessary to test the fairness of a coin?
Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations.
The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deals with the procedures used to get answers.

Objective A: Calculate values of finite arithmetic series, and both finite and infinite geometric series. (Lessons 13-1, 13-2, 13-4)

In 1–6, evaluate the series.

1. $3 + 7 + 11 + \ldots + 87$
2. $2^0 + 2^1 + 2^2 + \ldots + 2^{19}$
3. $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$
4. $50 - 10 + 2 - \frac{2}{3} + \ldots$
5. the sum of the first 60 positive integers
6. the sum of the first 10 terms of the sequence defined by the following:
   \[
   \begin{align*}
   t_1 &= 100 \\
   t_{n+1} &= t_n - 5 \quad \text{for } n \geq 1
   \end{align*}
   \]
7. a. Write the repeating decimal $0.354354\ldots$ as an infinite series.
   b. Find a simple fraction in lowest terms equal to this decimal.
8. The sum of the integers $1 + 2 + 3 + \ldots + k$ is 630. What is $k$?

Objective B: Use summation ($\Sigma$) or factorial (!) notation. (Lessons 13-3, 13-9)

In 9 and 10, (a) write the terms of the series; and (b) evaluate.

9. $\sum_{n=1}^{6} (2n - 5)$
10. $\sum_{i=-2}^{3} (7 \cdot 10^i)$
11. Multiple choice Which equals the sum of squares $1 + 4 + 9 + 16 + \ldots + 100$?
   (a) $\sum_{n=1}^{10} n$  (b) $\sum_{n=1}^{10} 2^n$  (c) $\sum_{n=1}^{10} n^2$  (d) $\sum_{n=1}^{100} n^2$
12. Suppose $a_1 = 15, a_2 = 16, a_3 = 16, a_4 = 17, a_5 = 18$. Evaluate $\sum_{i=1}^{5} a_i$.

In 13 and 14, rewrite using $\Sigma$-notation.

13. $2 + 4 + 6 + \ldots + 144$
14. $M = \frac{x_1 + x_2 + \ldots + x_n}{n}$

15. If $f(n) = n!$, calculate $f(2) + f(6)$.
16. Multiple choice $\frac{(n + 1)!}{n!} =$
   (a) 1  (b) $n$  (c) $n + 1$  (d) $n - 1$

Objective C: Calculate entries in Pascal’s Triangle and the number of subsets of a given set. (Lessons 13-5, 13-7)

17. Rows 0 to 2 of Pascal’s Triangle are given at the right. Write the next three rows.

18. Copy and complete The symbol $\binom{n}{r}$ represents the __ element in the __ row of Pascal’s Triangle.

In 19–22, evaluate.

19. $\binom{10}{5}$  
20. $\binom{4}{4}$
21. $7C_0$  
22. $100C_{99}$
23. Multiple choice The quantity $\frac{12!}{9!3!}$ equals all but which of the following?
   (a) $12C_3$  (b) $\binom{12}{3}$
   (c) $\binom{12}{9}$  (d) $12 \cdot 11 \cdot 10$
In 24 and 25, consider the set of letters in the English alphabet, \{A, B, C, \ldots, Y, Z\}.

24. How many subsets have
   (a) 1 element  
   (b) 3 elements  
   (c) 20 elements?

25. What is the total number of subsets that can be formed?

**Objective D: Expand binomials. (Lesson 13-6)**

In 26–29, expand.

26. \((x + y)^4\)  
27. \((p - 8)^7\)

28. \((3n^2 - 4)^3\)  
29. \(\left(\frac{a}{2} + 2b\right)^5\)

In 30 and 31, true or false.

30. The first term of the binomial expansion of \((8x + y)^{17}\) is \((8x)^{17}\).

31. The second term of the binomial expansion of \((4n - p)^{10}\) is \(\binom{10}{2}(4n)^8(-p)^2\).

**Objective E: Calculate descriptive statistics for a data set. (Lesson 13-9)**

In 33–35, consider the test scores:
90, 68, 75, 80, 90, 68, 99, 87. Find the:

33. mean, median and mode

34. possible values for the mean, median, and mode if one more score (ranging from 0 to 100) is added to the data set

35. standard deviation

36. Repeat Questions 33–35 for the scores:
88, 90, 90, 90, 80.

**Properties** deal with the principles behind the mathematics.

**Objective F: State whether or not an infinite geometric series has a limit. (Lesson 13-4)**

37. Under what condition(s) is it true that
   \[ g_1 + g_1r + g_1r^2 + g_1r^3 + \ldots = \frac{g_1}{1 - r}. \]

38. **a.** Does the infinite geometric series
   \[ 1 - 3 + 9 - 27 + \ldots \]
   have a value?

   **b.** If so, what is it? If not, why not?

In 39 and 40, (a) give the next 3 terms for each geometric series and (b) tell whether the infinite series has a limit.

39. 100 + 75 + \ldots

40. 1 - 2 + 4 \ldots

**Objective G: State properties of Pascal’s Triangle. (Lesson 13-5)**

In 41 and 42, true or false.

41. The first and last number in each row of Pascal’s Triangle is 1.

42. For all positive integers \(n\), \(\binom{n}{1} = \binom{n}{n-1}\).

In 43–45, consider the top row in Pascal’s Triangle to be the 0th row.

<table>
<thead>
<tr>
<th>row</th>
<th>0th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

43. What is the sum of the numbers in the 5th row?

44. What is the sum in the \(n\)th row?

45. Which entry of row \(n\) in Pascal’s Triangle is the coefficient of \(a^{n-1}b^n\) in the binomial expansion of \((a + b)^n\)?
**Objective H:** Solve applied problems using arithmetic or geometric series. (Lessons 13-1, 13-2, 13-4)

46. Suppose on January 1 you deposit $1.00 in an empty Piggy Bank. On January 8 you deposit $1.50. On January 15, you deposit $2.00; on each week thereafter you deposit $.50 more than the previous week.
   a. What kind of sequence do the individual deposits generate?
   b. What amount should you deposit in the 52nd week?
   c. What is the total at the end of 52 weeks? (Assume no withdrawals and no interest payments.)

In 47 and 48, a hiker walks 12 mi the first day and 0.6 mi less each succeeding day.

47. How far will the hiker walk in 5 days?
48. After how many days will the hiker have completed 99 miles?

49. Carla’s Clothing Shop opened 8 years ago. The first year she made $3000 profit. Each year thereafter her profits were about 50% greater than the previous year.
   a. How much profit did Carla earn during her 8th year in business?
   b. What is the total amount earned in the 8 years?

50. A ball on a pendulum moves 50 cm on its first swing. Each succeeding swing it moves .9 the distance of the previous swing.
   a. Write the first four terms of the series generated.
   b. Assuming this pattern continues indefinitely, how far will the ball travel before coming to a rest?

51. Congruent boxes are used to make a staircase as pictured above. If there are $n$ steps, how many boxes are needed?

**Objective I:** Use permutations, combinations, or probability to solve problems. (Lessons 13-7, 13-8)

52. There are 12 notes in a musical octave: $A, A\#, B, C, C\#, D, D\#, E, F, F\#, G,$ and $G\#$. In some twelve-tone music, a theme uses each of these notes exactly once. Ignoring rhythm, how many themes are possible?

53. In how many ways can the letters of the word NICELY be arranged?

In 54–56, consider that in the 100th Congress there were 54 Democratic and 46 Republican senators.

54. How many seven person committees could be formed with members from either party?
55. How many four member committees could be formed entirely Democratic?
56. What is the total number of possible committees that are entirely Republican? (Do not include the empty set.)

57. Ten people are in a room. Each decides to shake hands with everyone else exactly once. How many handshakes will take place?

In 58–60, consider that a fair coin is tossed 5 times. Calculate the probability of each event.

58. exactly 1 head
59. exactly 3 heads
60. exactly 5 tails

61. Assume a student takes a true-false test with 10 questions and that the student guesses on each question. Find the probability of getting:
   a. exactly 7 items correct
   b. 7 or more items correct.
**Objective J:** Use measures of central tendency or dispersion to describe data or distributions.  
(Lessons 13-9, 13-10)

In 62–64, consider the populations of the ten largest cities in the world. (Source: 1988 World Almanac; data rounded to the nearest 100,000).

<table>
<thead>
<tr>
<th>City</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokyo-Yokohama</td>
<td>25,400,000</td>
</tr>
<tr>
<td>Mexico City</td>
<td>16,900,000</td>
</tr>
<tr>
<td>Sao Paulo</td>
<td>14,900,000</td>
</tr>
<tr>
<td>New York</td>
<td>14,600,000</td>
</tr>
<tr>
<td>Seoul</td>
<td>13,700,000</td>
</tr>
<tr>
<td>Osaka-Kobe-Kyoto</td>
<td>13,600,000</td>
</tr>
<tr>
<td>Buenos Aires</td>
<td>10,800,000</td>
</tr>
<tr>
<td>Calcutta</td>
<td>10,500,000</td>
</tr>
<tr>
<td>Bombay</td>
<td>10,100,000</td>
</tr>
<tr>
<td>Rio de Janeiro</td>
<td>10,100,000</td>
</tr>
</tbody>
</table>

For this data set find the:

62. mean  
63. median  
64. mode.

In 65–67, consider the following heights of the starting five on a basketball team: 6'8", 6'10", 6'4", 6'8", 6'1". For this data set find the:

65. mode  
66. mean  
67. standard deviation.

68. John played one round of golf each day during his vacation. If the first six days his average was 90, what would he need to score on the seventh day to bring his average to 88?

In 69–70, use the following data reported by the College Entrance Examination Board.

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>mean</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>juniors</td>
<td>580,981</td>
<td>484</td>
<td>111</td>
</tr>
<tr>
<td>seniors</td>
<td>799,861</td>
<td>457</td>
<td>115</td>
</tr>
</tbody>
</table>

69. Which group, juniors or seniors, shows a greater dispersion of scores?

70. The mean score of all students taking this test was 467, which is not the average of the means of the juniors and seniors. Why not?

71. Assume mathematics aptitude is normally distributed among juniors. Within what interval would you expect the middle 68% of scores of juniors to occur?

**Objective K:** Give reasons for sampling.  
(Lesson 13-11)

72. What is an advantage of using a sample that is random?

73. To find the ratings of a television show in a small town, a network uses a sample rather than the population.
   a. What is the population in this situation?
   b. Why might a sample be preferred over the population?

**Representations** deal with pictures, graphs, or objects that illustrate concepts.

**Objective L:** Graph and analyze binomial and normal distributions. (Lesson 13-10)

74. Below is pictured a normal distribution with mean $m$ and standard deviation $s$.

![Normal Distribution](image)

75. Consider the probability function $P(n) = \frac{\binom{8}{n}}{2^8}$.

   a. Evaluate $P(n)$ for integers 0, 1, ..., 8.
   b. Graph this function.
   c. What name is given to this function?
CHAPTER 14

Dimensions and Space

14-1: Three-Dimensional Coordinates
14-2: Equations of Planes
14-3: Solving Systems in 3-Space
14-4: Distance and Spheres
14-5: Solids and Surfaces of Revolution
14-6: Higher Dimensions
14-7: Solving Higher Dimensional Systems
14-8: Fractals
A line has one dimension. Each point on it can be located with a unique real number \( x \). For this reason the real number line is often called *1-space*.

A plane has two dimensions. Each point on it can be located with a unique pair of real numbers \((x, y)\). Thus, the coordinate plane is often called *2-space*.

What is called space in most geometry texts has three dimensions and is often called *3-space*. In this chapter, you will see how every point in 3-space can be located with an *ordered triple* of real numbers \((x, y, z)\).

Many real-world situations can be described by the mathematics of lines (1-space), planes (2-space), or solids (3-space). However, some real situations are best described by mathematical models with four or more dimensions, and others by models with dimensions that are not whole numbers. For instance, Einstein’s theory of special relativity requires a 4-dimensional model, and in Lesson 14-8 you will see why a coastline such as that of Britain (shown at the left) has a fractional dimension between 1 and 2.

In this chapter you will learn to draw and analyze graphs and to solve equations in 3-space. You will also learn about uses of the mathematics of fractional dimensions and of dimensions greater than 3.
Points in a room can be located with a three-dimensional coordinate system. It is convenient to let the origin be a corner of the room where two walls and the floor intersect. Then with two coordinates $x$ and $y$ you can describe the location of an object on the floor. However, to locate an object in the room which is not on the floor (such as a point on a light hanging from the ceiling), you need a number to indicate the height from the floor. This is the $z$-direction. Thus, if the point $P$ on the light is 6 ft from the origin in the $x$-direction, 8 ft in the $y$-direction, and 9 ft in the $z$-direction (up), you could specify the position of the point uniquely by the ordered triple $(6, 8, 9)$. The $x$-coordinate is 6, the $y$-coordinate is 8, and the $z$-coordinate is 9.

The three axes are called the $x$-axis, the $y$-axis, and the $z$-axis. The positive direction is shown on each axis by a single arrowhead. By extending each axis, as shown in the figure at the right, any point can be described by the ordered triple $(x, y, z)$.
Example 1  Plot the point \( R = (-2, 4, 6) \) on a three-dimensional coordinate system.

Solution
1. Since the \( x \)-coordinate is -2, slide 2 units back (in a negative direction) on the \( x \)-axis.
2. Since the \( y \)-coordinate is 4, move 4 units in a positive direction parallel to the \( y \)-axis.
3. Since the \( z \)-coordinate is 6, go 6 units up parallel to the \( z \)-axis.

It helps to think of the point as the back, upper right vertex of a box with base dimensions 2 and 4 and height 6.

Recall from geometry that there is exactly one plane through two intersecting lines. The \( x \)-axis and the \( y \)-axis determine the \( xy \)-plane. Similarly, the \( x \)-axis and the \( z \)-axis determine the \( xz \)-plane, and the \( y \)-axis and \( z \)-axis determine the \( yz \)-plane. The \( xy \)-, \( xz \)-, and \( yz \)-planes are called coordinate planes. The three coordinate planes separate 3-space into eight regions called octants.

In the classroom coordinate system that opened the lesson, the entire classroom is in the upper right front octant, the one where all the coordinates are positive.
Example 2  What are the signs of the coordinates of a point in the lower right front octant?

Solution  Examine the adjectives "lower right front". Lower means a negative z-value. Right means a positive y-value. Front means a positive x-value. If the coordinates of the point in this octant are \((a, b, c)\), then \(a\) and \(b\) are positive and \(c\) is negative. Below are two representations of the point in 3-space.

![Diagram of coordinates](image)

The point \((2, -4, 0)\), shown at the right, is on the \(xy\)-plane. For any point on a coordinate plane, one of the coordinates is zero. This is similar to what happens in two dimensions: for any point on an axis, one of the coordinates is 0.

Questions

1. Any point in 3-space can be located with an ordered _?_.

In 2–4, true or false.

2. The number line can be called 1-space.

3. Only integers can be coordinates of points in 3-space.

4. The intersection of two walls and the floor of a room can represent the origin of a coordinate system in 3-space.

5. Refer to the picture of the room in the lesson. Which coordinate plane represents the wall with the window?

In 6–8, match the axis to its direction as pictured in this lesson.

6. \(x\)-axis  
   (a) up-down

7. \(y\)-axis  
   (b) forward-backward

8. \(z\)-axis  
   (c) left-right
9. Draw a coordinate system and plot the points $B = (5, -3, 4)$ and $C = (4, 0, -2)$.

10. The coordinate planes divide 3-space into eight regions called ____. 

11. Point $(a, b, c)$ is in the upper left back octant. What are the signs of $a$, $b$, and $c$? 

12. Multiple choice For any point $(a, b, c)$ on the $xz$-plane:
   (a) $a = 0$
   (b) $b = 0$
   (c) $c = 0$
   (d) none of these

13. Suppose the point $(6, 8, 9)$ locates the bottom of the wire connecting the lamp to the ceiling in the classroom mentioned in this lesson. If the wire is one foot long, what are the coordinates of the top point of the wire?

14. A point whose $x$- and $y$-coordinates are both zero must lie on the __-axis.

15. A box has the following vertices: the origin, $(0, 8, 0)$, $(2, 0, 0)$, $(2, 8, 0)$, $(2, 8, 5)$, $(2, 0, 5)$, $(0, 0, 5)$, and $(0, 8, 5)$.
   a. Draw the box on a 3-dimensional coordinate system.
   b. Determine its volume.
   c. Determine its surface area.

16. A cube which has sides of length 1 has one vertex at the origin. None of the coordinates of any vertex is negative. Find the coordinates of the other vertices of the cube.

17. In the rectangular box at the right, $D = (13, 2, 5)$ and $F$ is the origin.
   a. Find the coordinates of the points $A$, $B$, $C$, $E$, $G$, and $H$.
   b. Determine the volume of the box.
   c. Determine its surface area.

18. $p$ varies directly as the square of $t$. If $t = -2$, then $p = -12$. Find $p$ when $t = 4$. (Lesson 2-1)

In 19–21, suppose $y$ is a function of $x$. Describe what happens in each function below when $x$ is tripled. (Lesson 2-3)

19. $y = \frac{k}{x}$

20. $y = kx^{2}$

21. $y = 5x^{3}$
22. Graph: (a) \((x, y): x = 8\); (b) \((x, y): y = 8\). (Lesson 3-4)

In 23–26, match each graph with an appropriate equation. (Lessons 10-9, 9-7, 2-7, 2-4)

<table>
<thead>
<tr>
<th></th>
<th>a. (y = \sin x)</th>
<th>b. (y = \frac{x}{4})</th>
<th>c. (y = \frac{4}{x^2})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>d. (y = \frac{4}{x})</td>
<td>e. (y = e^x)</td>
<td>f. (y = (.7)^x)</td>
</tr>
</tbody>
</table>

23. \[\text{Graph: } (a) \]  
   \[\text{Graph: } (b) \]

24. \[\text{Graph: } (c) \]

25. \[\text{Graph: } (d) \]

26. \[\text{Graph: } (e) \]

27. Graph \(-2y < 3x + 6\). (Lesson 3-9)

28. Suppose one donut costs $.50 and one muffin costs $.75.
   a. Write an expression for the cost of \(d\) donuts and \(m\) muffins.
   b. State three ways in which you might spend exactly $6.00 on donuts and muffins.
   c. Graph all points \((d, m)\) that represent ways to spend exactly $6.00 on donuts and muffins. (Lessons 1-1, 3-3, 3-4)

29. Air traffic controllers must locate planes in the air. Explain how they can locate the position with three numbers.

30. Imagine your classroom with a three-dimensional coordinate system.
   a. What point would you pick as the origin?
   b. Estimate the coordinates of a point in the middle of your desk.
   c. Estimate the coordinates of other key points in the room.
Consider the plane graphed below. It is four units to the right of the \(xz\)-plane and parallel to it. Every point on the plane has the same \(y\)-coordinate, 4. Thus \(y = 4\) is an equation for this plane.

In 3-space every plane parallel to a coordinate plane has an equation similar to the one shown above. These equations and their graphs are analogous to the equations of lines parallel to the coordinate axes in 2-space. This is summarized below.

<table>
<thead>
<tr>
<th>2-space</th>
<th>3-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line parallel to (y)-axis: (x = a)</td>
<td>Plane parallel to (yz)-plane: (x = a)</td>
</tr>
<tr>
<td>Line parallel to (x)-axis: (y = b)</td>
<td>Plane parallel to (xz)-plane: (y = b)</td>
</tr>
<tr>
<td></td>
<td>Plane parallel to (xy)-plane: (z = c)</td>
</tr>
</tbody>
</table>

Example 1 shows how you can use this idea to describe points in 3-space.

**Example 1** Describe the set of all points 4 units away from the \(xy\)-coordinate plane.

**Solution** These points consist of two parallel planes \(P\) and \(Q\) where plane \(P\) is 4 units above the \(xy\)-coordinate plane and plane \(Q\) is 4 units below the \(xy\)-coordinate plane. An equation for plane \(P\) is \(z = 4\). An equation for plane \(Q\) is \(z = -4\).
The description of these points in 3-space is analogous to a situation in 2-space.

**2-space**
The locus of points \( r \) units from a line \( l \) is two lines parallel to \( l \) a distance \( r \) from it.

**3-space**
The locus of points \( r \) units from a plane \( P \) is two planes parallel to \( P \) at a distance \( r \).

You have learned that in 2-space all lines have an equation equivalent to the form \( Ax + By = C \). There is a similar equation for all planes in 3-space. It can be deduced using methods that are not difficult, but are beyond the scope of this book.

**Theorem:**

The set of points \((x, y, z)\) satisfying

\[
Ax + By + Cz = D,
\]

where not all of \( A, B, \) and \( C \) are zero, is a plane.

The equation \( Ax + By + Cz = D \) is called the **standard form of the equation of a plane**.

Planes parallel to coordinate planes are special cases of this theorem. For the plane \( y = 4 \) mentioned at the start of the lesson, \( A = 0, C = 0, \) and \( D = 4 \). In general, when two of the constants \( A, B, \) and \( C \) are zero, the plane is parallel to a coordinate plane.

Recall that in 2-space the \( x \)-intercept is the \( x \)-value of the point (or points) where the graph crosses the \( x \)-axis. The same definition applies in 3-space. Because three noncollinear points determine a plane, the three intercepts can be used to graph planes. Example 2 shows how to graph a triangular part of a plane when there are three intercepts.
Graph the triangular region determined by the intercepts of the plane with equation \(2x + 3y + 6z = 6\).

**Solution** First find the intercepts. To find the \(x\)-intercept, let \(y = 0\) and \(z = 0\). Then \(x = 3\), so \((3, 0, 0)\) is on the plane. To find the \(y\)-intercept, let \(x = 0\) and \(z = 0\). Then \(y = 2\) and so \((0, 2, 0)\) is on the plane. To find the \(z\)-intercept, let \(x = 0\) and \(y = 0\). Then \(z = 1\), and so \((0, 0, 1)\) is on the plane. Plot these points. Draw the triangle formed by these points. This triangular region is part of the plane determined by \(2x + 3y + 6z = 6\).

Not all planes have three intercepts. If exactly one of the constants \(A\), \(B\), and \(C\) is zero in the standard form of the equation of a plane, the plane is parallel to a coordinate axis. Example 3 illustrates this.

In 3-space, graph \(4x + 3z = 12\).

**Solution** As in Example 2 find the intercepts. The \(x\)-intercept is 3 and the \(z\)-intercept is 4. However, there is no \(y\)-intercept because when substituting \(x = 0\) and \(z = 0\), you get

\[4(0) + 3(0) = 12, \text{ or } 0 = 12.\]

So, the plane does not intersect the \(y\)-axis and is therefore parallel to it. The rectangular region graphed at the right is part of the plane determined by \(4x + 3z = 12\).
Just as the intersection of two lines in 2-space represents the solution to a 2 × 2 linear system, the intersection of three planes in 3-space represents the solution to a 3 × 3 linear system. Each of the equations \( \begin{cases} x = 3 \\ y = 8 \\ z = -1 \end{cases} \) represents a plane parallel to a coordinate plane.

If these three graphs are placed on the same coordinate system, the lower right graph is obtained. There is a single point of intersection, \((3, 8, -1)\). The coordinates of this point satisfy all three equations.

Because graphing planes in 3-space is time consuming and may be difficult, most people do not solve systems of three linear equations by graphing. However, computers can easily graph even more complicated equations and are increasingly being used to study complex systems.

Questions

1. **True or false** The plane with equation \( x = 5 \) is parallel to the \( xz \)-plane.

2. a. Graph the set of points 10 units away from the \( yz \)-plane.

   b. Write an equation for the set of points in part a.
In 3 and 4, consider the plane determined by the equation 
\( Ax + By + Cz = D \).

3. If exactly one of the constants \( A, B, \) and \( C \) is zero, the plane is parallel to a coordinate ___.

4. If exactly two of the constants \( A, B, \) and \( C \) are zero, the plane is parallel to a coordinate ___.

5. Define: \( z \)-intercept of a plane.

6. Plane \( P \), shown at the right, is parallel to the \( yz \)-plane and contains \((4, 0, 0)\). Give an equation for \( P \).

7. Graph the triangular region determined by the intercepts of the plane with equation \( 4x - 2y + 3z = 6 \).

8. The solution to the system 
\[
\begin{align*}
    x &= 4 \\
    y &= 2 \\
    z &= 7
\end{align*}
\]

is the point of intersection of the three planes ___, ___, and ___.

9. Refer to the graph at the right.
   a. What system is graphed?
   b. Find the solution to the system.

10. Graph the plane \( x - 3y = 6 \).

11. Give an equation for the plane containing the point \((0, 5, 0)\) and parallel to the \( xz \)-coordinate plane.

12. Graph the triangular region determined by the intercepts of the plane with equation \( 12x - 3y + 8z = 24 \).

13. A 100-point test has \( x \) questions worth 2 points each, \( y \) questions worth 4 points each, and \( z \) questions worth 5 points each.
   a. Write an equation that describes all possible numbers of questions.
   b. What is a suitable domain for \( x, y, \) and \( z \)?
   c. Graph the triangular region determined by the intercepts of this equation.
   d. Name 3 ordered triples \((x, y, z)\) that satisfy the equation from part a. Are they in the region?
14. Give equations for the six planes which contain the faces of the box drawn at the right.

15. Draw a coordinate system and plot the point $A = (-3, -4, -2)$. (Lesson 14-1)

16. True or false If point $(a, b, c)$ lies in the lower right front octant, then $a$ is negative and $b$ and $c$ are positive. (Lesson 14-1)

17. In electricity, impedance for an alternating current is similar to resistance in a direct current. The rule for combining impedances $z_1$ and $z_2$ in series is simply $z_1 + z_2$. Find the impedance of $z_1$ and $z_2$ in series given that $z_1 = 2 + 3i$ and $z_2 = 1 - 5i$. (Lesson 6-9)

18. The pressure of a liquid on a submerged object varies jointly as the depth of the object and the density of the liquid. When an object is 50 in. below the surface of a liquid with density 1.5 lb/in.\(^3\) the pressure on it is 93.4 psi. Find the pressure on an object 125 in. below the surface of a liquid with density 2.3 lb/in\(^3\). (Lesson 2-10)
In 19–21, consider the following equations describing relations in \((x, y)\) or \((t, A)\). (Lessons 9-7, 8-2, 6-5, 6-4, 3-4, 3-2, 2-7, 2-6, 2-4)

\[
\begin{align*}
y &= kx \\
y &= mx + b \\
y &= kx^2 \\
y &= ax^2 + bx + c \\
y &= \frac{k}{x} \\
A &= Pe^{rt} \\
A &= P\left(1 + \frac{r}{n}\right)^n \\
A &= Pl + \frac{P}{n}
\end{align*}
\]

19. Which equations represent lines when graphed?

20. Which equations represent parabolas when graphed?

21. Which equation would you use to calculate interest earned in a bank account compounded daily?

22. Solve \[
\begin{align*}
3x + 4y &= 29 \\
2x - 5y &= -42
\end{align*}
\] (Lesson 5.3)

23. a. What is the probability of getting exactly 2 heads in 8 tosses of a fair coin?
   b. Multiple choice  To which binomial expansion is part a related?
      (i) \((x + y)^8\)  (ii) \((x + y)^6\)  (iii) \((x + y)^4\)  (iv) \((x + y)^2\)
   c. To which combination is part a related?
   d. To which row of Pascal’s Triangle is part a related?
      (Lessons 13-5, 13-6, 13-7, 13-9)

24. The three-dimensional coordinate system used in this book is an example of a right-handed system. Find out what is meant by a left-handed coordinate system. Draw the coordinate axes of a left-handed system.
In a plane, two lines may intersect in 0 points, exactly 1 point, or in a line (if they are the same line). Any point of intersection is a solution to the system of equations of the lines. In space, two planes either are parallel or they intersect in a line. Therefore, a system of 2 equations in 3 variables cannot have exactly one solution. At least three equations are needed to solve for 3 variables. With 3 equations in 3 variables, there are four possible situations. They correspond to the possible relative positions of the 3 planes.

<table>
<thead>
<tr>
<th>Possible intersections of 3 planes</th>
<th>Instance</th>
</tr>
</thead>
<tbody>
<tr>
<td>no points</td>
<td>planes containing the floor, ceiling, and one wall of a room</td>
</tr>
<tr>
<td>exactly 1 point</td>
<td>planes containing the floor and two intersecting walls intersect at a corner</td>
</tr>
<tr>
<td>a line</td>
<td>planes containing three pages of an open book</td>
</tr>
<tr>
<td>a plane</td>
<td>three identical (coincident) planes</td>
</tr>
</tbody>
</table>

Any point of intersection is a solution to the system of equations of the planes.

Such systems arise in many settings.

**Example 1**

In looking back over its records, a publishing company noted the following costs for publishing books.

Job 1: 60 hr design, 100 hr writing, 200 hr production
Total cost: $23,000

Job 2: 30 hr design, 300 hr writing, 400 hr production
Total cost: $49,500

Job 3: 40 hr design, 80 hr writing, 150 hr production
Total cost: $17,400

Write a system that could be solved to find how much the company charged per hour for design, writing, and production.
Solution  There are three variables. Let
\[ d = \text{the hourly charge for design}, \]
\[ w = \text{the hourly charge for writing}, \]
\[ p = \text{the hourly charge for production}. \]

Note that each job represents a linear combination of \( d, w, \) and \( p. \) So, the system is as follows:
\[
\begin{align*}
60d + 100w + 200p &= 23000 \\
30d + 300w + 400p &= 49500 \\
40d + 80w + 150p &= 17400
\end{align*}
\]

In Chapter 5 you learned how to solve 3 equations with 3 variables by using matrices. However, when the inverse of a 3 by 3 matrix is not given, there is another algebraic method to solve such systems. This method is an extension of the linear combination method you have used to solve a 2 \( \times \) 2 system.

**Example 2**  Solve the system of Example 1. That is, find the hourly charge for design, writing, and production.

**Solution**

**Step 1:** First simplify each equation, if possible. The first equation can be divided by 20, all others by 10. Then number them for reference.

(1): \( 3d + 5w + 10p = 1150 \)
(2): \( 3d + 30w + 40p = 4950 \)
(3): \( 4d + 8w + 15p = 1740 \)

**Step 2:** Find a variable to eliminate. Two coefficients of \( d \) are the same, so \( d \) is a good candidate for elimination.

**Step 3:** Subtract (1) from (2).

\[
(2) - (1): \quad 25w + 30p = 3800
\]

Divide by 5 to simplify. We call this (4).

(4): \( 5w + 6p = 760 \)

Another equation in \( w \) and \( p \) is needed. We use (1) and (3) and again eliminate \( d. \)

\[
\begin{align*}
4 \cdot (1): \quad 12d + 20w + 40p &= 4600 \\
3 \cdot (3): \quad 12d + 24w + 45p &= 5220
\end{align*}
\]

Subtract to get \( -4w - 5p = -620 \)

Multiply by -1 to simplify. Call it (5).

(5): \( 4w + 5p = 620 \)
Step 4: Use (4) and (5) to solve for $w$ and $p$ as you normally would.

(4): $5w + 6p = 760$
(5): $4w + 5p = 620$

$4 \cdot (4): \quad 20w + 24p = 3040$
$5 \cdot (5): \quad 20w + 25p = 3100$

So $p = 60$.

Step 5: Substitute in (4) or (5) to find $w = 80$. Then substitute in (1), (2), or (3) to find that $d = 50$.

The company seems to have charged $50/hr for design, $80/hr for writing, and $60/hr for production.

Check Substitute $d = 50$, $w = 80$, and $p = 60$ into each equation of the original system.

Does $60(50) + 100(80) + 200(60) = 23000$?
Yes, $3000 + 8000 + 12000 = 23000$.

Does $30(50) + 300(80) + 400(60) = 49500$?
Yes, $1500 + 24000 + 24000 = 49500$.

Does $40(50) + 80(80) + 150(60) = 17400$?
Yes, $2000 + 6400 + 9000 = 17400$.

The solution is correct.

Here is the general strategy for the method of linear combinations.

1. Simplify the equations, if possible.
2. Choose a variable to eliminate.
3. Take any two equations and eliminate the chosen variable. Then take another two equations and eliminate the same variable.
4. Solve the resulting system of two equations in two variables using techniques which you already know.
5. Substitute your solution from Step 4 into one of the three original equations to solve for the third variable.
6. Check that your solution works in all equations of the system.

In the second step of the solution, any of the three variables can be chosen for elimination. Sometimes a specific choice will allow you to eliminate two variables at once.
Example 3  Use the method of linear combinations to solve the following system:

\[
\begin{align*}
&x + 2y + z = 11 \\
&5x + y + 4z = 73 \\
&3x + 2y + z = 31 \\
\end{align*}
\]

Solution

Step 1: Again we number the equations for easy reference. The equations are already simplified.

(1): \( x + 2y + z = 11 \)
(2): \( 5x + y + 4z = 73 \)
(3): \( 3x + 2y + z = 31 \)

Step 2: Notice that in the first and third equations, the coefficients of \( y \) and \( z \) are equal. This means that if we subtract the equations we eliminate two variables at once.

Step 3: \( (3) - (1): \)

\[
2x = 20
\]

Thus, \( x = 10 \). Call this equation (4).

Step 4: Determine an equation with \( x \) and one other variable. That is, eliminate either \( y \) or \( z \). We eliminate \( z \), by multiplying the first equation by 4 and subtracting the second.

4 \cdot (1): \( 4x + 8y + 4z = 44 \)
(2): \( 5x + y + 4z = 73 \)
(5): \( -x + 7y = -29 \)

Now solve the 2 \( \times \) 2 system

(4): \( x = 10 \)
(5): \( -x + 7y = -29 \)

by adding (4) and (5).

\[
7y = -19 \\
y = -\frac{19}{7}
\]

Step 5: Substitute \( x = 10 \) and \( y = -\frac{19}{7} \) into one of the original equations and solve for \( z \). We use (1) because it has the smallest coefficients.

\[
10 + 2\left(-\frac{19}{7}\right) + z = 11 \\
\frac{32}{7} + z = 11 \\
z = \frac{45}{7}
\]

The solution to the system is \( x = 10, y = -\frac{19}{7}, z = \frac{45}{7} \). You should check this.
Remember that not all $3 \times 3$ systems have unique solutions. For instance, in the system

\[
\begin{align*}
(1): & \quad 3x + y - 2z = 6 \\
(2): & \quad x + 2y + z = 7 \\
(3): & \quad 6x + 2y - 4z = 12
\end{align*}
\]

notice that the third equation is a multiple of the first. Suppose we choose to eliminate $x$ by using these equations. We would multiply the first equation by 2 and subtract the third equation from the result.

\[
2 \cdot (1): \quad 6x + 2y - 4z = 12
\]
\[
(3): \quad \underline{6x + 2y - 4z = 12}
\]
\[
0 = 0
\]

As with $2 \times 2$ systems, a result such as $0 = 0$, which is always true, means that there are infinitely many solutions. In two-space this means that the two equations represent the same line. In three space there can be one of three possible interpretations: all three equations may represent the same plane; two equations represent the same plane which intersects the third plane in a line; or all three planes intersect in the same line. In the system above, equations (1) and (3) name the same plane. This plane intersects the plane named by the second equation in a line. This line yields an infinite number of solutions to the system.

**Questions**

**Covering the Reading**

In 1 and 2, *true or false*. Refer to the general linear combination strategy for solving a system of three equations in three variables.

1. The goal at first is to obtain a system with fewer variables.

2. It is sufficient to check a solution in two of the three equations of a $3 \times 3$ system.

3. Suppose a competitor to the publishing company discussed in Examples 1 and 2 quotes the following prices for the same number of hours.

<table>
<thead>
<tr>
<th>Job</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$21,200</td>
</tr>
<tr>
<td>2</td>
<td>$46,850</td>
</tr>
<tr>
<td>3</td>
<td>$16,100</td>
</tr>
</tbody>
</table>

a. Write a system of equations that could be used to determine the hourly charge for design, writing, and production.

b. Solve the system in part a.
4. Consider the system
\[
\begin{align*}
5x + y + 6z &= 3 \\
x - y + 10z &= 9 \\
5x + y - 2z &= -9
\end{align*}
\]

a. Which equations can be added or subtracted to eliminate two variables at once?
b. Solve this system, and check your solution.
c. What does the solution mean geometrically?

5. Refer to the following system.
\[
\begin{align*}
5x + 4y + 8z &= 170 \\
4x + 3y + 6z &= 132 \\
3x + 5y + 4z &= 130
\end{align*}
\]

a. You decide to eliminate \( x \) by adding two equations. By what numbers should you multiply both sides of the first two equations?
\begin{align*}
5x + 4y + 8z &= 170 & \text{multiply by } 2 \\
4x + 3y + 6z &= 132 & \text{multiply by } 2
\end{align*}
b. Use the second and third equations to eliminate \( x \).
c. Solve the original system using your results from parts a and b.

6. a. If you obtain a statement such as \( 0 = 0 \) while solving a \( 3 \times 3 \) system, then the system has how many solutions?
b. What does the solution to the system mean geometrically?

7. Solve and check:
\[
\begin{align*}
2x - 3y + 4z &= -16 \\
5x + 2y - 2z &= 15 \\
x + y - z &= 6
\end{align*}
\]

8. If, while solving a system, you get a false statement such as \( 5 = 0 \), what can you conclude about the solution to the system?

9. A bicycle, three tricycles, and a unicycle cost $208. Seven bicycles and a tricycle cost $399. Five unicycles, two bicycles and seven tricycles cost $657. What is the cost of one bicycle?

10. An experiment was conducted to find the height above ground, \( y \), of an object \( t \) seconds after being dropped from a tall building. The following results were obtained.
\[
\begin{array}{c|cccccc}
\hline
t & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
y & 800 & 784 & 736 & 656 & 544 & 400 \\
\hline
\end{array}
\]

a. It was expected that a polynomial model would describe the relationship between \( y \) and \( t \). Use the method of finite differences to test this hypothesis. If the hypothesis is correct, determine the degree of the polynomial.
b. Write a system of equations to find the polynomial. Solve the system using the methods of this lesson and write the polynomial model for the data.
11. Write the equation of the plane parallel to the $xz$-plane which contains $(5, -6, -15)$. \textit{(Lesson 14-2)}

12. a. Graph the system in 3-space: $x = 7, z = 2, z = -3$.
   
b. \textit{Multiple choice} The solution to the system in part a is
      (a) a point (b) a line (c) a plane (d) none of (a)–(c). \textit{(Lesson 14-2)}

13. Graph $(x - 3)^2 + (y + 4)^2 = 16$ on a coordinate plane. \textit{(Lesson 12-1)}

14. \textit{Multiple choice} Which situation(s) below can be modeled by $y = mx + b, b \neq 0$? \textit{(Lessons 3-1, 2-4)}
    (a) The depreciation of a car which retains 97% of its value each year.
    (b) The weight of a man on a diet who loses 2 pounds per week for six weeks.
    (c) The height of a rock thrown into the air after $x$ seconds.
    (d) The perimeter of a square varies directly as the length of the side.

15. A ball is dropped to the ground from a height of 10 meters and bounces up to 60% of its height each time it bounces. Answer all parts to the nearest tenth of a meter.
   a. How far does it travel before hitting the ground the second time?
   b. How far does it travel before hitting the ground the eighth time?
   c. How far does it travel before stopping? \textit{(Lessons 13-2, 13-4)}

16. Consider the functions: $f(x) = \cos x, g(x) = \log x, \text{and } h(x) = |x|$. \textit{(Lessons 10-9, 9-4, 7-5, 7-3)}
    a. Which function has an inverse which is also a function?
    b. How can you tell from the graph?

17. Consider the equation \( \frac{x^2}{121} + \frac{y^2}{100} = 1 \).
    a. Which conic does it describe?
    b. Name its vertices. \textit{(Lesson 12-4)}

18. a. Find equations for three different planes that contain both the points $(1, 2, 3)$ and $(4, 6, 8)$.
   b. Solve the system of three equations you found in part a to determine the intersection of these planes.
   c. Interpret your solution.
In 1-space, on a number line, the distance between two points \(x_1\) and \(x_2\) is \(|x_1 - x_2|\).

In 2-space, to determine the distance between two points which lie on a line parallel to the \(x\)-axis, you find the absolute value of the difference of the \(x\)-coordinates. For instance, if \(A = (-4, 3)\) and \(B = (8, 3)\),

\[
AB = |-4 - 8| = 12.
\]

In general, if \(A\) and \(B\) are the points \((x_1, y_1)\) and \((x_2, y_2)\) on a line parallel to the \(x\)-axis,

\[
AB = |x_1 - x_2|.
\]

Similarly, if \(C\) and \(D\) are two points on a line parallel to the \(y\)-axis (as pictured above),

\[
CD = |y_1 - y_2|.
\]

Thus, in both 1-space and 2-space the distance between two points on a line parallel to a coordinate axis is the absolute value of the difference of the unequal coordinates. This is true also in 3-space.
Example 1

If \( R = (-5, 3, 2), \ S = (-5, 1, 2), \) and \( T = (-5, 3, -4), \) find

a. \( RS; \)

b. \( RT. \)

Solution

a. \( R \) and \( S \) lie on a line parallel to the \( y \)-axis. The \( y \)-coordinates are unequal. So
\[
RS = |y_1 - y_2| = |3 - 1| = 2.
\]

b. The line through \( R \) and \( T \) is parallel to the \( z \)-axis. Since the \( z \)-coordinates are not equal,
\[
RT = |z_1 - z_2| = |2 - (-4)| = 6.
\]

For distances in 2-space, when two points lie on an oblique line it is necessary to use the Distance Formula. Let \( E \) and \( F \) be points in 2-space with coordinates \((x_1, y_1)\) and \((x_2, y_2)\), respectively.

Then,
\[
EF = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\]

You can think of \( EF \) as the length of a diagonal of a rectangle.

Similarly, to find \( AB \) if \( A = (x_1, y_1, z_1) \) and \( B = (x_2, y_2, z_2) \), we first draw the rectangular box with base parallel to the \( xy \)-plane. Then \( AB \) is the longest diagonal of the box, and is also the hypotenuse of right triangle \( ABC \). By the Pythagorean Theorem,
\[
AB^2 = AC^2 + BC^2.
\]
Since $\overline{AC}$ is the hypotenuse of right triangle $ACD$, by substitution we have

$$AB^2 = CD^2 + AD^2 + BC^2.$$ 

Now $CD = |x_1 - x_2|$, $AD = |y_1 - y_2|$, and $BC = |z_1 - z_2|$. So by substitution,

$$AB^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

Take the square root of both sides:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This proves the following theorem.

**The Distance Formula in 3-Space**

The distance $d$ between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

**Example 2** Find the distance $d$ between points $P = (-2, 4, 6)$ and $Q = (0, 3, -5)$.

**Solution**

$$d = \sqrt{(0 - (-2))^2 + (3 - 4)^2 + (-5 - 6)^2}$$

$$= \sqrt{4 + 1 + 121}$$

$$= \sqrt{126}$$

$$\approx 11.2$$

In 2-space the set of points at a given distance from a fixed point is a circle with the fixed point as center and the given distance as the radius. In 3-space the set of points at a given distance from a fixed point is a sphere. Like a circle, a sphere is determined by its center (the fixed point) and its radius (the given distance).
In 2-space the circle with center \((0, 0)\) and radius \(r\) has equation \(x^2 + y^2 = r^2\). Now consider the sphere on a three-dimensional graph with center at the origin \((0, 0, 0)\) and a radius \(r\). If \((x, y, z)\) is any point on the sphere, then using the Distance Formula:

\[
r = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}
\]

\[
r = \sqrt{x^2 + y^2 + z^2}
\]

So \(r^2 = x^2 + y^2 + z^2\).

This proves the following theorem.

**Theorem**

The sphere with center \((0, 0, 0)\) and radius \(r\) has equation

\[
x^2 + y^2 + z^2 = r^2.
\]

**Example 3** Find an equation for the sphere with center at the origin and a radius of 8.

**Solution** Here \(r = 8\). So an equation is

\[
x^2 + y^2 + z^2 = 8^2
\]

or

\[
x^2 + y^2 + z^2 = 64.
\]

**Check** Graph the sphere by locating the points eight units on each axis in either direction from \((0, 0, 0)\). There are six such points: \((8, 0, 0), (-8, 0, 0), (0, 8, 0), (0, -8, 0), (0, 0, 8), \) and \((0, 0, -8)\). All of these triples should satisfy the equation

\[
x^2 + y^2 + z^2 = 64.
\]
Equations for spheres whose centers are not at the origin are also possible. You might even be able to guess what they are. Question 15 asks you to think about this idea.

Questions

1. In 2-space, if points $C = (x_1, y_1)$ and $D = (x_2, y_2)$ lie on a line parallel to the $x$-axis, then $CD = \_\_\_\_\_\.$

2. Let $P = (3, 4, 5)$ and $Q = (3, -2, 5)$.
   a. To which axis is $PQ$ parallel?
   b. Find $PQ$.

3. In words, explain how to find the length of a segment parallel to the $z$-axis.

4. Refer to the proof of the Distance Formula for 3-space.
   a. Why is $AB^2 = AC^2 + BC^2$?
   b. Which is the right angle in $\triangle ACD$?
   c. Which is the right angle in $\triangle ABC$?

In 5–8, find the distance between the points to the nearest tenth.

5. $M = (1, 2, 11)$ and $N = (7, 2, 9)$

6. $O = (0, 0, 0)$ and $T = (-1, 2, 2)$

7. $P = (0, -16, 4.3)$ and $Q = (-1.2, 6, -3.1)$

8. $(-2, -2, -2)$ and $(3, 3, 3)$

9. Finish this definition. A sphere is the set of points $\_\_\_\_\_\_\_\_\_\_\_\_\_\.$

10. Give an equation for the sphere with radius 7 and center at the origin.

11. Graph: $x^2 + y^2 + z^2 = 4$.

12. Triangle $ABC$ has vertices $A = (2, -1, 7)$, $B = (4, 0, -5)$, and $C = (-11, 8, 2)$. Find the perimeter of $\triangle ABC$.

13. Use the drawing at the right.
   a. Find an equation for the sphere with center $(0, 0, 0)$ and radius 5.
   b. Give the coordinates of a point on the sphere that is not on any of the axes.
14. Describe the set of points that satisfy the sentence
   \[ x^2 + y^2 + z^2 \leq 144. \]

15. The set of points satisfying the equation
   \[ (x - 3)^2 + (y - 2)^2 + (z + 5)^2 = 36 \]
   is a sphere whose center is not at the origin.
   a. Where is the center of the sphere?
   b. What is the radius of the sphere?
   c. Give the coordinates of two points on the sphere.
   d. Find an equation for the sphere with center \((a, b, c)\) and radius \(r\).

16. Question 15 suggests a 3-space analogue to the Graph Translation Theorem. State a Graph Translation Theorem for 3-space.

17. A daredevil is going to propel a motorized bike down a tightrope. The rope extends from a platform 30 meters high, 150 meters east and 40 meters north of its end, which is another platform 10 meters high. How long a rope is needed?

18. An office furniture company has a sale on file cabinets. Luke’s company bought 3 two-drawer, 3 three-drawer, and 6 four-drawer cabinets for $704.88. Obie’s company bought 4 two-drawer, 1 three-drawer, and 8 four-drawer cabinets for $774.87. Ben’s company bought 1 two-drawer, 1 three-drawer, and 12 four-drawer cabinets for $934.86.
   a. What is the cost of each type of cabinet? (Lesson 14-3)
   b. Use your answers to part a to find an equation which relates the price \(P\) to the number of drawers \(n\). (Lesson 3-5)
   c. According to your equation in part b, how much would a five-drawer cabinet cost? (Lesson 1-1)
19. Refer to the appendix of geometry formulas if necessary.
   a. Find the area of the base of a right circular cylinder whose volume is 108π cm³ if the height is 9 cm.
   b. Find the radius of the base of the cylinder in part a.
   c. Find the radius of a sphere with the same volume as the cylinder in part a.
   d. Which has more surface area, the cylinder in part a or the sphere in part c? (Lessons 8-1, 1-2, Previous Course)

20. a. Solve using the linear combination method. (Lessons 5-3, 12-10)
   \[
   \begin{align*}
   9x^2 - 6y^2 &= 291 \\
   6x^2 - 4y^2 &= 394
   \end{align*}
   \]
   b. Part a finds the points of intersection of what curves?
   (Lessons 12-7, 12-8)

21. A **lattice point in 3-space** is a point (x, y, z) in which x, y, and z are integers. Find ten lattice points on the sphere with equation \(x^2 + y^2 + z^2 = 66\).

22. In 2-space, the midpoint of the segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) is \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).
   a. Conjecture a formula for the midpoint of the segment with endpoints \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).
   b. Test your conjecture by finding the midpoint of the segment with endpoints \((5, -1, 6)\) and \((3, 5, -10)\).
   c. Is the point you got in part b equidistant from these endpoints? If not, revise your conjecture.
A *lathe* is a machine which holds a solid piece of wood or metal and spins it around at high speeds. By having a tool dig into the piece while it spins, circular cuts can be made in the piece. The result is a *solid of revolution*.

Some common figures are solids or surfaces of revolution. Begin with the circle in the $xy$-plane with radius $r$ centered at the origin. Imagine rotating the circle about the $x$-axis. Each point on the circle, except those on the $x$-axis, moves about the $x$-axis in a circular path. The entire circle traces, or *generates*, a sphere. An equation for the sphere is $x^2 + y^2 + z^2 = r^2$. 

$$x^2 + y^2 = r^2 \quad \text{in the } xy\text{-plane}$$
The surface which is generated by rotating a curve in a plane about a line is called a **surface of revolution**. The line about which the curve is rotated is the **axis of rotation**. In the previous diagram, the \( x \)-axis is the axis of rotation. Each cross section perpendicular to the axis of rotation is a circle. When a lamp base or leg of a chair is made with a lathe, the axis of rotation is vertical and the horizontal cross-sections are **disks** (unions of circles and their interiors).

The sphere could have been generated by rotating only a semicircle. The semicircle defined by \( x^2 + y^2 = r^2 \) for \( y \geq 0 \) (in the first and second quadrants of the \( xy \)-plane) could generate the whole sphere. The half-disk shaded would generate a solid ball.

Example 1 The line segment \( \overline{AB} \), with \( A = (0, 4, 6) \) and \( B = (0, 4, 1) \), is rotated about the \( z \)-axis. Describe the surface which is generated.

**Solution** Plot \( \overline{AB} \). Make a sketch of its path as it rotates about the \( z \)-axis. Each point on \( \overline{AB} \) moves in a circular path about the \( z \)-axis. So the cross section parallel to the \( xy \)-plane is a circle. Because \( \overline{AB} \) is parallel to the \( z \)-axis, each circle has the same radius, 4. Also, \( AB = \sqrt{6} \). So, the surface of revolution is a cylinder with radius 4 and height 5.
In Example 1, when we identified the surface of revolution as a cylinder we described it geometrically. We can also describe the surface algebraically. Choose any cross section of the cylinder, such as the one at the right, with \( C \) in the \( yz \)-plane. Then \( C \) has coordinates \((0, 4, z)\), where \(1 \leq z \leq 6\). If \( D \) is the center of this cross section, its coordinates are \((0, 0, z)\). Let \( E \) be any other point on the same cross section. Then \( E \) has coordinates \((x, y, z)\). Thus,

\[
DE = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - z)^2} = \sqrt{x^2 + y^2}.
\]

Because the cross section of the cylinder is a circle, radii \( DE \) and \( DC \) have the same length; i.e.,

\[
DE = DC.
\]

Since \( DC = 4 \), substituting into both sides we have

\[
\sqrt{x^2 + y^2} = 4.
\]

Square both sides of this equation to get

\[
x^2 + y^2 = 16.
\]

So, an analytic description of this cylinder is

\[
x^2 + y^2 = 16, \quad 1 \leq z \leq 6.
\]

In the \( xy \)-plane, \( x^2 + y^2 = 16 \) is a circle. Any cross section of the cylinder of Example 1 is the circle \( x^2 + y^2 = 16 \) in the plane \( z = c \) where \(1 \leq c \leq 6\). The cylinder also can be considered as the surface traced by sliding \( x^2 + y^2 = 16 \) in the plane \( z = 1 \) to \( x^2 + y^2 = 16 \) in the plane \( z = 6 \).

In Example 1, a vertical line in the \( yz \)-plane was rotated about the \( z \)-axis. Example 2 shows what results when an oblique line through the origin is rotated about this axis.

**Example 2**

Let \( P = (0, 0, 0) \) and \( Q = (0, 5, 8) \). Suppose \( PQ \) is rotated about the \( z \)-axis. Geometrically describe the surface which is generated.
Solution  If you spin $\overline{PQ}$ around the z-axis, you get a cone. Call $C$ the center of the base of the cone. Then $C = (0, 0, 8), CQ = 5$ and $CP = 8$. So the surface of revolution is a cone of radius 5 and height 8.

Notice that each cross section of the cone perpendicular to its axis is a circle. You would expect this both from the properties of conic sections and from the properties of surfaces of revolution.

You can rotate other curves to generate surfaces of revolution. Recall that in 2-space a parabola is the set of points equidistant from a point $F$ (the focus) and a line $l$ (the directrix). In 3-space a paraboloid is the set of points equidistant from a point $F$ (the focus) and a plane $P$. If a parabola is rotated about its axis of symmetry, the paraboloid generated has the same focus as the parabola.

The paraboloid possesses an incredible reflection property. Any ray parallel to the axis of symmetry of a paraboloid (or parabolic) mirror is reflected into the focus. For this reason, paraboloids are the shape of radio telescope receivers and solar collectors to concentrate faint incoming parallel waves. This reflection property is used in reverse to make automobile headlights and search lights. A light is placed at the focus and reflects off the parabolic mirror to form a cylindrical beam of light. Paraboloids are sometimes used in both ways; the focus of a radar transmitter can alternately receive or send waves.

Questions

1. **a.** What is a lathe?  
   **b.** Give an example of an object made with a lathe.

2. A curve in a plane is rotated about a coordinate axis. What path does each point follow?

3. **a.** Define: surface of revolution.  
   **b.** Give an example of a surface of revolution.
4. a. What does it mean to describe a surface algebraically or analytically?
b. What does it mean to describe a surface geometrically?

5. The graph of \( y^2 + z^2 = 50, z \geq 0 \), is a semicircle in the \( yz \)-plane. The curve is rotated about the \( y \)-axis.
   a. What geometric figure is generated?
b. Find an equation for the surface of revolution.

6. Let \( M = (3, 0, 2) \) and \( N = (10, 0, 2) \). \( MN \) is rotated about the \( x \)-axis.
   a. Sketch the surface generated.
b. Describe it geometrically.

7. The cylinder in Example 1 can be traced by sliding \( x^2 + y^2 = 16 \) from the plane \( z = 1 \) to the plane \( ? \).

8. In Example 2, suppose \( PQ \) were rotated about the \( y \)-axis.
   a. Draw the surface of revolution.
b. Describe the surface of revolution in as much detail as you can.


10. a. Where can you find paraboloids in the world?
b. What property causes them to have so many uses?

11. The circle \( x^2 + y^2 = 36 \) is rotated around the \( y \)-axis.
   a. What figure is generated?
b. What is its volume?
c. What is its surface area?

12. The line \( y = x \) in the \( xy \)-plane is rotated about the \( x \)-axis. Describe geometrically the surface generated.

In 13–15, refer to the diagram at the right. Sketch the graph of the surface generated by rotating the curve from \( A \) to \( B \), where \( B = (1,2,3) \), around the

13. \( z \)-axis
14. \( y \)-axis
15. \( x \)-axis.

16. Consider the cylinder defined by \( x^2 + y^2 = 36 \) for \( 0 \leq z \leq 8 \).
   a. Sketch its graph.
b. Find its volume.

17. The curve \( (x - 3)^2 + (y - 2)^2 = 1 \) is rotated about the \( x \)-axis. What figure is traced?
18. Find the distance between (-3, 4, 8) and (5, 2, 8). (Lesson 14-4)

19. True or false The sphere \((x - 3)^2 + (y - 3)^2 + (z - 3)^2 = 4\) is contained entirely within the octant where all three coordinates are positive. (Lesson 14-4)

20. Solve the following system: \[
\begin{align*}
x &= y + z + 100 \\
y &= x + z + 100 \\
z &= x + y + 100
\end{align*}
\] (Lesson 14-3)

21. The discriminant of a quadratic equation \(ax^2 + bx + c = 0\) is -900. What does this tell you about the graph of \(y = ax^2 + bx + c\)? (Lesson 6-7)

22. Some students make earrings and pendants in their spare time and sell all that they make. Every week they have available 10 kg of metal and 20 hours to work. It takes 48 g of metal to make an earring and 210 g to make a pendant. Each earring takes 30 minutes to make and each pendant takes 20 minutes. The profit on each earring is $3.00, and the profit on each pendant is $2.25. The students want to earn as much money as possible. Because you are taking this course, they ask you to give them advice. What numbers of earrings and pendants should they make each week? (Lesson 5-8)

23. True or false If matrix A has dimensions \(3 \times 4\), then it has 3 columns and 4 rows. (Lesson 4-1)

24. A tennis ball is thrown upwards at 35 ft/sec from a height of 6.5 feet. When will it hit the ground? (Remember: \(h = -16t^2 + v_0t + h_0\).) (Lesson 6-2)

25. Multiple choice What kind of sampling is usually used in obtaining TV ratings? (Lesson 13-12)
   (a) normal sampling  (b) random sampling
   (c) stratified sampling  (d) Sampling is not used.

26. The Earth is (more or less) a solid of revolution called an oblate spheroid. What exactly is an "oblate spheroid"?
You know that a line segment is a one-dimensional figure, a square has two dimensions, and a cube has three. You may be surprised to learn that there is a four-dimensional figure analogous to these others. Such a figure is called a hypercube.

Reasoning by analogy from the figures in lower dimensions enables us to visualize a hypercube and to study some of its properties. Note that a square can be formed by first translating a line segment of length $s$ a distance of $s$ units in a direction perpendicular to the segment, and then connecting corresponding vertices. Because the segment has two vertices, the square has four: two from the preimage segment and two from the image segment. The square also has four edges: two are the preimage and image segments, and two connect the corresponding vertices.

Similarly, the edges of a cube of side $s$ can be formed by translating a square of side $s$ a distance of $s$ units in a direction perpendicular to the plane of the square, and then connecting corresponding vertices. The number of vertices and edges of a cube can be found from the number of vertices and edges of a square. The cube has 8 vertices: 4 from the preimage and 4 from the image. It has 12 edges: 8 are sides of the preimage and image squares and 4 join the corresponding vertices. Notice that to draw a cube on a page, many of the right angles on its faces must be distorted.

The four-dimensional hypercube is formed by generalizing the above process. Imagine a cube of side $s$ being translated $s$ units in a direction perpendicular to its space, and connecting corresponding vertices. Unfortunately, as in the case of picturing 3-dimensional
objects on a 2-dimensional page, a picture of a hypercube distorts perpendicular lines and planes. Although it may be difficult to visualize a hypercube, we know many of its properties. For instance, it has 16 vertices: 8 from each of the preimage and image cubes. The hypercube has 32 edges: 24 from the preimage and image cubes, and 8 formed by joining the corresponding vertices of the cube.

Four-dimensional figures can also be studied through coordinates. A single real number \( x \) locates a point on a number line (1-space). An ordered pair \((x, y)\) of real numbers locates a point in 2-space; and an ordered triple \((x, y, z)\) of real numbers locates a point in 3-space. We define 4-space as the set of ordered 4-tuples \((x, y, z, w)\) of real numbers. The point \((0, 0, 0, 0)\) is the origin of this four-dimensional coordinate system; the point \((0, -4, 0, 0)\) is on the \( y \)-axis. It may seem strange that the coordinates are not written in alphabetical order, but the order \( x, y, z, w \) maintains \( x \) as first, \( y \) as second, and \( z \) as third.

The distance between points in 4-space can be determined in ways similar to distance in 2- or 3-space. In any space, the distance between two points on a line parallel to one of the coordinate axes is the absolute value of the difference of the unequal coordinates. This can be generalized to find the distance between any two points in 4-space.

**The Distance Formula in 4-Space**

The distance \( d \) between the points \((x_1, y_1, z_1, w_1)\) and \((x_2, y_2, z_2, w_2)\) is given by

\[
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2}.
\]
Example 1  Find the distance between the points in 4-space with coordinates (1, 10, 4, 3) and (1, 7, -2, 5).

Solution  Apply the distance formula for 4-space.

\[ d = \sqrt{(1 - 1)^2 + (10 - 7)^2 + (4 - (-2))^2 + (3 - 5)^2} \]
\[ = \sqrt{0 + 9 + 36 + 4} \]
\[ = \sqrt{49} \]
\[ = 7 \]

The distance between the points (1, 10, 4, 3) and (1, 7, -2, 5) is 7.

In 4-space, a **hypersphere** is the set of all points at a given distance from a fixed point. Thus, the set of points \((x, y, z, w)\) at a distance \(r\) from the origin satisfies the equation

\[ r = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2 + (w - 0)^2}. \]

**Theorem**

In 4-space, an equation for a hypersphere with center at \((0, 0, 0, 0)\) and radius \(r\) is

\[ x^2 + y^2 + z^2 + w^2 = r^2. \]

Example 2  Find an equation for the hypersphere in 4-space with center at the origin and radius 10.

Solution  Apply the theorem above. Here \(r = 10\).

An equation for this hypersphere is

\[ x^2 + y^2 + z^2 + w^2 = 100. \]

One point on this sphere is \((7, 1, 7, -1)\). Can you find others?

In the Questions you will see how a hypercube can be coordinatized. Besides ordered 4-tuples, points in 4-space can be represented as matrices. Just as the matrix \( \begin{bmatrix} x \\ y \end{bmatrix} \) can be used to represent the point \((x, y)\) in 2-space and the matrix \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) to represent \((x, y, z)\) in 3-space, the matrix \( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \) represents the point \((x, y, z, w)\) in 4-space.
Other figures in 4-space can then be represented by matrices with four rows. By applying matrix operations, computers are able to generate images of four-dimensional objects on two-dimensional computer screens.

Four-dimensional coordinate systems have many applications. For instance, a school that keeps records of a student’s social security number $S$, year of graduation $Y$, grade point or grade average $G$, and rank in class $R$ can store these data as an ordered 4-tuple $(S, Y, G, R)$. Then the records of many students can be stored in a matrix with four rows. In fact, any system of equations or inequalities with four variables can be considered as an algebraic problem in 4-space.

In Einstein’s theory of special relativity, $x$, $y$, and $z$ represent the ordinary rectangular coordinates of 3-space and $t$ represents a time coordinate. Each 4-tuple $(x, y, z, t)$ represents the position of an object in space and time. Time is thus the fourth dimension in Einstein’s coordinate system.

The concept of point as an ordered $n$-tuple generalizes to more than four dimensions. For instance, 6-space is a world in which each point is located by an ordered 6-tuple

$$(x_1, x_2, x_3, x_4, x_5, x_6).$$

Geometric models for such higher dimensional spaces are hard to visualize but through the use of coordinates and computers, problems in such spaces can be solved and figures we cannot visualize can be analyzed.

Questions

1. Name two four-dimensional figures described in the lesson.

2. State the number of vertices and edges for each figure.
   a. segment
   b. square
   c. cube
   d. hypercube

3. In 4-space, the distance $d$ between the points $(x_1, y_1, z_1, w_1)$ and $(x_2, y_2, z_2, w_2)$ is given by the formula $d = \sqrt{?}$.

In 4 and 5, find the distance between $P$ and $Q$.

4. $P = (3, -2, 2, 0)$  \hspace{1cm} $Q = (3, 5, 2, 0)$

5. $P = (1, 2, 3, 4)$  \hspace{1cm} $Q = (0, 3, 2, 3)$

6. Find an equation for a four-dimensional hypersphere with center at the origin and radius 8.5.
7. What is the radius of the hypersphere whose equation is 
   \( x^2 + y^2 + z^2 + w^2 = 16 \)?

8. What do the four coordinates \((x, y, z, t)\) in Einstein's theory of special 
   relativity represent?

9. True or false The point \((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})\) 
   represents a point in 10-space.

10. In 2-space the midpoint of the line segment whose endpoints have 
    coordinates \((x_1, y_1)\) and \((x_2, y_2)\) has coordinates 
    \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).
    
    a. State a conjecture about the coordinates of the midpoint of a 
       segment in 4-space whose endpoints have coordinates 
       \((x_1, y_1, z_1, w_1)\) and \((x_2, y_2, z_2, w_2)\).

    b. Using your conjecture in part a calculate the coordinates of the 
       midpoint \(M\) of \(PQ\) where \(P = (1, 10, 4, 3)\) and \(Q = (1, 7, -2, 5)\).

    c. Using your answer to part b, find \(PM\) and \(MQ\). Does each distance 
       equal \(\frac{1}{2} PQ\)?

In 11 and 12, use the sixteen points given here. They are vertices of a 
hypercube.

\[
\begin{align*}
P_1 &= (0, 0, 0, 0) & P_9 &= (0, 1, 1, 0) \\
P_2 &= (1, 0, 0, 0) & P_{10} &= (0, 1, 0, 1) \\
P_3 &= (0, 1, 0, 0) & P_{11} &= (0, 0, 1, 1) \\
P_4 &= (0, 0, 1, 0) & P_{12} &= (1, 1, 1, 0) \\
P_5 &= (0, 0, 0, 1) & P_{13} &= (1, 1, 0, 1) \\
P_6 &= (1, 1, 0, 0) & P_{14} &= (1, 0, 1, 1) \\
P_7 &= (1, 0, 1, 0) & P_{15} &= (0, 1, 1, 1) \\
P_8 &= (1, 0, 0, 1) & P_{16} &= (1, 1, 1, 1)
\end{align*}
\]

11. True or false The points \(P_1, P_2, P_3, P_4\) are the vertices of a square. 
    If true, justify your answer. If false, find four points from the list 
    which are the vertices of a square with side 1.

12. A diagonal of a hypercube is the longest line segment connecting its 
    vertices. Find the length of a diagonal of this hypercube.

13. Use your answers to Question 2 to make a conjecture relating the 
    dimension \(n\) of the figure and the number of vertices \(v\) of an 
    \(n\)-dimensional cube.

14. Refer to the figure at the right, which shows another way to draw a hypercube by drawing 
    a cube within a cube and connecting corresponding vertices with line segments.
    
    a. How many line segments must be drawn 
       to make the hypercube?

    b. Copy the figure and join corresponding 
       vertices.

    c. Explain how a 3-dimensional cube can be 
       accurately drawn in a similar manner.
15. a. Describe the intersection of the $xy$-plane and the sphere with center at the origin and radius 8.

b. Give an equation in the $xy$-plane for the cross section of part a. 

(Lesson 14-5)

16. The line segment from $(2, 5, 3)$ to the origin is rotated around the $y$-axis.

a. Describe the surface generated.

b. Find the volume of the surface. (Lesson 14-5)

17. Three ounces of cheddar cheese, ten seedless grapes, and one large carrot were eaten by Vivian at a party, and she recorded 181 calories. Jan had one ounce of cheese, two grapes, and two carrots for a total of 94 calories. Bob had twenty grapes, three carrots, and a half ounce of cheese and a total of 211 calories. What can be deduced from this information? (Lesson 14-3)

18. If a test is standardized with a mean of 500 and a standard deviation of 100, about what percent of test-takers would be expected to score between 100 and 600? (Lesson 13-10)

19. Give an example of normalized scores whose mean is 500 and whose standard deviation is 100. (Lesson 13-10)

In 20–25, describe the graph in one word. Do not actually draw the graph. 

(Lessons 2-7, 3-4, 6-2, 12-6, 14-2)

20. $\{(x, y): y = 4x^2 - 4x - 4\}$

21. $\{(x, y): x = \frac{7}{y}\}$

22. $\{(x, y): x + y = 0\}$  

23. $\{(x, y): x^2 + y^2 = 0\}$

24. $\{(x, y, z): y = 4\}$  

25. $\{(x, y): y = 4\}$

26. Solve for $x$: $3^x = 21$ (Lesson 9-9)

27. Find an equation of a polynomial with zeros at $(0, 0)$, $(-2, 0)$, and $(4, 0)$. (Lesson 11-5)

28. a. Solve the system

\[
\begin{align*}
2x - 7y &= 15 \\
-5x + y &= 1
\end{align*}
\]

using matrices.

b. Check by solving using some other method.

(Lessons 5-2, 5-3, 5-4, 5-6)

29. The book Flatland by Edwin Abbott Abbott (his real name) describes a world inhabited by two-dimensional beings who have the misfortune of being visited by a sphere from the third dimension. Read this book and describe how we as inhabitants of a three-dimensional world might perceive a visit by a four-dimensional hypersphere.
A system with \( m \) equations and \( n \) variables is said to have dimensions \( m \times n \). The numbers \( m \) and \( n \) may be any positive integers. In the last lesson, you read that 4-tuples can represent figures in space and time or four-category records of students. In earlier lessons, you learned about linear programming applications with hundreds or thousands of variables. The linear combination and substitution methods useful with systems containing 2 or 3 variables can be used to solve systems where \( m \) and \( n \) are greater than 3. But there is a great deal of writing; a more efficient means is needed.

Consider this system.

\[
\begin{align*}
(1) & \quad 4x - 2y + 3z + 7w = 1 \\
(2) & \quad -5y + 2z - w = -32 \\
(3) & \quad z + w = 5 \\
(4) & \quad 6w = -6
\end{align*}
\]

It is easy to solve if you begin with equation (4). From \( w = -1 \), substituting in equation (3) yields \( z = 6 \). Then substitute for \( z \) and \( w \) in equation (2) to get \( y = 9 \). Finally, substitute for \( y \), \( z \), and \( w \) in equation (1) to find \( x = 2 \).

The coefficient matrix is \( 4 \times 4 \), the same as the dimensions of the system. It is called a **triangular matrix** because there are all zeros in the lower left corner below the diagonal. If a coefficient matrix is in triangular form, the system is easy to solve. So to solve a system easily we wish to convert it into one whose coefficients form a triangular matrix.

To do this, rewrite the system as an **augmented matrix** consisting of the coefficients and the constants. For instance,

\[
\begin{align*}
5a + 4b + 8c - 2d &= 28 \\
-3a - 4b + 2c + 3d &= 16 \\
a + b + c &= 4 \\
2a + 3b - 2c + 5d &= 1
\end{align*}
\]

becomes

\[
\begin{bmatrix}
5 & 4 & 8 & -2 & 28 \\
-3 & -4 & 2 & 3 & 16 \\
1 & 1 & 1 & 0 & 4 \\
2 & 3 & -2 & 5 & 1
\end{bmatrix}
\]

Everything that can be done with the system can be done with the matrix; and it is less work not to have to write the \( a\), \( b\), \( c\), and \( d\). Multiplying both sides of an equation by \( k \) means multiplying a row by \( k \). Adding or subtracting equations corresponds to adding or subtracting rows. Switching the order of equations means switching rows. These are the legal **row operations** for matrices representing systems:

- a. Any row can be multiplied by any non-zero number.
- b. Any two rows may be added and one of the rows replaced with the sum.
- c. Any two rows may be switched.
Using the row operations wisely, any system can be converted into an augmented triangular matrix. Here is how that is done with the above matrix. Notice how we systematically convert the matrix into one in which the lower left hand corner is a triangular array of 0s.

Step 1: Switch rows (R1) and (R3) to get the number 1 in the upper left corner.

Step 2: Multiply row (R1) by 3 and add the result to row (R2). This makes the first element in row (R2) equal to 0.

Step 3: Repeat the idea of Step 2 with rows (R3) and (R4). That is, multiply row (R1) by -5 and add it to row (R3). Then multiply row (R1) by -2 and add it to row (R4).

Step 4: Switch rows (R2) and (R4). This gets a 1 in the second element of row (R2).

Step 5: Now add row (R2) to row (R3) for the new row (R3). Then add row (R2) to row (R4) for the new (R4).

Step 6: Switch rows (R3) and (R4).

Step 7: Add row (R3) to row (R4). The result is a matrix in augmented triangular form.

The matrix from step 7 corresponds to the following system:

\[
\begin{align*}
  a + b + c &= 4 \\
  b - 4c + 5d &= -7 \\
  c + 8d &= 21 \\
  11d &= 22
\end{align*}
\]

From the bottom, \( d = 2 \), then \( c = 5 \), \( b = 3 \), and \( a = -4 \).
Geometrically, an equation of the form \( Ax + By + Cz + Dw = E \) is a hyperplane. Solving the above system obtains the unique point of intersection of four hyperplanes.

Although it takes a lot of work to solve a system using augmented matrices, the advantages of this method are that it is easily done by computer and that it can be used with any linear system.

Example

Use augmented matrices to solve this system:

\[
\begin{align*}
3u + 3v - w &= 1 \\
4u - 2v + 4w &= 3 \\
5u + 8v - 2w &= 2 \\
\end{align*}
\]

Solution

1. Write the system as an augmented matrix.

   \[
   \begin{bmatrix}
   3 & 3 & -1 & 1 \\
   4 & -2 & 4 & 3 \\
   5 & 8 & -2 & 2 \\
   \end{bmatrix}
   \]

2. In order to make the first entry in the second row a 0, multiply the first row by \(-4/3\) and add it to the second row.

   \[
   \frac{-4}{3} \cdot (R1) + (R2):
   \begin{bmatrix}
   3 & 3 & -1 & 1 \\
   0 & -6 & \frac{16}{3} & \frac{5}{3} \\
   5 & 8 & -2 & 2 \\
   \end{bmatrix}
   \]

3. In order to make the first entry in the 3rd row 0, multiply the first row by \(-5/3\) and add it to the third row.

   \[
   \frac{-5}{3} \cdot (R1) + (R3):
   \begin{bmatrix}
   3 & 3 & -1 & 1 \\
   0 & -6 & \frac{16}{3} & \frac{5}{3} \\
   0 & 3 & -1 & 1 \\
   \end{bmatrix}
   \]

4. To make the second entry in the 3rd row 0, multiply the second row by \(1/2\) and add it to the third row.

   \[
   \frac{1}{2} \cdot (R2) + (R3):
   \begin{bmatrix}
   3 & 3 & -1 & 1 \\
   0 & -6 & \frac{16}{3} & \frac{5}{3} \\
   0 & 0 & 7 & 7 \\
   \end{bmatrix}
   \]

   The matrix is triangular but has fractions.

5. To clear the fractions, multiply the second row by 3 and the third row by 6.

   \[
   \begin{bmatrix}
   3 & 3 & -1 & 1 \\
   0 & -18 & 16 & 5 \\
   0 & 0 & 14 & 7 \\
   \end{bmatrix}
   \]

   The last matrix corresponds to the system:

\[
\begin{align*}
3u + 3v - w &= 1 \\
-18v + 16w &= 5 \\
14w &= 7
\end{align*}
\]

From the last equation, \( w = \frac{1}{2} \). Substituting in the second equation, \( v = \frac{1}{6} \). Then, substituting in the first equation, \( u = \frac{1}{3} \).
It is very difficult to solve a large system by hand without making an error. To help locate an error, remember that each row of an augmented matrix corresponds to an equation the solution must satisfy. For instance, in step 3 of the Example, the bottom row corresponds to $3v - \frac{1}{3}w = \frac{1}{3}$. This is satisfied by $v = \frac{1}{6}$ and $w = \frac{1}{2}$. If there were an error in the solution, somewhere there would be rows not satisfied by it. By locating where the errors begin, you can find where the arithmetic is faulty.

Questions

1. Solve the following system:
   \[
   \begin{align*}
   -x + 2y + 3z + w &= 50 \\
   5y - 4z - 6w &= 6 \\
   2z + 3w &= 19 \\
   4w &= 4
   \end{align*}
   \]

2. Geometrically, the system of Question 1 can be interpreted as finding the point of intersection of four ___.

3. If the matrix at the right is a triangular matrix, which elements equal 0?

   \[\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\]

4. **Multiple choice** Which is not a legal row operation?
   (a) Multiply each element of a row by 2.
   (b) Add 3 to each element of a row.
   (c) Add two rows and replace the first row with the sum.
   (d) Switch two rows.

5. a. Write the augmented matrix for the system at the right:
   \[
   \begin{align*}
   x - 4y + z &= 1 \\
   3x - 2y - 3z &= 15 \\
   2x + y - z &= 8
   \end{align*}
   \]
b. Solve the system using augmented matrices.

6. Name an advantage of the use of augmented matrices.

7. The sequence $1, 4, 10, 20, 35, 56, \ldots$ has the recursive formula
   \[
   \begin{align*}
   a_1 &= 1 \\
   a_n &= a_{n-1} + \frac{n(n + 1)}{2} \text{ for } n > 1.
   \end{align*}
   \]
   An explicit formula for this sequence is of the form
   \[
   a_n = an^3 + bn^2 + cn + d,
   \]
   where $a, b, c, d$ are the solutions to the system at the right:
   \[
   \begin{align*}
   a + b + c + d &= 1 \\
   8a + 4b + 2c + d &= 4 \\
   27a + 9b + 3c + d &= 10 \\
   64a + 16b + 4c + d &= 20
   \end{align*}
   \]
   Solve this system using augmented matrices.
8. Here are some quantities of foods and the total protein (in grams) in them.

<table>
<thead>
<tr>
<th>Milk (ounces)</th>
<th>Whole-wheat bread (slices)</th>
<th>Roast beef (ounces)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>4</td>
<td>43</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>6</td>
<td>66</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>8</td>
<td>78</td>
</tr>
</tbody>
</table>

Solve a system using augmented matrices to find the amount of protein in 1 ounce of milk.

9. Find the distance between the points in 4-space with coordinates (2, 5, -7, -2) and (3, 5, 7, -1). (Lesson 14-6)

10. State the number of vertices of a 4-dimensional hypercube. (Lesson 14-6)

11. How long a walking stick can fit diagonally into a box with dimensions 80 cm by 10 cm by 20 cm? (Lesson 14-4)

12. What simple fraction equals the repeating decimal 2.53? (Lesson 13-4)

13. Expand \((x + 2y)^5\). (Lesson 13-6)

14. Here are mean temperatures for Fairbanks, Alaska and Minneapolis, Minnesota by month.

<table>
<thead>
<tr>
<th></th>
<th>J</th>
<th>F</th>
<th>M</th>
<th>A</th>
<th>M</th>
<th>J</th>
<th>A</th>
<th>S</th>
<th>O</th>
<th>N</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fairbanks</td>
<td>-13</td>
<td>-4</td>
<td>9</td>
<td>30</td>
<td>48</td>
<td>59</td>
<td>62</td>
<td>57</td>
<td>45</td>
<td>25</td>
<td>4</td>
</tr>
<tr>
<td>Minneapolis</td>
<td>11</td>
<td>18</td>
<td>29</td>
<td>46</td>
<td>59</td>
<td>68</td>
<td>73</td>
<td>71</td>
<td>61</td>
<td>50</td>
<td>33</td>
</tr>
</tbody>
</table>

a. What is the yearly mean temperature for each city?
b. Which city has the greater variation in temperature, as measured by standard deviation?
c. Over a period of many years, what function might best approximate those mean temperatures, a sine wave, an absolute value function, or a parabola? (Lessons 6-2, 7-3, 10-9)

In 15–18, find all real solutions.

15. \(x^{3/4} = 64\) (Lesson 8-6)

16. \(\sin t = 0\) (Lesson 10-9)

17. \(3 \log m = 6\) (Lesson 9-4)

18. \(12x^3 + 6x^2 - 3x = 0\) (Lessons 6-6, 6-7, 11-4, 11-6)

19. Factor \(4p^5 - 108p^2\) completely. (Lesson 11-3)

20. Make up a 5 \(\times\) 5 system for which you know the solution. Show that the system can be solved using augmented matrices.
Below is a curve called the Mandelbrot set, named after Benoit Mandelbrot, a French-born American mathematician who works for IBM. It was brought to the attention of the world in the 1970s, and is created using ideas related to powers of complex numbers beyond the scope of this book.

The Mandelbrot set is an example of a fractal. Fractal objects are self-similar, that is, they do not change their appearance significantly when viewed under a microscope of arbitrary magnifying power. The word fractal is derived from the Latin word fractus, which means fragmented, broken, or irregular. Fractals often have very irregular, infinitely long boundaries.

Fractals may be abstract mathematical objects such as the Mandelbrot set or the pyramid shown below; or they may occur naturally as in the bark of a tree or the irregular coastline of Great Britain shown on the first page of the chapter.
Both theoretical and natural fractals may have whole number dimensions. The fractal pyramid at the bottom of the previous page has dimension 2. But many fractals have dimensions that are not whole numbers.

To explain how fractals occur and how their dimensions are calculated, we consider a question asked by Mandelbrot: How long is the coast of Great Britain? There is no obvious way to answer this question. Coastlines move in and out. If a river goes to the sea, how far do you go in before you are inland and not on the coast?

Despite these difficulties, there is a reasonable way to measure a coastline. Consider the curve below which represents a coast. First, pick a unit (a mile or a kilometer, perhaps). Then put a stake in the ground at some point A along the coast. Now imagine drawing a circle with center at the stake and radius equal to the unit.

Go along the coast to B, where the imaginary circle intersects the coastline, and put another stake at B. Repeat this process to find C, D, and other points along the coastline. Count the stakes and you get the length of the coastline in that unit.

If you pick a smaller unit, the circles will be smaller. Then the points where the stakes are placed take the ins and outs of the coastline more into account. For instance, here is a unit \( \frac{1}{3} \) the size of the previous unit, but on the same coastline.

In this case, you get 4 times as many stakes even though the unit is only \( \frac{1}{3} \) the length.

For instance, suppose the first unit was yards, and the size of the coastline is \( y \) yards. Then the unit \( \frac{1}{3} \) as long would be feet, and the size of the coastline is \( 4y \) feet.
Now let us bring in the idea of dimension: length is 1-dimensional; area is 2-dimensional; volume is 3-dimensional. Of what dimension is the above coastline? We make a table comparing units with yards to units with feet in different dimensions.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Dimension</th>
<th>Relationship between yards and feet in that dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>1</td>
<td>1 yard = 3 feet = $3^1$ feet</td>
</tr>
<tr>
<td>Area</td>
<td>2</td>
<td>1 square yard = 9 square feet = $3^2$ square feet</td>
</tr>
<tr>
<td>Volume</td>
<td>3</td>
<td>1 cubic yard = 27 cubic feet = $3^3$ cubic feet</td>
</tr>
</tbody>
</table>

The general pattern implies that in dimension $D$, to convert yards to feet, multiply by $3^D$. The base of 3 is due to the original unit being 3 times the smaller unit. In the unknown dimension of this coastline, to convert yards to feet, we multiplied by 4.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Dimension</th>
<th>Relationship between yards and feet in that dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coastline</td>
<td>$D$</td>
<td>1 coastline yard = 4 coastline feet $= 3^D$ coastline feet</td>
</tr>
</tbody>
</table>

So by solving $3^D = 4$ we can get the dimension of this coastline.

$$3^D = 4 \quad \Rightarrow \quad D = \frac{\log 4}{\log 3} \quad \Rightarrow \quad D \approx 1.26$$

The piece of the coastline drawn on the previous page has a dimension of about 1.26.

For a perfectly smooth coastline, there would be 3 coastline feet for each coastline yard. Solving $3^D = 3$ gives $D = 1$; the dimension is 1 as you would expect for a smooth coastline. Rougher coastlines, those that go in and out more, have dimensions nearer 2. The dimension of Great Britain’s coastline is about 1.25.

**Example** Consider a figure generated recursively as follows. Begin with an equilateral $\triangle ABC$. Split each side of the triangle into five congruent parts. On two of those parts draw equilateral triangles as shown below. Then repeat this process on the smaller segments again and again. That is, each time replace $\triangle$ by $\triangle \square \triangle$. The result is called a *snowflake curve*. What is the dimension of the “infinite-sided” boundary that arises?
**Solution**  Think of measuring the boundary first with a unit equal to the length of $\overline{AB}$. The boundary is 3 of these units. Now move to a smaller unit $\frac{1}{3}$ the length. Along the boundary, the distance from $A$ to $B$, $B$ to $C$, or $A$ to $C$ is now 7 of these smaller units in size. So the total boundary is 21 of the smaller units. At each stage, the boundary is multiplied by 7 when the unit is $\frac{1}{3}$ the size. Let $D$ be the dimension of the boundary. Then, because the original segment is 5 times the next smaller segment,

$$5^D = 7.$$  

Solving yields  

$$D = \frac{\log 7}{\log 5} \approx 1.21.$$  

Thus, this snowflake fractal has dimension about 1.21.

In general, given a self-similar object of $N$ parts scaled by a ratio $r$ from the whole, its **fractal dimension** $D$ is the solution to the equation  

$$\left(\frac{1}{r}\right)^D = N \text{ or } D = \frac{\log N}{\log (1/r)}.$$  

In the previous Example, $N = 7$ and $r = \frac{1}{5}$.

Mandelbrot and others have shown that cloud formations, holes in Swiss cheese, radio static, the motion of molecules, and even the shape of galaxies can be modeled by fractals. The concept of fractal dimensions, like the concept of dimensions higher than 3, is now seriously used by applied mathematicians. Fractals provide a striking example of how mathematical ideas continue to be invented, and on the next page, show the beauty of mathematics. It is a fitting way to end this book.
Picture of a Discretized Boundary Value Problem using
\( f(\mu) = \mu - \mu^3 \) in 6-space and represented on the window
\(-4 \leq x \leq 4, \ -3 \leq y \leq 3 \) in 2-space.
Questions

1. What is a fractal?
2. Who first introduced the idea of fractals, and when?
3. True or false There are some curves which have a non-integral dimension.
4. Multiple choice There are about 1.6 km in a mile. A coastline that is measured in miles to be 10 miles long will therefore be measured in km to be:
   (a) less than 16 km long   (b) exactly 16 km long
   (c) at least 16 km long   (d) cannot be determined.
5. a. To change yards to feet, multiply by __?
   b. To change square yards to square feet, multiply by __?
   c. To change cubic yards to cubic feet, multiply by __?
   d. To change "dimension-D" yards to "dimension-D" feet, multiply by __?
6. If one coastline yard equals 4.5 coastline feet, what is the dimension of the coastline?
7. State two phenomena that can be modeled by fractals.

Applying the Mathematics

In 8 and 9, consider the figures defined as follows.
Begin with a line segment.

Figure 1

To produce figure \( n \) divide each segment of figure \( n - 1 \) in four parts and replace the middle two by congruent segments placed as shown below.

Figure 2

8. Draw Figure 3.
9. Here is a sketch of the result of several more iterations of the above procedure. Find the fractal dimension of the figure that results if this procedure is applied indefinitely.
In 10 and 11, a figure is formed recursively so that at each stage

\[ \text{is replaced by} \]

10. Draw the next figure in this pattern.

11. Prove that when this pattern is repeated over and over again, the figure generated has dimension 2.

12. At the right is an equilateral triangle. Each side has been split into three parts and on the middle part an equilateral triangle has been drawn. This is then repeated again and again. What is the dimension of the final boundary?

13. Use augmented matrices to solve the following system:

\[
\begin{align*}
2a + b + c &= 3 \\
-5a - b + c &= -9 \\
8a + 2b - c &= 14
\end{align*}
\]

(Lesson 14-7)

14. Graph the triangular region determined by the intercepts of the plane with equation \(4x + 4y + 20z = 20\). (Lesson 14-2)

15. In a lottery, you must match 6 numbers chosen from 50.
   a. How many different combinations of such numbers are there?
   b. What are your chances of winning? (Lesson 13-7)

16. Find the sum of the first 100 terms of the arithmetic sequence that begins 1000, 980, \ldots. (Lesson 13-1)

17. What conic section is generated by intersecting a cone with a plane which is parallel to one edge of the cone? (Lesson 12-1)

In 18 and 19, solve for \( \theta \). (Lessons 10-2, 10-6)

18.

19.

20. Rewrite \( \sqrt[5]{-161051a^{11}b^{19}} \). (Lesson 8-8)

21. Find an explicit formula for \( t_n \), the \( n \)th term of the geometric sequence 12, 6, 3, 1.5, .75, \ldots. (Lesson 8-3)
22. A rental car company charges $39 for a weekend rental plus $0.15/mi. Suppose these are the only charges.
   a. Let \( m \) = the number of miles driven in one weekend, and \( C \) = the cost of renting the car. Write a formula that gives \( C \) as a function of \( m \).
   b. If the cost of a weekend's rental is $96, how many miles was the car driven? \((Lessons \ 3-1, \ 1-7)\)

In 23 and 24, refer to the quadratic function graphed below.\((Lessons \ 6-3, \ 6-4, \ 11-3, \ 11-6)\)

![Graph of a quadratic function with points (0, 9), (3, 0), and (6, 0).]

23. State an equation for its axis of symmetry.

24. Find an equation for the curve.

25. Some reference books about fractals are *Fractals: Form, Chance and Dimension* and *The Fractal Geometry of Nature* by Benoit Mandelbrot, and *The Beauty of Fractals: Images of Complex Dynamical Systems* by H. O. Peitgen and P. H. Richter. Find examples of fractals different from those mentioned in this lesson with both whole-number and fractional dimensions.
We think of the physical world we live in as 3-dimensional. With ordered 3-tuples, points can be located in that space, and equations for common figures such as planes, spheres, and cylinders can be derived.

However, the world can be viewed as 4-dimensional, with time as the fourth dimension. And many phenomena can be modeled by fractals, figures which may have fractional dimension.

To work in these dimensions, we use analogies from 2-space. The Distance Formula, equations for lines and circles, and the solving of $2 \times 2$ systems can all be extended to 3-space; the result is another Distance Formula, equations for planes and spheres, and the solving of $3 \times 3$ systems. Further extensions to 4-space yield still another Distance Formula, equations for hyperplanes and hyperspheres, and ways to solve higher dimensional systems.

Below are the most important terms and phrases for this chapter. You should be able to give a definition for those terms marked with a *.

**Lesson 14-1**
ordered triple
3-dimensional coordinate system
z-coordinate, z-axis
coordinate plane, xy-plane, yz-plane, xz-plane
octant

**Lesson 14-2**
standard form of the equation of a plane

**Lesson 14-3**
extended linear combination method

**Lesson 14-4**
Distance Formula in 3-Space
* sphere
lattice point in 3-space

**Lesson 14-5**
analytic description
axis of rotation
disk
* surface of revolution
solid of revolution
* paraboloid

**Lesson 14-6**
hypercube
hypersphere
4-space
* ordered 4-tuple
Distance Formula in 4-Space
equation for a hypersphere

**Lesson 14-7**
dimensions of a system
triangular matrix
augmented matrix
row operations
hyperplane

**Lesson 14-8**
fractal, fractional dimension
self-similar
snowflake curve
Directions: Take this test as you would take a test in class. Use graph paper and a calculator. Then check your work with the solutions in the Selected Answer section in the back of the book.

1. Draw one 3-dimensional coordinate system and plot these points on it.
   a. (−1, 2, 0)
   b. (2, −2, 3)

2. Give the coordinates of point S.

3. Find $UH$.

4. Give the coordinates of a point in the upper left front octant of a 3-dimensional coordinate system.

5. A box has the following vertices:
   (2, 0, 0), (2, 0, 3), (2, 4, 3), (2, 4, 0),
   (0, 0, 3), (0, 4, 0), (0, 4, 3), and (0, 0, 0).
   Find the volume of the box.

6. a. Write an algebraic description of the points 3.5 units from the $yz$-plane.
   b. Either sketch a graph of your response to part a or describe it in words.

7. A sphere has a radius of 13 units and center at the origin. Give an equation for the sphere.

8. Give the augmented matrix for the system.

9. Solve the system using any method.

In 10 and 11, multiple choice.

10. Which plane is $\parallel$ to the $z$-axis in 3-space?
    a. $y = 7$
    b. $3z - x - y = 5$
    c. $12y - 60z = 1$
    d. $x - y = 0$

11. The surface generated when a segment, parallel to the $x$-axis in the $xy$-plane, is rotated around the $x$-axis is a
    a. circle
    b. cylinder
    c. cone
    d. paraboloid

In 12 and 13, consider the plane with equation $4x + y + 2z = 4$.

12. True or false This plane is parallel to one of the coordinate planes.

13. Graph the triangular region determined by the intercepts of the given plane.

14. Find an equation for a four-dimensional hypersphere with center at (0, 0, 0, 0) and radius 12.

15. State the possible numbers of points in which 3 planes can intersect.

16. Describe the intersection of a sphere with equation $x^2 + y^2 + z^2 = 49$ and the $yz$-plane.

In 17 and 18, consider the figures generated by the following recursive procedure. Begin with a square.

To construct Figure $n$, trisect each segment of Figure $n - 1$ and on the middle segment construct a square extending out.

That is, replace $\underline{\hspace{2cm} }$ with $\underline{\hspace{2cm} }$

17. Draw Figure 3.

18. If this process is continued indefinitely, what is the dimension of the figure produced?

    a. Let $a =$ the number of adults’ tickets bought, $c =$ the number of children’s tickets bought, and $s =$ the number of senior citizens’ tickets bought. Write a system of equations that can be used to determine the cost of each type of ticket.
    b. How much should Pablo pay for one of each type of ticket?
CHAPTER 14

Chapter Review

Questions on SPUR Objectives

SPUR stands for Skills, Properties, Uses, and Representations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS deal with the procedures used to get answers.

Objective A: Solve $3 \times 3$ and $4 \times 4$ systems of equations. (Lessons 14-3, 14-7)

1. **Multiple choice** Which step gives you $13y + 5z = -21$ from the following system:
   \[
   \begin{align*}
   2x + 3y - z &= -1 \\
   -x + 5y + 3z &= -10 \\
   3x - y - 6z &= 5
   \end{align*}
   \]
   (a) Multiply the first equation by 3 and add it to the second equation.
   (b) Multiply the third equation by 5 and add it to the second equation.
   (c) Multiply the second equation by 2 and add it to the first equation.
   (d) Multiply the first equation by -6 and add it to the third equation.

In 2 and 3, solve each system using linear combinations.

2. \[
\begin{align*}
5x - y + z &= 5 \\
3x + y - z &= 3 \\
x + 2y - z &= 3
\end{align*}
\]

3. \[
\begin{align*}
r + 2s + t &= 5 \\
2r - s + t &= 4 \\
3r + s + 4t &= 1
\end{align*}
\]

Objective B: Find distances between points in 3- and 4-space. (Lessons 14-4, 14-6)

In 6—9, find the distance between the points. Round your answer to the nearest integer.

6. (2, -8, 6) and (-3, 9, 11)
7. (3, 1.2, .4) and (-1.7, -.1, 4)
8. (1, -1, 2, 3) and (1, -1, 5, 3)
9. (7 $\frac{1}{3}$, 10, -6, 4) and (8, 22, -6, -15)

In 10 and 11, refer to the sphere shown at the right.

10. $FA$;
11. the length of any diagonal of the cube.

Objective C: Write and analyze equations for spheres and hyperspheres. (Lessons 14-4, 14-6)

12. Write an equation for the sphere with center at (0, 0, 0) and radius 9.
13. Describe the graph of the equation $x^2 + y^2 + z^2 = 50$ in 3-space.
14. What is an equation for a hypersphere with center at the origin and radius 7?
15. Consider the equation $x^2 + y^2 + z^2 + w^2 = 729$ of a hypersphere. Name its center and radius.
**Objective D:** Identify properties of planes in 3-space. (Lesson 14-2)

16. Give an equation for the xz-coordinate plane.
17. What is an equation of the plane containing the point (0, 0, -7) and parallel to the xy-plane?
18. **Multiple choice** The plane $x = 7$ is
   (a) parallel to the x-axis
   (b) contained in the yz-plane
   (c) perpendicular to the plane $y = 10$
   (d) none of these.
19. The standard form of the equation of a plane is $\underline{?}$. 
20. $7y - 2z = 4$ is an equation for a plane parallel to the $\underline{?}$-axis.
21. a. How many solutions has the following system:
   \[
   \begin{align*}
   -x + y - 5z &= 6 \\
   x - 2y + 7z &= 14 \\
   2x - 2y + 10z &= -12
   \end{align*}
   \]
   b. Interpret part a geometrically.

**Objective E:** Extend 2- and 3-dimensional ideas to higher or fractional dimensions. (Lessons 14-7, 14-8)

22. A point in 5-space would have coordinates $\underline{?}$.
23. What is the distance between $(a, b, c, d)$ and $(e, f, g, h)$?
24. Give an equation for the hypersphere with center $(0, 0, 0, 0)$ and radius 10.
25. What is a fractal?
26. If 1 coastline yd equals 5 coastline ft, what is the dimension of the coastline?
In 27 and 28, consider figures generated by the following recursive procedure. Begin with a line segment.
Figure 1

To construct Figure $n$ divide each segment of Figure $n - 1$ into 5 congruent parts and on the 2nd and 4th construct squares extending out.
Figure 2

27. Draw Figure 3.
28. Suppose this process is continued indefinitely. Find the dimension of the figure that results.

**USES** deal with applications of mathematics in real situations.

**Objective F:** Use $3 \times 3$ or $4 \times 4$ linear systems to solve real world problems. (Lessons 14-3, 14-7)

29. A test contains multiple-choice, fill-in-the-blank, short answer, and essay questions. All questions of a given type are worth the same number of points. Here are the number of questions correct for 4 students, and each student’s total score.

<table>
<thead>
<tr>
<th>Student</th>
<th>m.c.</th>
<th>fill-in</th>
<th>s.a.</th>
<th>essay</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neil</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>46</td>
</tr>
<tr>
<td>Ida</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>68</td>
</tr>
<tr>
<td>Carol</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>Evan</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>47</td>
</tr>
</tbody>
</table>
30. After two tests, Gordon's average in math was 76. After three tests, it was 83. If his teacher dropped the lowest test grade, Gordon's average would be 88. What were Gordon's three test scores?

31. A travel agent books three charter groups to go on a weekend cruise. A group of surgeons reserves 9 1st class, 22 2nd class, and 15 3rd class rooms for $15,380. A group of journalists has 4 1st class, 13 2nd class, and 8 3rd class rooms for $8,330. And finally an association of teachers books 5 1st class, 7 2nd class, and 25 3rd class rooms for $11,300. What is the charge for a room in 1st, 2nd, and 3rd class?

REPRESENTATIONS deal with pictures, graphs, or objects that illustrate concepts.

Objective G: Graph sets of points in 3-space.
(Lessons 14-1, 14-2, 14-4)

32. Plot these vertices of a box: (10, -3, 8), (-2, -3, 8), (-2, 11, 8), (-2, 11, -1), (10, 11, -1), (10, 11, 8), (-2, -3, -1), and (10, -3, -1).

33. Graph the triangular region determined by the intercepts of the plane with equation $12x + 2y - 9z = 36$.

34. Graph the plane with equation $x + z = 8$.

35. Graph the set of points satisfying $x^2 + y^2 + z^2 = 1$.

36. Give an equation for plane $P$ at the right which is parallel to the $xz$-plane.

Objective H: Describe cross-sections or the surface of revolution generated by rotating a set of points. (Lesson 14-5)

In 37–39, (a) sketch and (b) describe in words the surface of revolution which is generated by rotating:

37. a circle with radius 3 centered at the origin in the $yz$-plane around the $y$-axis.

38. the segment with endpoints $A = (0, -8, 7)$ and $B = (0, -8, 12)$ around the $z$-axis.

39. the line with equation $y = x$ around the $x$-axis.

40. How is a paraboloid generated?

41. Triangle $ABO$ is revolved in space around the $x$-axis.

a. What solid figure is formed?
b. What is the volume of that figure?

In 42–45, describe the cross-section when the cylinder below is intersected by a plane:

42. parallel to the bases and between them.

43. not parallel to the bases but between them.

44. perpendicular to the bases and containing the center of one of them.

45. perpendicular to the bases and not containing the center of either of them.
Models for Operations

A model for an operation is a pattern that describes many of the uses of that operation.

**Models for addition:** $x + y$ can stand for:

1. (Putting together model) the result of putting together quantities $x$ and $y$ when there is no overlap. Example: If you have $2m$ dollars and I have $n$ dollars, our total is $2m + n$ dollars.
2. (Slide model) the result of a slide $x$ followed by a slide $y$. Example: If the temperature changes $5^\circ$ and then changes $c^\circ$, the total change is $5 + c$ degrees.

**Models for subtraction:** $x - y$ can stand for:

3. (Take-away model) the result when a quantity $y$ is taken away from a quantity $x$. Example: The measure of $\angle ABC$ pictured below is $90 - x^\circ$. (An angle with measure $x^\circ$ is taken away from an angle of measure $90^\circ$.)

4. (Comparison model) how much more $x$ is than $y$, or the difference between $x$ and $y$. Example: Millie is 16 years old. Mel is $Y$ years old. Millie is $16 - Y$ years older than Mel.

**Models for multiplication:** $xy$ can stand for:

5. (Area model) the area of a rectangle with length $x$ and width $y$. Example: A rectangular array of dots with $m$ rows and $n$ columns has $mn$ dots in all.
6. (Size change model) the result of taking a quantity $y$ and enlarging or contracting it by a scale factor of $x$. Example: If an insect leg has length $L$ and the leg is magnified 50 times, then the image will have length $50L$.
7. (Rate factor model) the total when a rate $x$ is applied to a quantity $y$. Example: A person walks $2\frac{1}{2}$ miles an hour for $3\frac{1}{2}$ hours. Then the total distance is $2\frac{1}{2}$ miles $\cdot \frac{3}{2}$ hours, or $8\frac{3}{4}$ miles.

**Models for division:** $\frac{x}{y}$ can stand for:

8. (Splitting-up model) the result of splitting up $x$ things into $y$ parts, or $x$ things into parts with $y$ per part. Example: If there are $x$ muffins with 2 muffins per guest, then $\frac{x}{2}$ people can be served.
9. (Rate model) the rate $x$ per $y$. Example: If there are 1500 students in a 4-year high school, then on the average there are $\frac{1500}{4}$ or 375 students per year.
10. (Ratio comparison model) how many times $x$ is bigger than $y$. Example: In similar triangles, if the ratio of similitude is $\frac{3}{2}$, the first triangle is 1.5 times the second in linear dimensions.

**Model for powering:** $x^y$ can stand for:

11. (Growth model) the amount by which a quantity is multiplied in a time $y$ if it is multiplied by $x$ in each unit time. Example: If the population is growing at a rate by which it doubles every 35 years, then in 10 years it will be multiplied by $2^{10/35}$.
In this book, we use many measurement formulas. The following symbols are used.

- $A =$ area
- $a =$ length of apothem
- $a$, $b$, and $c$ are lengths of sides
  (when they appear together)
- $b_1$ and $b_2$ are lengths of bases
- $B =$ area of base
- $C =$ circumference
- $d =$ diameter
- $d_1$ and $d_2$ are lengths of diagonals
- $h =$ height
- $L =$ lateral area
- $l =$ length or slant height
- $n =$ number of sides
- $P =$ perimeter of base
- $r =$ radius
- $S =$ total surface area
- $s =$ side
- $\theta =$ measure of angle
- $T =$ sum of measures of angles
- $V =$ volume
- $w =$ width

## Two-Dimensional Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Perimeter, Length, and Angle Measure</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$-gon</td>
<td>$T = 180(n - 2)$</td>
<td>$A = \frac{1}{2}ap$</td>
</tr>
</tbody>
</table>
| regular $n$-gon | $p = ns$ \[
\theta = \frac{180(n - 2)}{n}
\] | $A = \frac{1}{2}bh$ |
| triangle | $p = a + b + c$ | $A = \frac{1}{2}ab$ |
| right triangle | $c^2 = a^2 + b^2$ (Pythagorean theorem) | $A = \frac{\sqrt{3}}{4}s^2$ |
| equilateral triangle | $p = 3s$ | $A = \frac{1}{2}h(b_1 + b_2)$ |
| trapezoid | | $A = bh$ |
| parallelogram | | $A = lh$ |
| rhombus | $p = 4s$ | $A = \frac{1}{2}d_1d_2$ |
| rectangle | $p = 2l + 2w$ | $A = lw$ |
### Two-Dimensional Figures

<table>
<thead>
<tr>
<th>Perimeter, Length, and Angle Measure</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>square</td>
<td>$p = 4s$</td>
</tr>
<tr>
<td>circle</td>
<td>$C = \pi d = 2\pi r$</td>
</tr>
</tbody>
</table>

### Three-Dimensional Figures

<table>
<thead>
<tr>
<th>Lateral and Total Surface Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>prism</td>
<td>$V = Bh$</td>
</tr>
<tr>
<td>right prism</td>
<td>$L = Ph$ $S = Ph + 2B$</td>
</tr>
<tr>
<td>box</td>
<td>$S = 2(lw + lh + hw)$</td>
</tr>
<tr>
<td>cube</td>
<td>$S = 6s^2$</td>
</tr>
<tr>
<td>pyramid</td>
<td></td>
</tr>
<tr>
<td>regular pyramid</td>
<td>$L = \frac{Pl}{2}$ $S = \frac{Pl}{2} + B$</td>
</tr>
<tr>
<td>cylinder</td>
<td></td>
</tr>
<tr>
<td>right circular cylinder</td>
<td>$L = 2\pi rh$ $S = 2\pi rh + 2\pi r^2$</td>
</tr>
<tr>
<td>cone</td>
<td></td>
</tr>
<tr>
<td>right circular cone</td>
<td>$L = \pi rl$ $S = \pi rl + \pi r^2$</td>
</tr>
<tr>
<td>sphere</td>
<td>$S \approx 4\pi r^2$</td>
</tr>
</tbody>
</table>
The BASIC commands used in this course and examples of their uses are given below.

**LET ...**
A value is assigned to a given variable. Some versions of BASIC allow you to omit the word LET in the assignment statement.

- LET \( X = 5 \)  
  The number 5 is stored in a memory location called \( X \).
- LET \( N = N + 2 \)  
  The value in the memory location called \( N \) is increased by 2 and then restored in the location called \( N \).

**PRINT ...**
The computer prints on the screen what follows the PRINT command. If what follows is a constant or variable, the computer prints the value of that constant or variable. If what follows is in quotes, the computer prints exactly that quote.

- PRINT \( X \)  
  The computer prints the number stored in memory location \( X \).
- PRINT "X-VALUES"  
  The computer prints the phrase X-VALUES.

**INPUT ...**
The computer asks for a value of the variable named, and stores that value.

- INPUT \( X \)  
  When the program is run, the computer will prompt you to give it a value by printing a question mark, and then store that value in memory location \( X \).
- INPUT "HOW OLD?"; \( AGE \)  
  The computer prints HOW OLD? and stores your response in memory location \( AGE \).

**REM ...**
This command allows remarks to be inserted in a program. These may describe what the variables represent, what the program does or how it works. REM statements are often used in long complex programs or programs others will use.

- REM PYTHAGOREAN THEOREM  
  The statement appears when the LIST command is given, but it has no effect on the program.

**FOR ... NEXT ... STEP ...**
The FOR command assigns a beginning and ending value to a variable. The first time through the loop, the variable has the beginning value in the FOR command. When the computer hits the line reading NEXT, the value of the variable is increased by the amount indicated by STEP. The commands between FOR and NEXT are then repeated.

```
10 FOR \( N = 3 \) TO 6 STEP 2  
20   PRINT \( N \)  
30 NEXT \( N \)  
40 END
```

The computer assigns 3 to \( N \) and then prints the value of \( N \). On reaching NEXT, the computer increases \( N \) by 2 (the STEP amount), and prints 5. The next \( N \) would be 7 which is too large. The computer executes the command after NEXT, ending the program.
IF ... THEN ...

The computer performs the consequent (the THEN part) only if the antecedent (the IF part) is true. When the antecedent is false, the computer ignores the consequent and goes directly to the next line of the program.

IF X > 100 THEN END PRINT X

If the X value is less than or equal to 100, the computer ignores "END," goes to the next line, and prints the value stored in X. If the X value is greater than 100, the program goes to the END statement.

GO TO ...

The computer goes to whatever line of the program is indicated. GOTO statements are generally avoided because they interrupt program flow and make programs hard to interpret.

GOTO 70

The computer goes to line 70 and executes that command.

END ...

The computer stops running the program. No program should have more than one END statement.

### Functions

The following built-in functions are available in most versions of BASIC. Each function name must be followed by a variable or constant enclosed in parentheses.

**ABS** The absolute value of the number that follows is calculated.

\[ \text{LET } X = \text{ABS} (-10) \]

The computer calculates \(|-10| = 10\) and assigns the value 10 to memory location X.

**INT** The greatest integer less than or equal to the number that follows is calculated.

\[ X = \text{INT} (N + .5) \]

The computer adds .5 to the value of N, calculates \([N + .5]\), and stores the result in X.

**LOG** The natural logarithm, i.e., the log to base e, of the number that follows is calculated.

\[ \text{LET } J = \text{LOG}(6) \]

The computer calculates \(\ln 6\) and assigns that value 1.791759 to memory location J.

**SQR** The square root of the number or expression that follows is calculated.

\[ C = \text{SQR} (A \times A + B \times B) \]

The computer calculates \(\sqrt{A^2 + B^2}\) using the values stored in A and B and stores the result in C.
## Field Postulates

For any real numbers $a$, $b$, and $c$:

<table>
<thead>
<tr>
<th>Property</th>
<th>Addition</th>
<th>Multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closure properties</td>
<td>$a + b$ is a real number.</td>
<td>$ab$ is a real number.</td>
</tr>
<tr>
<td>Commutative properties</td>
<td>$a + b = b + a$</td>
<td>$ab = ba$</td>
</tr>
<tr>
<td>Associative properties</td>
<td>$(a + b) + c = a + (b + c)$</td>
<td>$(ab)c = a(bc)$</td>
</tr>
<tr>
<td>Identity properties</td>
<td>There is a real number $0$ with $0 + a = a + 0 = a$.</td>
<td>There is a real number $1$ with $1 \cdot a = a \cdot 1 = a$.</td>
</tr>
<tr>
<td>Inverse properties</td>
<td>There is a real number $-a$ with $a + (-a) = -a + a = 0$.</td>
<td>If $a \neq 0$, there is a real number $\frac{1}{a}$ with $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.</td>
</tr>
<tr>
<td>Distributive property</td>
<td>$a(b + c) = ab + ac$</td>
<td></td>
</tr>
</tbody>
</table>

## Other Postulates

**Equality:** For any real numbers $a$, $b$, and $c$:

- Reflexive property $a = a$
- Symmetric property If $a = b$, then $b = a$.
- Transitive property If $a = b$ and $b = c$, then $a = c$.
- Substitution property If $a = b$, then $a$ may be substituted for $b$ in any arithmetic or algebraic expression.
- Addition property If $a = b$, then $a + c = b + c$.
- Multiplication property If $a = b$, then $ac = bc$.

**Inequality:** For any real numbers $a$, $b$, and $c$:

- Trichotomy property Either $a < b$, $a = b$, or $a > b$.
- Transitive property If $a < b$ and $b < c$, then $a < c$.
- Addition property If $a < b$, then $a + c < b + c$.
- Multiplication property If $a < b$ and $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$.

**Powers:** For any nonnegative bases and real exponents, or any nonzero bases and integer exponents:

- Product of Powers property $b^m \cdot b^n = b^{m+n}$
- Power of a Power property $(b^m)^n = b^{mn}$
- Power of a Product property $(ab)^m = a^m b^m$
- Quotient of Powers property $\frac{b^m}{b^n} = b^{m-n}$, for $b \neq 0$
- Power of a Quotient property $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, for $b \neq 0$
Addition and Multiplication: For all real numbers \( a, b, \) and \( c: \)

- Multiplication Property of 0: \( 0 \cdot a = 0 \)
- Multiplication Property of -1: \( -1 \cdot a = -a \)
- Opposite of an Opposite property: \( -(a) = a \)
- Opposite of a Sum: \( -(b + c) = -b + -c \)
- Distributive Property of Multiplication over Subtraction: \( a(b - c) = ab - ac \)
- Addition of Like Terms: \( ac + bc = (a + b)c \)
- Addition of Fractions: \( \frac{a}{c} + \frac{b}{c} = \frac{a + b}{c} \), for \( c \neq 0 \)

Powers and Roots:

- Zero Exponent: If \( b \) is a nonzero real number, \( b^0 = 1. \)
- Negative Exponent: If \( x > 0 \), then \( x^{-n} = \frac{1}{x^n}. \)
- \( \frac{1}{n} \) Exponent: When \( x \geq 0 \), \( x^{1/n} \) is the \( n \)th root of \( x \).
- Rational Exponent: For any positive real number \( x \) and positive integers \( m \) and \( n \), \( x^{m/n} = (x^{1/n})^m \), the \( m \)th power of the positive \( n \)th root of \( x \), and
  \( \sqrt[n]{x^m} = (x^m)^{1/n}, \) the positive \( n \)th root of the \( m \)th power of \( x \).
- \( n \)th Root of \( n \)th Power: For all real numbers \( x \), and integers \( n \geq 2 \):
  - if \( n \) is odd, \( \sqrt[n]{x^n} = x \), and
  - if \( n \) is even, \( \sqrt[n]{x^n} = |x| \).

Logarithm: For any positive real number base \( b \neq 1 \), and any positive real numbers \( x \) and \( y: \)

- \( \log_b 1 = 0 \)
- \( \log_b b^n = n \)
- Log of a Product property: \( \log_b(xy) = \log_b x + \log_b y \)
- Log of a Quotient property: \( \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \)
- Log of a Power property: \( \log_b (x^n) = n \log_b x \)

Trigonometry:

In any triangle \( ABC: \)

- Law of Sines: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
- Law of Cosines: \( c^2 = a^2 + b^2 - 2ab \cos C \)

For all real numbers \( \theta: \)

- Complements property: \( \sin \theta = \cos(90^\circ - \theta) \) and \( \cos \theta = \sin(90^\circ - \theta) \)
- Supplements property: \( \sin \theta = \sin(180^\circ - \theta) \)
- Pythagorean property: \( \cos^2 \theta + \sin^2 \theta = 1 \)
- Periodicity property: \( \sin \theta = \sin(\theta + 360n^\circ) \), for all integers \( n \)
  \( \cos \theta = \cos(\theta + 360n^\circ) \), for all integers \( n \)
SELECTED ANSWERS

LESSON 1-1 (pp. 2-8)
3. expression 7, \( p < y \) 9. 25 + 3w 13. a. 10ce b. mce
15. is not equal to 17. sample: 7x = 8 \( \leq \) 3x + 2 19. x + y
21. xyv 23. \( S = T \) liters 25. \( E + F \) eggs 27. a. 1150 = 2x
b. (1150 - 2x)x c. x(1150 - 2x) \( \geq \) 60,000 29. quadrilateral
31. hexagon

LESSON 1-2 (pp. 9-13)
3. \( d = 20 \) See below. 5. a. samples: \( \pi \), 5.8, \( \frac{3}{2} \) b. samples:
\( \pi \), \( \sqrt{2} \), 0.101001000100001000001 \ldots 
7. c, b, a
9. 42.4 ft 11. a. no b. no 13. \( n = 3 \), 4, 5, \ldots
15. a. \( s > 0 \) b. 43.3 cm² 17. -13 19. \( \frac{1}{11} \) 21. \( a + b \) 23. \( ab \)
25. \( b - a \) 27. \( n + 6 < 60 \)

LESSON 1-3 (pp. 14-20)
3. See below. 5. 3, 6, 10, 15, 21, 28, 36, 45 9. the sequence of cubes of natural numbers
11. a. fourth b. 262,144
13. a. See below. b. \( S_n = n^2 \) 15. 1, 5, 12, 22
17. \$74,090.44 19. c 21. sample: 10 FOR \( N = 1 \) TO 200
20 PRINT N
30 NEXT N
40 END
23. 727 25. \( S \) is undefined for \( N = 88 \); \( S \) is negative for \( N > 88 \). 27. See below.

LESSON 1-4 (pp. 21-25)
1. 24, 36, 54, 81, 121. 5 3. 40 \( \frac{1}{5} \) \( \frac{1}{5} \) \( \frac{1}{5} \) \( \frac{1}{5} \) \( \frac{1}{5} \)
5. a. 11 b. \( T/2 \) or \( T \ast .5 \) 7. 7, 17, 27, 37, 47 9. 21, 34

LESSON 1-5 (pp. 26-31)
3. sample: \( 6(5 - 4) = 6(5 - 6) \) 5. sample: \(-10 \ast -1 = 10 \)
Prop. of Add. (v) Assoc. Prop. of Add. 13. \( 24x^2 + 60x - 36 \) 15. \( p = 2(L + W) \) 17. \( a_n = 2.4n - 8.4 \) 19. \( \frac{2x}{y} \)
21. \( 5x - 13 \) 23. \( \frac{3}{5} \) 25. \( a \) \( \frac{7}{9} \) 29. \$0.055d dollars

LESSON 1-6 (pp. 32-36)
7. a. c 11. "if \( 12 = 20 - 3t \), then \( t = \frac{8}{3} \)" and "if \( t = \frac{8}{3} \),
then \( 12 = 20 - 3t \)." 13. \( z = -33\frac{1}{2} \) 15. Add. Prop. of
Equation 17. 2.6(3a - 4) = 9.1 \( \Rightarrow \) 7.8a - 10.4 = 9.1 \( \Rightarrow \)
7.8a = 19.5 \( \Rightarrow \) a = 2.5; a = 2.5 \( \Rightarrow \) \( a = 3.5 \) \( \Rightarrow \)
2.6(3a - 4) = 2.6(3.5) = 9.1 19. sample: \( \frac{3 + 2}{3} \neq \frac{3}{2} \)
\( + 2^2 \) 21. a. -3, 2, 5, 7 b. The result is the original number.
c. \( n, n - 4, 3(n - 4), 3(n - 4) + 9, \frac{3(n - 4) + 9}{3} \)
\( \frac{3(n - 4) + 9}{3} + 1; \frac{3(n - 4) + 9}{3} + 1 = \frac{3}{3} \left( \frac{(n - 4) + 3}{1} \right) = \right)
\( \frac{(n - 4) + 3}{1} + 1 = n - 4 + 3 + 1 = n 23. 3, 1, 3, 1, 3 \)

LESSON 1-7 (pp. 37-41)
1. 33\frac{3}{10} m 3. to clear fractions 5. \( 0.06s + 0.08(20,000 - s) = \
1500; 0.06s + 1600 = 0.08s = 1500; -0.2s = -100; s = 5000 \)
7. a. \( 2x - 9 \) b. \( x = \frac{5}{3} \) 9. \( m = \frac{21}{10} \) 11. \( x = 10 \) 13. \( z = 0 \)
15. a. \( x = 75,000 \) b. A company invests \$100,000, some at
7\% and some at 5\%. If the annual total return is \$6,500, how
much is invested at 7\%? 17. the 97th 19. a. See below.
b. 5, 10, 15, 20 c. iv 21. a, b

19. a

LESSON 1-8 (pp. 42-46)
1. 266 mi 3. \( r = \frac{d}{t} \) 5. 48 cm 7. b 9. \( \frac{C}{r} = 2\pi \) 11. a. \( S = \
R \) b. 30 ft 13. \( \frac{a_n - a_1}{d} + 1 = n 15. 12 \) 17. a. \$0.06d
b. .94d 19. Each triangle with vertex V has area \( \frac{1}{2}d_1 \cdot \frac{1}{2}d_2 \), so the total area of the 4 triangles is \( 4 \left( \frac{1}{2}d_1 \cdot \frac{1}{2}d_2 \right) = \frac{1}{2}d_1d_2 \).

**LESSON 1-9 (pp. 47-50)**

1. d. 5. The set of all \( x \) less than -3 7. \( m < 7 \); See below.
2. a. If \( C \) is the circumference, then \( 0 < C < 28 \).
4. a. 13. \( x \geq \frac{1}{14} \). See below.
5. 15. a. \( T = 5000 + 50s \)
6. 60 sacks or fewer 17. \( b = \frac{2A}{h} \) 21. \( x = 15.5^\circ \), \( x + 5 = 20.5^\circ \), \( 4x = 62^\circ \)

**CHAPTER 1 PROGRESS SELF-TEST** (pp. 51-52)

1. If \( r_n = 5 + 7n \), then \( r_1 = 5 + 7(1) = 12 \), \( r_2 = 5 + 7(2) = 19 \), \( r_3 = 5 + 7(3) = 26 \), \( r_4 = 5 + 7(4) = 33 \).
2. If \( s_1 = 5 \) and \( s_n = s_{n-1} + 7 \), \( n > 1 \), then \( s_1 = 5 \), \( s_2 = s_2 = 7 \), \( s_3 = s_3 + 7 = 12 \), \( s_4 = s_4 + 7 = 19 \), \( s_5 = s_5 + 7 = 26 \), 3. \( t_n = 5 + 7(t) = 5 + 56 = 61 \). Since \( s_5 = s_5 + 7 = 26 + 7 = 33 \), \( s_6 = s_6 + 7 = 40 \), and \( s_7 = s_7 + 7 = 47 + 7 = 54 \). 5. If \( t_n = 5 + 7n \), then \( t_n - 5 = 7n \) and \( n = (t_n - 5)/7 \). 6. Since \( t_n = n + 40 \), \( t_1 = 3(1) + 40 = 43 \), \( t_2 = 3(2) + 40 = 46 \), \( t_3 = 3(3) + 40 = 49 \), \( t_4 = 3(4) + 40 = 52 \) the program will print 43, 46, 49, 52. 7. \( (2x - 3)(6x - 5) = (2x - 3)6x - (2x - 3)5 = 12x^2 - 18x - 10x + 15 = 12x^2 - 28x + 15 \). 8. \( 3(4 + a) = -25 - 25 = 12 + 3a - 25 \); \( t_a = 4a - 13 - 9 \). If \( d = \frac{1}{2}gr^2 \), then \( d = \left( \frac{1}{2} \right)(32)(3)^2 = (16)(9) = 144 \). 10. If \( 1.7y = 0.9 + .5y \), then \( 10(1.7y) = 10(0.9 + .5y) \), \( 17y = 9 + 5y \), \( 12y = 9 \), and \( y = \frac{9}{12} = .75 \).
11. If \( \frac{7}{x} = 3 \), then \( 3x = .7 \) and \( x = \frac{7}{3} = 0.23 \).
12. If \( .12x + .08(15,000 - x) = 1480 \), then \( 100[.12x + .08(15,000 - x)] = 100(1480) \), \( 12x + 8(15,000 - x) = 148,000 \), \( 12x + 120,000 - 8x = 148,000 \), \( 4x + 120,000 = 148,000 \), \( 4x = 28,000 \), and \( x = 7000 \). 13. If \( \frac{1}{2}p = 1 + p \), then \( \frac{1}{2}p = 1 \) and \( p \leq -2 \); See below.
14. b. See below. 14. Choices (b) and (c) are not formulas. 15. Choice (b) is solved for \( d \). 16. A counterexample is \( t = -3 \); \( t^2 = 9 \) but \( t \neq 3 \). 17. Commutative Property of Addition 18. Distributive Property 19. Definition of division

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 53-55 or to the Vocabulary (Voc.) on page 51. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

**Question** | 1 | 2 | 3 | 4 | 5 | 6 | 7-8 | 9 | 10-12 | 13
---|---|---|---|---|---|---|---|---|---|---
**Objective** | A | A | A | A | A | E | B | C | A | D
**Lesson** | 1-3 | 1-4 | 1-3 | 1-4 | 1-8 | 1-3 | 1-5 | 1-2 | 1-7 | 1-9

**Question** | 14 | 15 | 16 | 17-19 | 20 | 21 | 22-23 | 24 | 25 | 26
---|---|---|---|---|---|---|---|---|---|---
**Lesson** | 1-2 | 1-8 | 1-6 | 1-5 | 1-2 | 1-2 | 1-1 | 1-4 | 1-2 | 1-2

**CHAPTER 1 REVIEW** (pp. 53-55)

1. .119 3, 6973.57 5. 10, 3, -4, -11, -18 7. .3, .03, .003, .0003, .00003 9. a, 100 b, 30 c. NEXT N 11. \( y + x + 11 \) 13. \( 20 - 5a = 5b \) 15. \( ab + ad + bc + bd = 17 \)
19. \( \frac{1}{2} 21 \), \( w = 5 \) 23. \( 32,500 \) 25. \( n = \frac{27}{n} \) 27. 3 29. Sample:
49. of a Sum c. Opp-Op d. Comm. Prop. of Add. e. def. of subt. 39. \( n > 0 \); \( n \) an integer 41. a, b, 26,000; 27,560; 29,212,60; 30,966.42; 32,824.40 b, 78,665.59 43. \( T = 7 \) 45. \( x = 7 \) 47. a, \( 4000 + 200d = 14,000 \) b. \( d \geq 50 \)
51. c
LESSON 2-1 (pp. 58-62)
3. b 5. sample: Let \( n \) = the number of cans and \( r \) = the refund on \( n \) cans at 5¢ per can. 7. -120 9. 100 ft 11. a. 3 b. 300,000 13. \( d = kr \) 15. a. sample: if \( s = 10 \), then \( d = 6.25 \) ft b. 25 ft c. 4 (no matter what values were chosen in a. and b.) 17. 1985 19. See below. 21. 36

LESSON 2-2 (pp. 63-67)
3. a. 5 b. 300 ft \( \approx 4.4 \) ft 9. 80 11. \( I = k/D^2 \) 13. directly 15. inversely 17. 94 lb 19. \( y \leq \frac{2}{3} \) 21. \( 8x^3 \)

LESSON 2-3 (pp. 68-72)
3. multiplied by \( c^n \) 5. Nathan is \( \frac{1}{3} \) as far from the pivot as Oprah. 7. \( y \) is divided by 81 9. \( y \) is divided by 4. 11. In a direct variation, doubling \( x \) multiplies \( y \) by \( 2^n \), whereas in an inverse variation, \( y \) is divided by \( 2^n \). 13. \( \frac{4}{3} \) 15. \( y_2 = \frac{k}{(cx)^n} = \frac{k}{c^n x^n} = \frac{1}{c^n} \frac{k}{x^n} = \frac{1}{c^n} \cdot y_1 = \frac{y_1}{c^n} \) 17. about 5.86 19. a. \( x = \pm 7 \) b. \( x = \pm \frac{7}{3} \) c. \( x = \pm \frac{7}{2} \)

LESSON 2-4 (pp. 73-78)
3. \( \frac{1}{5} \) 5. a line; \( k; (0, 0) \) 7. 0.06 9. -\( \frac{1}{3} \) 11. positive, negative 13. a. \( k \) b. The slope of the line whose equation is the form \( y = kx \) is \( k \). 15. a. \( W \) is divided by 3. b. \( W \) is multiplied by 2. 17. a. \( x = \frac{2y}{3} \) b. \( x = \frac{2y}{3} \) c. \( x = \frac{2y}{3} + 2 \) d. \( x = \frac{2y}{3} - 1 \)

LESSON 2-5 (pp. 79-85)
3. False 5. If the parabola is folded about the y-axis, the halves of the parabola coincide. 7. a. See below. b. See below. 9. \( a \cdot y = 3x^2 \) b. \( y = \frac{1}{2}x^2 \) c. \( y = -2x \) d. \( y = -x^2 \) 11. a. 11 b. -5, 25 k c. 5, 25 k d. 21 pairs of the form \( x, kx^2 \) where \( x \) goes from -5 to 5 by increments of \( \frac{1}{2} \). 13. a. \( l = k/d^2 \) b. \( \frac{1}{16} \) 15. See below. 17. a. \( \triangle DEF; \triangle KLM \) b. ASA, SSS

LESSON 2-6 (pp. 86-91)
3. the graph in the interval \(-50 \leq x \leq 30\) and \(-10 \leq y \leq 4\) 5. yes 7. a. the first b. the first 9. a. See below. b. -6 11. a. See below. b. See below. c. See below. 13. See below. 15. a. 1.5 b. 216 c. 1.5 d. -108 17. a. \( 6x + 8 \) b. \(-2x - 2\) c. \( 8x^2 + 22x + 15 \) 19. a. See below. b. See below. c. Many answers are possible. The largest value of \( b \) on our grapher is 370.

9. a. (i) Many possible answers.
LESSON 2-7 (pp. 92-98)
3. a. $-\frac{3}{4}$ b. $-\frac{3}{16}$ 5. $\frac{1}{3}$ 7. II and IV 9. b 11. a. See below.
b. -2 c. See below. d. $-\frac{2}{5}$ e. the graph of $y = \frac{24}{x^2}$ 13. a. 2
b. $y = x$, $y = -x$ c. Yes 15. $y = -16x^2$ 17. Multiplication
Property of Equality, Distributive Property 19. a. Yes b. by
SAS or SSS (use Pythagorean Theorem) to show $EC = CA$)

LESSON 2-8 (pp. 99-104)
5. substitute 40 for $P$. $\frac{400}{4^2} = 20.75$, not 42, so the formula
is not a good model. 7. 92 ft$^3$ 9. a. See below. b. Quadrant
III is part of “all solutions” but not part of real-world
applications. 11. d 13. -1.36 15. c 17. a 19. $k/r$

LESSON 2-9 (pp. 105-109)
3. The points appeared to lie on a line through the origin. 5. a
hyperbola 7. Using $MAXWT = \frac{K}{d^2}$, with $K = 800$. Using
this value of $K$ when $d = 4$, $\frac{800}{16} = 50$. Yet
$MAXWT = 200$ when $d = 4$. So this is not a good model.
Using $MAXWT = \frac{K}{d^2}$, with $K = 800$. When $d = 4$, this
model predicts $MAXWT = \frac{800}{4} = 200$, which is correct.
9. a. See below. b. directly c. See below. d. inversely
9. e. $V = \frac{kT}{P}$ 11. a. 0 b. iii 13. a 64.6 m b. 1.22 15. 0 $\leq$
w $< 3.0$ m

LESSON 2-10 (pp. 110-114)
3. 66.7 ft$^2$/in$^3$ 5. $R = \frac{kL}{d^2}$ 7. About 430 BTU 9. $k =$
w$\frac{w}{y/xz}$ 11. about 63 min 13. It stays the same. 15. a. See be-
low. b. parabola c. 30 d. No 17. False 19. if $a < b$, then
$a + c < b + c$ 21. $x \geq -27$

15. a.
CHAPTER 2 PROGRESS SELF-TEST (pp. 117–118)

1. \( y = \frac{k}{x} \)
2. \( n = \frac{k}{d^2} \)
3. The two formulas are \( w = k_i d^4 \) and \( w = \frac{k_2}{L^2} \), so the formula for all 3 variables is \( w = \frac{kd^4}{L^2} \).
4. Using \( s = kp^2 \) with \( s = 10 \) and \( p = 3 \), find \( k \):
   \[ 10 = k(3)^2, \text{ so } k = \frac{10}{9} = 1.1. \text{ Then use the formula } s = \frac{1.1}{p^2} \text{ for } p = 8; s = 1.1(8)^2 = 71.1. \]
5. For \( y = 3x^2 \), when \( x \) is doubled, \( y \) is multiplied by \( 2^2 \) or 4.
6. For \( y = \frac{6}{x} \), when the \( x \)-value is multiplied by \( c \), the \( y \)-value is divided by \( c \).
7. For \((12, 18)\) and \((20, 30)\), \( (y_2 - y_1)/(x_2 - x_1) = \frac{18 - 15}{20 - 12} = \frac{3}{8} \).
8. False (counterexamples are \( y = \frac{k}{x} \) or \( y = \frac{k}{x^2} \), where \( x \) cannot be zero).
9. parabola, \( k \) is positive
10. The graph is not continuous.
11. a. neither inversely nor directly (Since \( SA = 4\pi r^2 \), the surface area of a sphere varies directly as the square of the radius.)
   b. inversely
12. See below.
13. See below.
14. Since the graph is symmetric to the \( y \)-axis, and is in quadrants III and IV, the form is \( y = \frac{-k}{x^2} \), which is option c. \( y = \frac{-3}{x^2} \).
15. d. See below.
16. a. Use \((3,620)\) to find a value for \( K \) in \( F = \frac{K}{L} \); \( 620 = \frac{k}{3} \), so \( k = 620 \cdot 3 = 1860 \).
   Test this model \( F = \frac{1860}{L} \) with other points: for \((5, 372)\), the model predicts \( F = \frac{1860}{5} = 372 \), which checks. The other ordered pairs also check, so the model is \( F = \frac{K}{L} \) or \( F = \frac{1860}{L} \).
17. The formula for \( V \) and \( g \) is \( V = k_i g^2 \), and the formula for \( V \) and \( h \) is \( V = k_i h \). The formula for all 3 variables is \( V = khg^2 \).
18. The model is \( s = kPt^4 \). Use the values \( s = .09604 \), \( r = .07 \) (since \( d = .14 \)), and \( P = 100 \) to calculate \( k: .09604 = k(100)(.07)^4 \) so \( k = \frac{.09604}{100(.07)^4} = \frac{.09604}{.02401} = 40 \).
   Then use the formula \( s = 40P/r^4 \) to calculate \( P \) for \( r = .05 \).

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 119–123 or to the Vocabulary (Voc.) on page 117. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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CHAPTER 2 REVIEW (pp. 119–123)

1. \( y = kx^2 \)
2. \( n = k/r^3 \)
3. \( U = \frac{kV^2}{LD} \)
4. \( 7. P = kmld^2 \)
9. jointly; \( s, t, \) and \( u \)
11. 21 13. -32 15. 1.8 17. -25
19. \(-\frac{7}{16} = -\frac{35}{4} \)
21. \( s \) is tripled.
23. \( p \) is divided by 4.
25. divided by \( c^4 \)
27. not affected
29. \((0, 0) \)
31. b
33. True
35. inversely
37. directly
39. a. See below.
40. \( L = k\overline{5} \)
41. See below.
42. \( P \) varies directly as \( R^3 \).
43. \( \$13.50 \)
45. 3 minutes
47. about 15,188 lb
49. See below.
53. See below.
55. c
57. b
59. See below.
61. d

since \( d = .1 \) and \( s = .09604; .09604 = 40P(.05)^4 \), so \( P = \frac{.09604}{(40)(.05)^4} = \frac{.09604}{.0005} = 384.16 \approx 384 \) units.
**LESSON 3-1** (pp. 124-131)

5. True 7. the initial condition 9. 53 days 11. a. 6 b. -5
13. a. 1 b. 3 15. a. (4, 10) is on the graph because it satisfies y = \(\frac{3}{4}x + 7\); y = \(\frac{3}{4}(4) + 7 = 3 + 7 = 10\). b. \(\frac{3}{4}\)
c. \(\frac{10}{4} - 7 = \frac{1}{4}\) d. 7 17. a. \(y = \frac{7}{2}x\) b. See below.
c. \(m = \frac{7}{2}\), \(b = 0\) d. constant increase 19. \(y = \frac{1}{2}x + \frac{5}{2}\)  
21. a. \(9\pi\) cm\(^2\)  b. \(\frac{1}{4}\pi x^2\) cm\(^2\)  23. a

17. b.

**LESSON 3-2** (pp. 132-137)

1. slope-intercept 3. b 5. Does 2(-1) = -3(4) + 10? Yes, -2 = -12 + 10. 7. \(\frac{1}{3}\) 9. \(\angle HGI \equiv \angle KJL; \angle GHI \equiv \angle JKL; \angle GHJ = JK 11. See below. 13. a. See below. b. \(y = \frac{2}{3}x - 6\)
c. \(x = 22\frac{1}{3}\); yes 15. a. \(y = -\frac{5}{8}x + 12\) b. slope: \(-\frac{5}{8}\); y-intercept: 12 c. See below. 17. a. See below. b. See below. 19. negative 21. neither (zero) 23. 5360.3825  25. \(x < 22.5\)

**LESSON 3-3** (pp. 138-142)

3. \(2W + 7\) 5. linear combination 7. a. 6S b. .9N c. 6S + .9N d. .6S + .9N = 18 e. \(N = 20 - .67S\) See below.  
f. 14 oz 9. a. nonnegative integers b. 5S + 10L = 70  
c. See below. d. 7 and 0, 6 and 2, 5 and 4, 4 and 6, 3 and 8, 2 and 10, 1 and 12, 0 and 14. 11. a. \(\frac{3}{2}\) b. -4 c. \(y = \frac{3}{2}x - 4\)  13 15 18 17. \(\sqrt{(a - c)^2 + (b - d)^2}\)  19. See
below. 21. a. \( R = 100 - \frac{3}{2}d \) b. 135 days

7. e.

9. c.

19.

LESSON 3-4 (pp. 143-144)
1. a 3 b 5 c 2 7. a. See below. b. The slope is undefined. 9. horizontal; 0 11. y 13. a. horizontal b. x-intercept: none; y-intercept: 4 c. See below. 15.a. vertical b. x-intercept: 8; y intercept: none 15.c. See below. 17. 10x - 5y = 1 19. a. 0.8S + .06R = 84 b. samples: (1000, 300); (600, 600); (200, 900) 21. b

7. a.

13. c.

15. c.

LESSON 3-5 (pp. 148-153)
3. The y-intercept is given. 5. True 7. a. point-slope b. slope-intercept c. point-slope 9. \( y - 1 = \frac{2}{3}(x - 7) \)

11. The point-slope form is \( y - 16 = -2(x - 5) \). This is equivalent to the standard form \( 2x + y = 26 \). 13. \( K = 273.15 \cdot \frac{5}{9}(F - 32) \) 15. a. See below. b. \( \overline{PQ} \): \( x = 3; \overline{QR} \): \( y = -5; \overline{RS} \): \( x = -2; \overline{SP} \): \( y = 4 \) c. 45 sq. units 17. \( L = 18 \) 19. \( A + 2B + 3C > 370 \)

LESSON 3-6 (pp. 154-159)
1. b. See below. 3. a. \( -\frac{3}{2}, 6, -\frac{5}{2} \) b. \( \frac{1}{2} \) c. See below.

7. a. \( a_n = 6 + (n - 1)9 \) b. 897 9. 16 rows 11. a. no b. The domain is the set of natural numbers. 13. \( \frac{5}{2} \) mi

15. 49th 17. \( y = \frac{3}{5}x - 4 \) 19. \( y = \frac{1}{3}x + \frac{51}{5} \) 21. c

1. b.

3. c.

LESSON 3-7 (pp. 160-163)
3. a. 1, 7, 13, 19 b. 10 LET A = 1

20 FOR N = 1 TO 25
30 PRINT A
40 LET A = A + 6
50 NEXT N
60 END

5. a. \( \{a_1 = 13 \}

\[ a_n = a_{n-1} + 6 \] for \( n > 1 \).

b. \( a_n = 13 + (n - 1)6 \) 7. \( a_1 = 13.2; a_n = a_{n-1} + 2.4 \) for \( n > 1 \) 9. \( a_n = -x + (n - 1)3x = 3xn - 4x \)

11. 10 FOR N = 1 TO 1000 13. 10 FOR N = 1 TO 13

20 PRINT 2 - N - 1 20 PRINT
30 NEXT N 30 NEXT N
40 END 40 END

15. 626 17. \( 2x + y = 0 \) 19. Use \( y = 3 - x \); (3-2, 1-1, E) a. See below. b. \( 0 \leq x \leq 3 \) c. (1.5, 1.5)
**Lesson 3-8 (pp. 164-168)**

3. $2\frac{1}{2}$ hr, 5. $14$ mph, 7. $\$42$ 9. $1.2$ mi 11. $\frac{1}{2}$ hr 13. See below.

15. a. See below. b. $c = 10.15$ for $0 \leq m \leq 120$; $c = 0.035m + 5.95$ for $m > 120$. 17. a. $a_n = 7 - 5n$

b. $a_1 = 2; a_n = a_{n-1} - 5$ for $n > 1$ 19. 13 rows

21. $y > \frac{3}{8}x + \frac{3}{4}$ 23. translation 25. rotation 27. rotation

**Chapter 3 Progress Self-Test (pg. 175)**

1. See below. 2. See below. 3. See below. 4. a. $m = \frac{-A}{B} = \frac{4}{5}$ b. x-intercept: $C = \frac{12}{4} = 3$; y-intercept: $C = \frac{12}{3} = \frac{12}{5}$

5. negative values of $m$ 6. $m = \frac{2 - \frac{3}{4}}{4 - (-5)} = \frac{1}{9}$

$y - 2 = \frac{3}{4}(x - 4); y - 2 = \frac{4}{9}x + \frac{5}{2}; m = \frac{3}{5}$

$y - (-1) = \frac{5}{3}(x - 5); y + 1 = \frac{5}{3}x - \frac{25}{3}; 3y + 3 = 5x - 25; 5x - 3y = 28$. 8. a. vertical lines b. horizontal lines 9. $36S + 48L$ in. 10. $3000 = 36(50) + 48L$; $3000 = 1800 + 48L$; $1200 = 48L$; $L = 25; 25$ long laces 11. $-40 + 0.8$ meters 12. $-10 = -40 + 0.8r; 30 = 0.8r; t = 37.5$ sec. 13. a. The program represents the recursive formula of a sequence. The formula is $a_n = a_{n-1} - 3$, for $n > 1$ and the first twelve terms of the sequence are: 1, 6, 11, 16, 21, 26, 31, 36, 41, 46, 51, 56 b. Yes 14. $3y = x + 6; x - 3y = -6$. 15. a. $a_1 = -7; d = -3; a_n = -7 - 3n + 3; a_n = -3n - 4$. 17. $a_1 = -7; d = -3; a_n = a_{n-1} - 3$, for $n > 1$.

18. a. $0 \not\equiv 0; No$. b. $3(2) - 5(-1) \not\equiv 8; 6 - (-5) \not\equiv 8$; $11 \not\equiv 8$; No 19. $a_1 = 20; d = 15; a_n = 110$; therefore $110 = 20 + (n - 1)15; 110 = 20 + 15n - 15$;

$105 = 15n; n = 7; 7$ weeks 20. a. $m = \frac{800 - 400}{400 + 360}; \frac{400}{760} = \frac{4}{3}$. b. $y = \frac{4}{3}x - 400$
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 176–179 or to the Vocabulary (Voc.) on page 174. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 3 REVIEW** (pp. 176–179)

1. a. 7 b. -2 3. a. 0 b. 4 5. a. \(x = -4.7\) b. none 7. \(y = 8x - 245\) \(y - 4 = \frac{3}{2}(x - 2)\) 11. \(3x + 2y = 1\) or \(y - 2 = \frac{1}{3}(x + 1)\) 13. \(y = \frac{1}{2}x + 2\) 15. \(2x - 3y = 5\)

17. a. \(a_n = 7 + 5(n - 1)\) b. \(a_1 = 7; a_n = a_{n-1} + 5\) for \(n > 1\) c. 377 19. \(a_1 = -9; a_n = a_{n-1} + 2\) for \(n > 1\) 21. a. 1000 b. 100, 101, 102, 103, 104 horizontal 25. vertical 29. The graph of \(y > 2x - 7\) is the half-plane to the left and above the line \(y = 2x - 7\). 31. no 33. positive 35. slope-intercept 37. horizontal 39. adding; constant difference 41. yes 43. yes 45. yes 47. no 49. \(w = 3 + .2n\)

51. 500 - 30w 53. $124,000 55. a. \(C = 3p + 1000\) b. $2500 57. a. \((A + B)\) gal b. \((.06A + .08B)\) gal c. \(.06A + .08B = 259\) 59. 5 years 61. See below. 63. See below. 65. See below. 67. zero 69. \(x + 2y = 4\) 71. 45 mph 73. See below.

**LESSON 4-1** (pp. 182–186)

3. a. 5 b. 4 c. 5 × 4 7. total number of shorts 9. \(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\); point 11. a. \(\begin{bmatrix} -1 & -4 & 3 & 6 & 4 \\ 6 & 1 & -2 & 2 & 7 \end{bmatrix}\) b. No; corresponding elements are not equal. 13. a. \(4 \times 2\) b. the total number of Navy personnel c. the total number of commissioned officers 15. 2.0 17. See below. 19. b. c. e. f

**LESSON 4-2** (pp. 187–191)

1. [-11] 3. a. \(2 \times 3\), \(2 \times 2\) b. No c. The number of columns in the first matrix does not equal the number of rows in the second matrix. 5. \(\begin{bmatrix} 22 & 50 \\ -12 & 3 \end{bmatrix}\) 7. when \(n = p\)

9. Matrix multiplication is associative. 11. \(\begin{bmatrix} 8 & -8 & 6 \\ -1 & 4 & 2 \\ -13 & 16 & -7 \end{bmatrix}\)

13. a. \(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\) b. \(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\) c. True 15. $921.50

17. \(\begin{bmatrix} 50 & 40 & 17 \\ 100 & 80 & 3 \end{bmatrix}\) 19. The distance between \((1, 5)\) and \((7, 6)\) is \(\sqrt{37}\); between \((1, 5)\) and \((-5, 4)\) is \(\sqrt{37}\); between \((7, 6)\) and \((-5, 4)\) is \(2\sqrt{37}\). The triangle is isosceles.

**LESSON 4-3** (pp. 193–198)

1. A size change of magnitude 3 maps \((3, 1)\) onto \((9, 3)\). 7. True 9. True 11. a. \(P' = (7.5, 10)\) b. \(y = \frac{4}{5}x\) 13. a. \(\frac{7}{2}\)
b. \(\frac{7}{2}\) c. Yes, because the slopes are equal. d. Yes, because the slopes are \(\frac{4}{6}\).

15. a. \[
\begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
2 & 3 & 4 \\
7 & 6.2 & 8.2
\end{bmatrix}
\]

c. twice as long 17. a. \(3 \times 4\) b. \[
\begin{bmatrix}
25 \\
70 \\
30
\end{bmatrix}
\]
c. Chicago:

\$2,105,000, Minneapolis: \$1,125,000. Syracuse: \$465,000

19. \(a = \frac{3}{2}b = \frac{7}{2}\) 21. See below.

21.

![Graph with labeled axes and a shaded area]

---

**LESSON 4-4** (pp. 199–203)

3. \(a = 2, b = \frac{1}{3}\) 5. shrink, stretch 7. a. \(\frac{7}{2}\) b. \(\frac{3}{2}\) c. No; the slopes are different. d. No 9. size change, \[
\begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
\]

11. a. rectangle b. \[
\begin{bmatrix}
0 & 12 & 12 \\
12 & 0 & 0
\end{bmatrix}
\]
c. square d. See below. 13. a. See below. b. \(45^\circ\)

15. \[
\begin{bmatrix}
9 & 3 & -12 \\
3 & 5 & -2 \\
-6 & 1 & 12
\end{bmatrix}
\]

17. a. \(x = 225\) b. \(y = 25\)

11. d.

13. a.

---

**LESSON 4-5** (pp. 204–209)

3. \[
\begin{bmatrix}
2 & 4 & 4 \\
1 & 1 & 2
\end{bmatrix}
\]
or \(A^1 = (2, 1), B^1 = (4, 1), C^1 = (4, 2)\);

11. a. \(F' = (a, c), S' = (b, d)\) b. column c. column d. The first column is the image of \((1, 0)\) and the second column is the image of \((0, 1)\).

13. \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

is the matrix for \(r_{\gamma-x}\). 15. The area is multiplied by 9. If \(A = bh\) and \(b^1 = 3b\) and \(h^1 = 3h\), then \(A^1 = b^1h^1 = (3b)(3h) = 9bh = 9A\).

17. \(110^\circ\) 19. See below.

---

**LESSON 4-6** (pp. 210–215)

3. sample: \[
\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 2 \\
2 & 3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & 2 \\
0 & 1
\end{bmatrix}
\]

5. a. \((AB)C = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\begin{bmatrix}
e & f \\
g & h
\end{bmatrix}\begin{bmatrix}
i & j \\
k & l
\end{bmatrix}
\]

\[
= \begin{bmatrix}
ae + bg & af + bh \\
cg + dh & c + dh
\end{bmatrix}\begin{bmatrix}
i & j \\
k & l
\end{bmatrix}
\]

b. \(A(BC) = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\begin{bmatrix}
e & f \\
g & h
\end{bmatrix}\begin{bmatrix}
i & j \\
k & l
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\begin{bmatrix}
ei + fk & ej + fl \\
gi + hk & gj + hl
\end{bmatrix}
\]

b. Size change transformations are commutative.

17. isosceles right triangle b. \[
\begin{bmatrix}
-28 & 28 & 0 \\
0 & 0 & 7
\end{bmatrix}
\]
c. isosceles d. 49 units\(^2\), 196 units\(^2\)

---

**LESSON 4-7** (pp. 217–221)

1. \(135^\circ\) 9. (-5, 3) 11. (5, -3) 13. \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

15. a. \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

b. \(r_x\) 17. a. See below. b. See below. c. \(R_{180}\)

19. \[
\begin{bmatrix}
1 & 0 \\
0 & 3
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
1 & 0 \\
0 & 3
\end{bmatrix}
\]

23. 6.94 25. Perpendicular lines are lines that meet to form equal adjacent angles; lines that meet to form \(90^\circ\) angles; lines that meet to form right angles.

27. a. The reflection over the y-axis

b. \[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}\begin{bmatrix}
-0 & -4 \\
-6 & 4
\end{bmatrix}
\]

= \[
\begin{bmatrix}
0 & 4 \\
6 & 4
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
3 & 3 \\
2.5 & 2
\end{bmatrix}
\]

c. \(r_x(x, y) = (-x, y)\)
5. a. 3  b. $y + 2 = 3(x - 7)$  7. $y - 3 = \frac{2}{7}(x + 1)$  9. a
11. $x = 6$  13. $a \parallel b \perp c \perp d \parallel$  15. $a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ b. $r$,
17. 1, 9, 36, 100, 225  19. 2307.7

LESSON 4-9 (pp. 228–233)
3. $\begin{bmatrix} 13 & 11 & 13 & 13 \\ 8 & 7 & 10 & 7 & 6 \\ 23 & 32 & 26 & 36 & 17 \\ 20 & 23 & 24 & 39 & 29 \end{bmatrix}$
5. $\begin{bmatrix} 6 & -6 \\ 18 & -16 \end{bmatrix}$  7. $\begin{bmatrix} 1 & -10 \\ -36 & 18 \end{bmatrix}$
9. a. $\begin{bmatrix} 8 & -1 \\ -8 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 8 & -1 \\ -8 & 1 \end{bmatrix}$ c. Addition of matrices is associative.
11. $a = 8, b = \frac{3}{2}, c = 21, d = 24.5$
13. a. $\begin{bmatrix} 10 & -4 & -6 & 14 \\ 8 & -6 & -2 & 14 \\ 9 & -12 & 3 & 21 \end{bmatrix}$ b. How many more points each

team had in 1983-84 than in 1982-83. c. How many more wins each team had in 1983-84 than in 1982-83.
15. a. $y - 0 = \frac{1}{2}(x - 3)$ b. $y - 0 = -2(x - 3)$  17. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
19. $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$  21. on $y - 21 = \frac{17}{13}(x - 5)$

CHAPTER 4 PROGRESS SELF-TEST (p. 240)
1. See below.  2. $\begin{bmatrix} 14 & 3 & 8 \\ 120 & 190 & 250 \end{bmatrix}$  3. Two matrices with
dimensions $m \times n$ and $n \times p$ can be multiplied—the number of columns in the left matrix must match the number of rows in the right matrix. $AB$ would be a $3 \times 2$ times a $2 \times 2$ matrix; the product would exist. $BA$ would be a $2 \times 2$ times a
$3 \times 2$ matrix; the product would not exist.
4. $BC = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 8 & 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 16 & 10 \end{bmatrix}$
5. $B - C = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 6 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 1 & 3 \end{bmatrix}$
6. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$  7. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 8 & 6 \end{bmatrix}$
8. $7B = 7 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & 35 \end{bmatrix}$
9. Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the result of multiplying any $2 \times 2$ matrix by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the original matrix. Another way to express that is the image of any point under that transformation is the same as the pre-image.
10. The slope of the line $y = 5x - 3$ is 5, so the slope of the line perpendicular to $y = 5x - 3$ is $-\frac{1}{5}$. Using the point (3, -2.5) and $y - y_1 = m(x - x_1)$, the desired equation is $y = -(2.5) - \frac{1}{5}(x - 3)$ or $y + 2.5 = -\frac{1}{5} (x - 3)$.
11. $r_x \cdot R_{270} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$  12. Each point is moved 8 units to the right and 8 units down, so the transformation is $T_{8, -8}$.
13. The product can be written as
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 241-243 or to the Vocabulary (Voc.) on page 239. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

### Question 1 2 3 4 5 6 7 8 9 10

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<thead>
<tr>
<th>Objective</th>
<th>H</th>
<th>D</th>
<th>C</th>
<th>A</th>
<th>A</th>
<th>G</th>
<th>G</th>
<th>G</th>
<th>C</th>
<th>B</th>
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### Lesson 4-7 4-5 4-2 4-8 4-2 4-4 4-7 4-10 4-7 4-3 4-5

### CHAPTER 4 REVIEW (pp. 241-243)

1. \([59] [512 \ 200] 5. \begin{bmatrix} 11 & 6 \\ 4 & -8 \\ 8 & 2 \end{bmatrix}

7. \begin{bmatrix} 2 & 33 \\ 13 & 3 \\ -7 & -13 \end{bmatrix}
9. \(a = -8; b = 5\)
11. \(a = 8; b = 5\)

13. \(y = 8\)
15. \(y - 2 = 4(x - 2)\)

17. a. Yes
b. See Question 5, Lesson 4-6.

19. a. \(TN\)
b. \(q \times p\)

21. \begin{bmatrix} 5 & 3 & 1 \\ 10 & 12 & 6 \end{bmatrix}
23. element in 2nd row and 1st column

25. Factory 1: $7,670,000; Factory 2: $4,390,000

27. \begin{bmatrix} 249 & 403.20 & 154.80 \\ 118.80 & 236.40 & 65.40 \end{bmatrix}
29. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}:

it is the identity transformation.

31. a. \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
b. \(r_3\)

33. c

35. e
37. \begin{bmatrix} 0 & 1 & 5 & 4 \\ 0 & -4 & -3 & 1 \end{bmatrix}
39. \(T_{2,-1}\) 41. \begin{bmatrix} 1 & 3 & 9 & 6 \\ 1 & 7 & 1 & -6 \end{bmatrix}

43. See below.
LESSON 5-1 (pp. 246–250)
3. See below. 7. \( \{ x : 0 < x < 10 \} = \{ x : x > 0 \} \cap \{ x : x < 10 \} \)
9. a. See below. b. See below. 13. a. ii b. iv c. i d. v
15. The conjunction should be "or." 17. a. See below.
19. a. 67, 68, 69, . . . , 83 b. 1, 2, 3, . . . , 100 21. a. \( y \) is divided by 4. b. \( y \) is multiplied by 8. 23. \( \begin{bmatrix} -1 & -4 & 3 \\ 2 & 0 & -3 \end{bmatrix} \):
See below.

3.

\[
\begin{array}{c|c|c|c}
\hline
& 6 & 11 & \hline
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
x & y & \hline
\end{array}
\]

LESSON 5-2 (pp. 251–257)
3. a. (6, 20) b. (6, 20) satisfies both equations.
7. sample: \( \begin{cases} x - y = 5 \\ -2x + 2y = -10 \end{cases} \)
13. a. two b. consistent
c. (2, 1), \((-\frac{1}{2}, -4)\) d. substitute both points into both sentences of the system: \(2(1) = 2, 2(-2) = 1 = 3; \)
\((-\frac{1}{2})(-4) = 2, 2(-\frac{1}{2}) = (-4) = 3. 15. a. See below. b. 2
c. about \((2.5, 2.5)\) & \((-4.25, 9.25)\) 17. a. \( y = 9x \) b. father: 9 m; daughter: 45 m c. 10 sec d. 90 m 19. See below.
21. See below. 23. \(5.99x + 6.25y + 7.99z = T\)

15. a.

\[
\begin{array}{c|c|c|c}
\hline
& \hline
\end{array}
\]

LESSON 5-3 (pp. 258–261)
3. We would like to have techniques for solving systems that always provide exact solutions. 5. \( \frac{1}{2} \) servings of stew; 1 slice of bread 7. \((1.5, .75)\) 9. infinitely many
11. \( (\frac{7}{23}, \frac{1}{25}) \) 13. consistent 15. inconsistent 17. a. \( N + S = 35 \) b. \( N(60) + S(80) = 25.2 \) c. \( S = 21, N = 14 \)
19. See below. 21. a. one b. consistent

19.

\[
\begin{array}{c|c|c|c}
\hline
x & y & \hline
\end{array}
\]

LESSON 5-4 (pp. 264–268)
3. one 5. \((15, -14.7)\) 9. b. d 11. a. iii b. \( H = 48,000; V = 12,000 \) 13. juice: .4 C; vinegar: .2 C; oil: 5.4 C
15. \((\frac{2}{3}, \frac{1}{9})\) 17. \( \begin{bmatrix} 2 & \sqrt{3} & -1 \\ 0 & 5 & 1 \\ -4 & 11 & -2 \end{bmatrix} \) 19. \( \angle ROC = 70^\circ, \angle CKR = 60^\circ, \angle KRO = 80^\circ, \angle OCK = 150^\circ \)

LESSON 5-5 (pp. 269–274)
9. a. 2 b. \( \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix} \) c. \( \begin{bmatrix} 1 & -2 \\ -1 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)
11. a. \( ab \) b. \( \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \) c. \( \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

13. a. 64 b. 9 c. 576 15. a. \( e = \frac{1}{6}, f = \frac{1}{3}, g = -\frac{1}{2}, h = 0; \)
\( \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} \end{bmatrix} \) b. \( ad - bc = 6; \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \)
17. (2, 8) 19. When at least one of the equations has been or can easily be solved for one variable, or when the system has one linear and one non-linear equation, or when there are more than two variables and equations. 21. See below.
21. $y = 3x - 6$

LESSON 5-6 (pp. 275–281)
3. $4 + 3(6) = 22$; $2(4) - 6 = 2$ 5. one 7. the $2 \times 2$ identity matrix 9. the $3 \times 3$ identity matrix 11. infinitely many solutions 13. $n = 14$ 15. a. $(-3, 4, -4, 5)$

LESSON 5-7 (pp. 282–288)
3. Is $-6 > -2(10) + 6$? Yes. Is $-6 \leq \frac{1}{2}(10) - 3$? Yes, so the solution checks. 5. Is $-2 = -2(4) + 6$? Yes. Is $-2 = \frac{1}{2}(4) - 3$? Yes, so this is a solution for the vertex of the feasible set. 11. a. $(4, 5)$ b. $x \leq 6; y \geq 0; x \geq 0; y \leq x + 1$

LESSON 5-8 (pp. 288–294)
3. True 5. $\$20,965$ 9. a. See below. b. in rounded values.

LESSON 5-9 (pp. 295–299)
3. $\$8.00$ 5. $6x + y \leq 150$ b. $T = (20, 30);$ $U = (25, 0)$ c. No; profit is still maximized at $Q$. 7. a. $h$: oz of hamburger; $p$: potatoes b. See below.

The inequalities are $h \geq 0; p \geq 0; 0.8h + 1.1p \geq 5; 10h + 0p \geq 30; 6.5h + 4p \geq 35$ c. See below. d. $11h + 0.5p$

e. $(3, 3.9)$ f. 3 oz of hamburger; 3.9 potatoes 9. 25% aluminum: 32 kg; 75% aluminum: 128 kg 11. See below.
7. b.

<table>
<thead>
<tr>
<th></th>
<th>Iron (mg)</th>
<th>Vit. A (units)</th>
<th>Protein (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>hamburger</td>
<td>.8</td>
<td>10</td>
<td>6.5</td>
</tr>
<tr>
<td>potato</td>
<td>1.1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>required</td>
<td>5</td>
<td>30</td>
<td>35</td>
</tr>
</tbody>
</table>

**CHAPTER 5 PROGRESS SELF-TEST (p. 301)**

1. See below. 2. From the graph, the solutions are approximately (-1.7, -2.9) and (1.2, -1.4). 3. a. The system is inconsistent. b. The equations represent distinct parallel lines.

4. Substitute the first two equations into the third: 3r + 11 - 8(4t) = 4, 3r + 33 - 32t = 4, -29t = -29, r = 1; s = 4t = 4(1) = 4; r = t + 11 = (1) + 11 = 12. (r, s, t) = (12, 4, 1). 5. Add twice equation 1 to three times equation 2: 2(-3x + 3y) + 3(-4x - 2y) = 2(2) + 3(3), -6x + 6y - 12x - 6y = 4 + 9, -18x = 13, x = 13/18, -3x + 3y = 2, -3(11/18) + 3y = 2, 11/6 + 3y = 12/6, 3y = 1/6, y = 1/18.

(x, y) = (13/18, 1/18). 6. If 2e + 2s = 2.78 and 3e + 4s = 4.99, multiply the first equation by -2 and add it to the second: -2(2e + 2s) + (3e + 4s) = -2(2.78) + (4.99), -e + 3s + 4e = -5.56 + 4.99, -e = -0.57, and e = -0.57.

One egg might be 57 cents. 7. Det \[
\begin{bmatrix}
8 & 3 \\
6 & 5
\end{bmatrix}
\] is (8)(5) - (3)(6) = 40 - 18 = 22. The inverse of \[
\begin{bmatrix}
8 & 3 \\
6 & 5
\end{bmatrix}
\] is \[
\begin{bmatrix}
5 & 3 \\
6 & 5
\end{bmatrix}
\]. 8. Since \[
\begin{bmatrix}
8 & 3 \\
6 & 5
\end{bmatrix}
\] [x [y]
\] = \[
\begin{bmatrix}
5 & 3 \\
6 & 5
\end{bmatrix}
\] [x [y]
\] = \[
\begin{bmatrix}
5 & 3 \\
6 & 5
\end{bmatrix}
\] [x [y]
\]. 9. The two inequalities are y ≤ x and x < z, which is option c.

**10.** If c and s represent the numbers of chairs and sofas, respectively, then the system of inequalities is: 7c + 4s ≤ 133, 2c + 6s ≤ 72, c ≥ 0, and s ≥ 0.

11. See below. There are four vertices: (0, 0), (0, 12), (19, 0), and the intersection of 2c + 6s = 72 and 7c + 4s = 133. Add twice equation one to -3 times equation 2: 2(2c + 6s) - 3(7c + 4s) = 2(72) - 3(133), 4c + 12s - 21c - 12s = 144 - 399, -17c = -255, c = 15; 2c + 6s = 72 so 2(15) + 6s = 72, 30 + 6s = 72, 6s = 42, s = 7. The fourth vertex is (15, 7). 12. Substitute each of the vertices, (0, 0), (0, 12), (19, 0), and (15, 7), into the profit formula \( p = 80c + 70s \). For (0, 0), \( p = 80(0) + 70(0) = 0 \); for (0, 12), \( p = 80(0) + 70(12) = 0 + 840 = 840 \); for (19, 0), \( p = 80(19) + 70(0) = 1520 + 0 = 1520 \); for (15, 7), \( p = 80(15) + 70(7) = 1200 + 490 = 1690 \). The profit is maximized at $1690 by making 15 chairs and 7 sofas per day.

13. \( y = 7 \)
CHAPTER 5 REVIEW (pp. 302-304)

1. a. 3, 7, -1 5. (-6.5, -10.5) 7. \(\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right)\) 9. a. substitution, graphing, linear combination, use of matrices  b. (-4, -20)

11. a. 2 b. \(\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \end{bmatrix}\) 13. a. 18 b. \(\begin{bmatrix} \frac{1}{3} & \frac{2}{9} \\ \frac{1}{6} & \frac{2}{18} \end{bmatrix}\)

15. \(ad - bc\) b. \(\begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}\) 17. (-1, 3)

19. \(\left(\frac{4}{7}, \frac{17}{5}\right)\) 21. \(x = 2\) and \(y = 8\) 23. a. inconsistent b. no solutions 25. a. consistent b. one solution 27. \(t = \frac{21}{4}\)

29. a. c 31. at the vertices 33. $1.50 35. Sugar-O’s: 85 g 37. a. Spanish: 180 min; algebra: 60 min b. Spanish: 3.6 points; algebra: 1.5 points 39. See below. 41. See below. 43. See below. 45. \(x \leq -3\) and \(y \leq -1\)

47. See below. 49. \(x \leq -3\) and \(y \leq -1\)

LESSON 6-1 (pp. 308–313)

1. d 3. 20, -20 5. 1250 + 150w + 4w² 7. \(x^2 - x - 2\) 9. \(x^2 + 2xy + y^2\) 11. \(25n^2 + 80np + 64p^2\) 13. \(4w^2 - 2w + \frac{1}{4}\) 15. \((x - y)^2 = (x - y)(x - y) = (x - y)x - (x - y)y = x^2 - yx - xy + y^2) = x^2 - xy - yx - y^2 = x^2 - 2xy - y^2\) 17. \(x\) 19. \(\sqrt{(d - b)^2 + (c - a)^2}\) 21. \(\sqrt{104}\)

23. \(x = 3\) 25. \(\frac{1}{2}n^2 + \frac{3}{2}n\) 27. \(4xy\) 29. \(r = \sqrt{\frac{140}{\pi}}\)

6.7 inches 31. See below. 33. Use a compass or dividers to transfer the distance and draw the line with a ruler and T-square, or use a construction method. 35. a. \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) b. inverses

31.

LESSON 6-2 (pp. 314–321)

3. \(\frac{9.8}{\sec^2}\) 5. 15 ft 7. \(t \approx \frac{1}{4}\) or \(t \approx 2.5\) 9. \(h = -1.609\); therefore ball is below ground. 11. See below. 13. See below.

15. a. \(h = -\frac{1}{2}(9.8)r^2 + 10r + 1\) b. 6.1 m c. See below. 17. a. 96 + 40w + 4w² 19. a. \(x^2 - 4x + 4\) b. \(3x^2 - 12x + 12\) c. \(3x^2 - 12x\)

21. (15, 30)

11.
15. b. $\sqrt{(x - 0)^2 + (y - 2)^2} = \sqrt{(x - x)^2 + (y - (-2))^2}$
   $x^2 + (y - 2)^2 = (y + 2)^2$
   $x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$
   $x^2 = 8y$
   For $(4, 2)$
   $16 = 16$

21.
\[
y = x^2 - 6
\]
\[
y = (x-5)^2 - 6
\]

**LESSON 6-4** (pp. 329–333)
1. $x = \pm 10$  b. $x = 10$  c. $x = 13, x = -13$  d. $y = (x - 8)^2$ is the graph of $y = x^2$ moved 8 units to the right.
5. a. $(6, 7\frac{1}{2})$  b. $y = 6\frac{11}{12}$  c. $y = -7$  d. $y = 9$
7. a. $x = 11$  b. $x = 33$ or $-35$  c. $x = 11$
13. $y = 5, y = -5(x - 2)^2$  15. A parabola is a set of all points in a plane whose distance from given point equals its distance from a given line and the point is not on the line.
17. a. $x^2 + 8x + 16$  b. $2x^2 + 16x + 32$
19. $\frac{1}{2}n^2 - \frac{1}{2}n$

7. d.

**LESSON 6-5** (pp. 334–339)
1. $y = x^2 + 6x + 11$  3. $y = 2x^2 - 16x + 31$  5. $81$
7. $b^2 = 4$  b. $\left(\frac{1}{3}, \frac{5}{3}\right)$
9. $\left(\frac{2}{3}, \frac{1}{3}\right)$  11. $\left(\frac{2}{3}, \frac{11}{3}\right)$  13. $\left(\frac{1}{8}, \frac{141}{7}\right)$  b. 35.25 feet
15. $(y - 3)^2 = (x - y)^2$  17. $x = -10, x = -22$
19. $z = \pm \sqrt{2} + 5$
21. $200 \frac{ft}{min}$
23. True  25. True

**LESSON 6-6** (pp. 340–345)
7. $-1, -\frac{3}{10}$  9. $h = \text{height}; x = \text{distance}$
11. 56.125 ft and 343.8 ft 13. $n + 1, n + 2$  15. a. $a = 11; b = -20; c = -4$
b. $x = 2 \text{ or } x = -\frac{2}{11}$  17. a. $a = 4; b = -12; c = 9$
b. \( m = \frac{3}{2} \)  
19. a. \( x = 0 \) or \( x = \frac{-b}{a} \)  
   b. \( y = 0 \) or \( y = -\frac{8}{3} \)  
21. First mistake: one must use \( \frac{1}{2} \) or \( \frac{3}{2} \). Second mistake: one must use two parentheses.  
   23. \( \frac{7}{13} \)  
25. 0  
   27. The two equations are equivalent.  

LESSON 6-7 (pp. 346-352)  
7. \( a = 9 \)  
   a. \( 11 \)  
   b. \( 13 \)  
   The square root of a negative number is not real—it’s imaginary.  
15. one real root  
   17. It does not intercept the \( x \)-axis.  
   19. 1, rational  
   21. False, if discriminant \( = 0 \), the graph has exactly 1 \( x \)-intercept.  
   23. \( k = 6 \) or \( k = -6 \)  
   25. a. error message in line 300  
   b. Add these two statements:  
   150 IF \( A = 0 \), GO TO 650  
   650 PRINT “NOT A QUADRATIC EQUATION”  
27. 7.64, 2.64  
   29. They have the same roots because the two equations are equivalent.  
   31. \( \sqrt{7} \)  

LESSON 6-8 (pp. 353-356)  
3. False  
   7. \( (i\sqrt{5})(i\sqrt{5}) = i^2 \cdot 5 = -5 \)  
   9. \( x = \pm 4\sqrt{7} \)  
11. \( i\sqrt{7} \)  
13. \( 2i \)  
15. \( -3\sqrt{2} \)  
19. \( 7i \)  
21. \( 13i \)  
23. 5  
25. \( 3\sqrt{2} \)  
27. a. False: \( 2i + (-2i) = 0 \)  
   b. False: \( (2i)(3i) = -6 \)  
29. \( c. x = \frac{-11 \pm \sqrt{41}}{20} \)  
31. positive  

LESSON 6-9 (pp. 357-361)  
3. \( a = 14 \), \( b = 5 \)  
   5. \( a = 0 \), \( b = 1 \)  
   7. 29 - 7i  
   9. 5  
   11. a. FOIL  
   b. Dist. prop.  
   c. \( i^2 = -1 \)  
   d. Comm. prop.  
   13. \( a = \)Dist. prop.  
15. \( 25 - 2i \)  
17. \( -4 + 17i \)  
   19. 252 + 64i  
   21. \( a = 0 + 0i \)  
   b. \( 1 + 0i \)  
   23. a. 2a  
   b. \( (a - bi)(a + bi) = a^2 - b^2 \)  
   c. \( a^2 + b^2 \) \( = \)  
   25. \( \frac{1}{13} + \frac{5}{13} \)  
27. a. \( (-8, -4) \)  
   b. \( -8 \pm \sqrt{2} \)  
   c. See below.  
   29. Girolamo Cardano  
   31. \( \frac{-3 \pm \sqrt{17}}{2} \)  

LESSON 6-10 (pp. 362-366)  
1. 2 + \( i, 2 - i \)  
   3. 2 + 5i, 2 - 5i  
   5. \( 10 + 5\sqrt{6} \)  
   7. \( -\frac{2 - \sqrt{2}}{3} \)  
   9. \( -\frac{b}{c} \)  
11. negative  
   13. \( -\frac{5 \pm i \sqrt{5}}{6} \)  
15. a. See below.  
   b. \( x = 5 \)  
   c. \( x = 5 \pm i\sqrt{5} \)  
   d. Average of roots = 5; axis of equation is \( x = 5 \).  
   e. The equation of the axis of symmetry of a parabola is \( x = k \), where \( k \) is the average of the roots.  
17. a. vertex (1, 6); \( x \)-intercepts (0, 3.45), (0, -1.45); axis of symmetry \( x = 1 \)  
   b. See below.  
   19. a. \( AB = \)  
   b. \( CB = \)  
   c. \( BC = \)  

15. a.  

17. b.  

CHAPTER 6 PROGRESS SELF-TEST (p. 369)  
1. Rewrite \( y = x^2 - 8x + 12 \) in the form \( y = k = \)  
   \( a(x - h)^2 \).  
   y = -12 = \( x^2 - 8x \), y = -12 + 16 = \( x^2 - 8x + 16 \), y = 4 = \( (x - 4)^2 \).  
2. The vertex is (4, 4).  
3. The \( x \)-intercepts are the value(s) of \( x \) for which \( y = 0 \)  
   \( = \)  
   (x - 6) \( x - 2 \), so \( x = 6 \), or \( x = 2 \).  
4. \( 2i \cdot i = k^2 \)  
   2(1) = 2  
   \( -15 - \sqrt{8} \cdot \sqrt{2} = \sqrt{8} \cdot \sqrt{2} = \sqrt{16} \)  
   \( = (-1)(4) = -6 \)  
   \( 4 + \sqrt{8} = 4 + \sqrt{8} = \frac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2} \)  

7. \( (3i + 2)(6i - 4) = (3i)(6i) + (3i)(-4) + (2)(6i) + \)  
   \( (2)(-4) = 18i^2 - 12i + 12i - 8 = 18(-1) - 8 = \)  
   \(-18 - 8 = -26 \)  
   8. \( z - w = (2 - 4i) - (1 + 5i) = \)  
   \(-11 - 4i = -11 + 4i \)  
   9. By the Graph Translation Theorem, the graph of \( y = -2 = -(x + 1)^2 \) is 1 unit to the left and 2 units above the graph of \( y = -x^2 \). The choice is c.  
10. See below.  
11. Since the directrix, \( y = 6 \), is one unit from the vertex (3, 5), the focus is also one unit from the vertex, directly below it. The focus is (3, 4).  
12. If \( 3x^2 + 14x - 5 = 0 \), then \( (3x - 1)(x + 5) = 0 \). If \( 3x - 1 = 0 \),
then $3x = 1$ and $x = \frac{1}{3}$; if $x + 5 = 0$, then $x = -5$.

The solutions are $x = \frac{1}{3}$, $-5$. 13. If $(m + 40)^2 = 2$,
then $m + 40 = \pm \sqrt{2}$ and $m = -40 \pm \sqrt{2}$.

14. a. The Discriminant Theorem states that if the discriminant is zero, there will be 1 real root, and therefore one x-intercept. b. Since the discriminant is positive, there are 2 real roots, and there are 2 x-intercepts. 15. a. The Discriminant Theorem states that there is 1 real root if the discriminant is 0 and since 0 is a perfect square, the root is rational. b. Since the discriminant is negative, there are two complex roots.

16. $(a - 3)^2 = a^2 - 6a + 9$ (using Binomial-Square Theorem) 17. $(8v + 1)^2 = 64v^2 + 16v + 1$ (using Binomial-Square Theorem) 18. General formula is $h = -4.9t^2 + v_0t + h_0$, $v_0 = 10$, $h_0 = 20$; therefore $h = -4.9t^2 + 10t + 20$. If $t = .5$, then $h = -16r^2 + 12r + 4 = -16(.5)^2 + 12(.5) + 4 = -4 + 6 + 4 = 6$. The ball is 6 ft high after .5 sec. 20. To find the value(s) of $t$ for which $h = 0$, $0 = -16r^2 + 12t + 4$, $0 = -4r^2 + 3r + 1$, and $0 = 4r^2 - 3r - 1$. If $4r^2 - 1 = 0$, then $t = \frac{1}{2}$. If $t - 1 = 0$, then $t = 1$. The ball hits the ground at $t = 1$, or after 1 second. 21. a. When the box is folded, the dimensions are $40 - 2s$, $30 - 2s$, and $s$. The volume is $(40 - 2s)(30 - 2s)(s)$. b. If $s = 2$, then $(40 - 2s)(30 - 2s)(s) = (40 - 4)(30 - 4)(2) = (36)(26)(2) = 1872$ cm$^3$. 22. a.-b. See below. 23. The coefficient of the squared term ($x^2$) is what affects the shape of the parabola. Choice a has a coefficient of 2 for the $x^2$ term, all the other parabolas have 1 so they are all the same shape. The choice is a.

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 370–373 or to the Vocabulary (Voc.) on page 367. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 6 REVIEW** (pp. 370–373)

1. $a^2 + 2ax + x^2$ 3. $9x^2 + 24x + 16$ 5. $9r^2 - 90r + 225$ 7. $y = 3x^2 + 12x + 2$ 9. $y + 31 = (x + 5)^2$ 11. b 13. 6i 15. 2i 17. -3 19. $\frac{1}{2} \pm \frac{1}{2}i$ 21. $\pm 4\sqrt{3}$ 23. $\pm 3i$

25. $x = \frac{6}{3}$, $-\frac{1}{2}$ 27. $a = \frac{-3 \pm \sqrt{23}}{8}$ 29. $x = 1 \pm 2\sqrt{2}$ 31. $x = \frac{-1 \pm \sqrt{2}}{2}$ 33. $n = \frac{1 \pm i\sqrt{59}}{3}$ 35. 0 + 2i

37. 37 + 5i 39. 8 - 6i 41. 6 + 11 43. d 45. $y = -0.1$ 47. $k = 7$, $-5$ 49. a. $\sqrt{111}$ 51. a. $\sqrt{10,400} = 20\sqrt{26}$ b. no real solutions 53. b. two real roots 55. c. irrational 57. a. real, rational 59. non-real 61. real, rational

63. 4.5 seconds 65. 150 ft 67. 55 by 11 meters 69. See below. 71. See below. 73. c 75. a. 0.6 second and $t = 3.4$ seconds 77. two 79. one 81. See below.
LEsson 7-1 (pp. 376-380)
3. B maps $x$ onto $\frac{x^2}{20}$. 5. 125 7. 10 9. 96 feet is not a realistic following distance, since the stopping distance needed is about 240 ft. 11. independent 13. 13 15. No; the x-values of 16 and 8 are each paired with two different y-values.
17. 2,173,000 19. 50% 21. a. $\frac{9}{16}$  b. $\frac{9}{16}$  c. $\frac{9}{16}$  d. $\frac{9}{16}$
23. a. inversely, w  b. $g(w) = \frac{k}{w}$
25. a. $50,000 = \frac{1}{2}E + \frac{1}{2}E + \frac{1}{2}E + 50,000 = E$
   b. $400,000 27. V = \frac{k}{T}$

LeSSON 7-2 (pp. 381-387)
1. The set of allowable substitutions for the independent variable 3. True 5. \{y: 0 \leq y \leq 240\} 7. No; vertical lines for $x > 0$ intersect the graph twice. 9. yes 11. a. See below.
   b. yes c. $D = \{2.3,5\}, R = \{4\}$ 13. a. See below.  b. yes c. $D = \{\text{all real numbers}\}, R = \{y: y \geq 0\}$ 15. a. See below.
   b. yes c. $D = \{x: x \leq 0\}, R = \{y: y \geq 0\}$ 17. 325° F
19. $12 \leq A(t) \leq 138$ 21. $t \approx 0.5, t \approx 10.2$ and $t \approx 15$
23. a. 9  b. -7  c. $1 + \frac{x^3}{8}$ 25. 212 27. 36

11. a.

13. a.

15. a.

LeSSON 7-3 (pp. 388-392)
1. $\$10,625  a. 1251  b. 147  c. $7x^2 + 7x - 63$
   d. $49x^2 + 7x - 9$ 9. $x \neq 3$ and $x \neq -3$ 11. 1.9 $\sqrt{3}$
13. a. $x = 0$  b. x = 0  15. a. $(r \circ s)(x) = 1000 + \sqrt{2(2500 + \sqrt{x})}$  b. $1095$ barracuda 17. $K^2$
19. a. 100 ft
   b. $D = \{r: 0 \leq t \leq 4\}$  c. $R = \{h: 0 \leq h \leq 100\}$

LeSSON 7-4 (pp. 393-397)
1. $\$4.25 5. 11 7. 8 9. $6.5 \leq N < 7.5$ 11. any value of $x$
13. b 15. a. 3  b. -1  c. 6  d. See below. 17. $3x^2 - 4$
19. See below. 21. range = \{positive even integers\}
23. a. $2\sqrt{5}$  b. $7\sqrt{3}$  c. 2

15. d.

19.

LeSSON 7-5 (pp. 398-404)
3. a. $-x$  b. $x$ 5. a. See below.  b. See below.  c. See below.
   d. See below. 9. True 11. Let $C = \text{SQR}(A^2 + B^2)$
13. $y = x^6$
15. a. See below.  b. domain = \{all real numbers\}, range = \{all nonpositive numbers\}
17. a. $0.015 = \frac{|p - 50.015|}{b}$  b. $50.000 \leq p \leq 50.030$
21. No 23. $t = \frac{y - m}{x}$ 25. $[\begin{array}{c}0 \\ 1 \\ 0 \end{array}]$

5. a.

5. b.
5. c.  
\[ y = x \]

5. d.  
\[ f(x) = -\sqrt{x} \]

15. a.  
\[ y = -|x| \]

19.  
\[ f(x) = [-x] \]

11. c.  
\[ y = x - 9 \]

LESSON 7-6 (pp. 405-410)

5. \( x = 3y \)  9. No 11. a. Yes, it is a function. b. \( y = \frac{x - 9}{4} \)
c. See below. d. The slopes are reciprocals. 13. a. \( f(x) = 1.33x \) b. \( g(y) = .75y \)
15. a. It is being raised by successive tugs on a rope. b. The flag is not always moving due to the regripping of the rope. 17. a. 15 b. x c. identity function
19. a. \( \frac{1}{2} \) b. the slope of \( y = f(x) \)

CHAPTER 7 PROGRESS SELF-TEST (p. 420)

1. \( f(3) = 9 \cdot 3^2 - 11 \cdot 3 = 9 \cdot 9 - 33 = 81 - 33 = 48 \)
2. \( 7(12) = \frac{12(12 + 1)}{2} = \frac{156}{2} = 78 \) 3. \( f(n + 1) = (n + 1)^2 = n^2 + 2n + 1 \)
4. Yes, because for each element in the domain there is associated exactly one range element
5. Yes, see answer to Question 4. 6. \( f(g(7)) = f(-8(7)) = f(-56) = \frac{-56}{2} = 3136 \) 7. \( g(f(x)) = g(x^2) = -8(x^2) = -8x^2 \)
8. No, because \( f(g(x)) \neq x \). 9. Because \( x^6 \) is non-negative for all values of \( x \), the range is \( \{y: y \geq 0\} \) which implies quadrants I and II. 10. \( \{1, 3, 5\} \) which is the set of first coordinates 11. \( \{(2, 1), (4, 3), (6, 5)\} \)
12. Replace \( g(x) \) by \( y \). \( y = 5x + 10 \). Switch \( x \) and \( y \) coordinates to get \( x = 5y + 10 \). Solve for \( y \) to get \( y = \frac{1}{5}(x - 10) = \frac{1}{5}x - 2 \). 13. Apply the vertical-line test to get (b), (c), and (d) 14. Apply the horizontal-line test to get (b) 15. a. See below. b. Refer to the graph to get \$14.50. 16. The function is not 1-1 since the horizontal-line test fails 17. Since the graph of the function is above the \( x \)-axis, the range is \( \{y: y > 0\} \). 18. See below. Reflect the original function about the line with equation \( y = x \). 19. sample: \( x > 0 \). Restrict any part of the domain so that the horizontal-line test checks. 20. a. The domain is the set of all real numbers since there are no undefined values for \( x \).

b. Since \( x^2 \) is always any non-negative real number with a minimum value of 0, \( x^2 + 2 \) has a range \( \{y: y \geq 2\} \).
21. \( g(x) = \{x + 1\} \) so \( g(4.3) = \{4.3 + 1\} = \{5.3\} = \{-4\} = 4 \) 22. See below. 23. a. Substitute the values into \( ABS(M - N) \). \( ABS(3.2 - 7) = ABS(-3.8) = 3.8 \) b. The output represents the distance between two numbers on a number line.

15. a. Cost (dollars) vs. time (minutes)

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c} 
\hline 
\text{time (minutes)} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline 
\text{cost (dollars)} & 16.25 & 14.50 & 12.75 & 11.00 & 9.25 & 7.50 & 5.75 & 4.00 \\
\hline 
\end{array} \]
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 421-423 or to the Vocabulary (Voc.) on page 419. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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CHAPTER 7 REVIEW (pp. 421-423)

1. \(5, 3, -243, 5, 40, 7, -3, 9, 14\)
2. \(11, g\)
3. \(13, x^2 - 12x + 37\)
4. \(15, 257\)
5. \(17, a. 26,000, b. 1,000\)
6. \(19, 20, \text{PRINT} 100 * \text{INT} ((N + 50)/100)\)
7. \(21, \frac{1}{3}(x - 7)\)
8. \(23, y = \frac{1}{3}x + \frac{1}{3}\)
9. \(25, \text{No, because -1 is mapped to both 1 and -1.}\)
10. \(27, \text{not a function}\)
11. \(29, \text{function}\)
12. \(31, \text{not a function}\)
13. \(33, x = 0\)
14. \(35, D = \{\text{all real nos}\}; R = \{\text{all non-negative nos}\}\)
15. \(37, D = \{\text{all real nos}\}; R = \{\text{all integers}\}\)
16. \(39, D = \{\text{all real nos}\}; R = \{\text{all reals} \geq -5\}\)
17. \(41, \text{Domain:} \{x: x = 6\}; \text{Range:} \{y: y \geq 0\}\)
18. \(43, \text{True}\)
19. \(45, B(1950) \text{is the population of Baltimore in 1950.}\)
20. \(47, a. 15,600, b. \text{The average yearly growth in Philadelphia from 1900 to 1950.}\)
21. \(49, c. 51, \text{See below}\)
22. \(51, \text{See below}\)
23. \(53, \text{See below}\)
24. \(55, \text{See below}\)
25. \(57, 1 \text{ and III}\)
26. \(59, D = \{\text{all real nos}\}; R = \{-1 \leq y \leq 1\}\)
27. \(61, D = \{-5 \leq x \leq 5\}; R = \{-4 \leq y \leq 4\}\)
28. \(63, b\)
29. \(65, \text{See below}\)
30. \(67, \text{They are reflection images over the line} y = x.\)
31. \(69, \text{See below}\)
LESSON 8-1 (pp. 426-431)
1. a. $6^3$ b. $6^2 \cdot 6^3 = (6 \cdot 6) \cdot (6 \cdot 6 \cdot 6)$
   c. $6^5$ 3. a. $4^{10}$
   b. $(4^2)^5 = (4 \cdot 4)^5$; $4^{10} = (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4)$
   c. $4^{10} = 4^{10}$
2. $5 \cdot 5 \cdot 5 \cdot 5 = 10 \cdot 10 \cdot 10 \cdot 10 = 100 \cdot 100 = 10,000$

LESSON 8-2 (pp. 432-437)
1. 1.08 $\times 3 = 3.216$; $2.247 \times 2 = 4.494$; $2.524$; $2.524$; $2.676$
2. Solution 1: $(1191.02)$. Solution 2: $(1100)(1.06)$; $(1000)(1.06)$
3. $F(3) = (1000)(1.06)^3$; $(4) = (1000)(1.06)^4$; $(5) = (1000)(1.06)^5$
4. Solution 2: $(F(4) = (1000)(1.06)^4)$
5. $F(4) = (1000)(1.06)^4$; $F(3) = (1000)(1.06)^3$; $(4) = (1000)(1.06)^3$; $(5) = (1000)(1.06)^5$
6. True b. False c. $677.25$ b. $1386.23$
7. a. $300$ b. $338.23$ c. $338.23$
8. a. $6561(3)^{10}$ b. $1024x^{10}$
9. $18^5$ 21. $x^18$ 23. $x = 6$
25. $n = 1, 2, 3, 4, 5, 6$
27. a. $T = kx^2$
15. a. 10 PRINT "A PROGRAM TO CALCULATE BANK BALANCE"
20 PRINT "PRINCIPAL, ANNUAL RATE, NO. OF YEARS", P, R, Y
30 PRINT "YEAR", "AMOUNT"
40 FOR C = 1 TO Y
50 A = P * (1 + R)^C
60 PRINT C, A
70 NEXT C
80 END
15. b. YEAR AMOUNT
   1 265.00
   2 280.90
   3 297.75
   4 315.62
   5 334.56
   6 354.63
   7 375.91
   8 398.46
   9 422.37
  10 447.71

LESSON 8-3 (pp. 438-442)
5. 6.4 $\times 3.5$, 32.27 7. $= 2.82$ ft
8. a. .16 b. $g_7 = 100.2^n - 1$
9. a. 280 b. 8.75 13. $= 4.471.12$
15. a. 4. $2\sqrt{2}$ b. yes; $r = \frac{1}{\sqrt{2}}$
17. G1 = 16;
R = .25; N = 5 19. $1077.28$
21. $20x^3$ 23. $\frac{2^4}{81}$
25. $m^2(2m^3 + 1)$
27. a. $y = \frac{x}{x+2}$
29. a. $\frac{x - 6}{3}$

LESSON 8-4 (pp. 443-447)
1. a. $b^{x+y}$ b. $b^x$ c. 1 3. a. 1 b. $\frac{1}{8}$ c. $\frac{1}{64}$
5. d 9. 10
13. 4y 15. $\frac{1}{4}$ 17. 6561 19. 32 21. 32 23. $1.5625 \times 10^{-5}$
25. $2.5 \times 10^{-7}$ cm or $2.5 \times 10^{-9}$ m
27. a. yes b. $g_7 = 6561(\frac{1}{3})^{n-1}$ c. $= 0.0014$
29. a. 1st yr = $1100$; 2nd yr = $1210$; 3rd yr = $1331$; 4th yr = $1464.10$; 5th yr = $1610.51$
   b. 1st yr = $100$; 2nd yr = $110$; 3rd yr = $121$; 4th yr = $133.10$; 5th yr = $146.41$
   c. 1st yr = $100$; 2nd yr = $110$; 3rd yr = $121$; 4th yr = $133.10$; 5th yr = $146.41$
   d. increasing 31. $x - \frac{6}{3}$

LESSON 8-5 (pp. 468-453)
3. 5. $x = 5$
9. a. 80 10. 1 12. 3 13. 80 14. 3 15. 80
b. 4.3 13. 12 15. 1.142 17. 1.995 19. > 21. < 23. >
25. = 27. $\geq 2059$; yes 29. a. $F_n = 440(2^{1/2})r - 1$ b. 880
31. $\frac{4}{m}$ 33. .16 or $\frac{4}{25}$ 35. 1 37. 1.5; 6.5$r - 1$ 39. $1054.41$
41. a. See below. b. $z = 3\sqrt{2}$ cm $= 4.24$ cm c. See below.
   d. $3\sqrt{3} = 5.196$ cm
LESSON 8-6 (pp. 454-458)
1. $a = 4^9$, $a, \sqrt[4]{100000^4}$ or $(\sqrt[4]{100000})^4$, or $(100000^4)^{\frac{1}{4}}$ or $(100000^4)^{\frac{1}{8}}$ b. the second c. $10,000$ 5. 8 7. 19.683 9. 161,051 11. $a = 6.31; 6.31^3 = 100.01$ 15. $268.0142; 268.0142^2 = 12$ 17. $= 22.5 \text{ cm}^2$ 19. a. -2 b. 2 c. no 21. $\frac{104000}{2111}$ 23. $x^2$ 25. $y^\frac{3}{4}$ 27. $= 0.862$ 29. a. bth; c. $\frac{1}{3}$ 33. $= 0.2214.84$ for $a \neq 0$ and $m$ and $n$ integers, $(a^m)^n = a^{mn}$. 33. $= 0.2214.84$

LESSON 8-7 (pp. 459-462)
1. $\frac{1}{3}, \frac{32}{25}, 5.0991, 7. \frac{81}{1}, 9. 4, 11.$ a. yes b. yes c. yes d. no e. yes f. yes 13. positive 15. $\frac{5}{4}$ 17. $-3x^{-1}$ 19. No, $F$ is directly proportional to $M \frac{1}{3}$ 21. a. 36 b. 338 23. $x = \frac{3}{5}; y = \frac{5}{6}$

LESSON 8-8 (pp. 463-467)
1. $\frac{1}{n^n}$ th 3. 6 7. $x^3$ 11. 2.15 13. 12.58 15. 20 17. $x^2$ 19. $3\sqrt{2}$ 21. (b) 23. $\sqrt{2} + \sqrt{3} > \sqrt{5}$ 25. a. 13.5 20. 25, 30, 375 b. $t_n = 4 \cdot \left(\frac{3}{5}\right)^{n-1}$ 27. $= 0.000001$ or $\frac{1}{10,000,000}$ 29. 343 31. 2, -3, 4 33. $x = \frac{1}{\sqrt{1573}} \approx 0.048$

CHAPTER 8 PROGRESS SELF-TEST (p. 481)
1. $(3)^4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$. -3^4 = -81; (-3)^-4 = \left(\frac{1}{-3}\right)^4 = \frac{1}{81}; 
2. $625 = 5^4$ 3. $\sqrt{625} = 25$ 3. $\sqrt{11,390,625} = 15$. A sequence of keystrokes is 11390625 [x^y] 6 [1/3] [x^y]. 4. $(\frac{1}{5})^{-\frac{5}{3}} = (32)^{\frac{5}{3}} = (\sqrt[3]{32})^{5} = 2^{5} = 64$ 5. $\frac{625}{x^{y^z}} = \frac{\sqrt[3]{5}}{\sqrt[3]{5}} = \frac{5}{xy}$ 6. $\sqrt[3]{\sqrt[5]{525^5}} = \sqrt[3]{\frac{\sqrt[5]{525^5}}{5}} = \sqrt[3]{\frac{\sqrt[5]{525^5}}{5}} = \sqrt[3]{\frac{\sqrt[5]{525^5}}{5}}$ 7. $x^a = 144$, then $x = 16$, $x = \pm 2$. 8. If $x^\frac{3}{5} = 64$, then $\left(x^\frac{3}{5}\right)^{\frac{5}{3}} = 64^{\frac{5}{3}}$ so $64^{\frac{5}{3}} = 4^2$. 9. $2^a = 5^b$, then $5^a + 21 = 5^b$ so $n + 21 = 29$ and $n = 8$. 10. If $T = 2\pi \sqrt{\frac{L}{g}}$, then solving for $L$: $\frac{T}{2\pi} = \sqrt{\frac{L}{g}}$, $\frac{T^2}{4\pi^2} = \frac{L}{g}$, and $L = \frac{T^2g}{4\pi^2}$. 11. Using the equation $V = 13,500(1.17)^n$ for $n = 3, V = 21,621.78$ which rounds to $21,620$. 12. Using $A = 200\left(1 + \frac{0.055^3}{100}\right)^{100}$, $A = 200(1.001157534)^{1825} \approx 200(1.333) = \$266.61$ 13. Any amount of money to double in $n$ years, $2 = (1 + r)^n$. If $n = 4$, $\sqrt[4]{2} = 1.1892 \approx 19\%$. 14. If $100(A - 5)^4 = 1600$, then $(A - 5)^4 = 16$. $A - 5 = \pm 16\frac{1}{4}$, $A = 7$ or $A = 3$. 15. If $\frac{1}{3}(20 - P)^{\frac{1}{2}}$, then $(20 - P)^{\frac{1}{2}} = 30$, $20 - P = 300^2 = 900$. 16. If $\sqrt[4]{343} = \sqrt[4]{\frac{125}{3}}$. Then $(\frac{125}{3})^{\frac{1}{4}} = \frac{5}{\sqrt[4]{3}}$ and $(\frac{5}{\sqrt[4]{3}})^{\frac{1}{4}} = \frac{5^{\frac{1}{4}}}{\sqrt[4]{3}}$. Thus $\frac{5}{\sqrt[4]{3}} = 1$ and $n = 3$. 17. $\frac{2.1 \cdot 10^2}{10^{-3}} = 2.1 \cdot 10^2 \cdot 10^3 = 2.1 \cdot 10^5 = 210,000$ 18. After 24 hours there are 48 half-hour periods, so there will be $5(2)^{48}$. 19. Each term is 4 times the previous term, and the initial term is 2, so $t_n = t_1(r^n - 1) = 2 \cdot 4^{n-1}$. 20. $a^\frac{1}{2} = \frac{1}{a^{\frac{1}{2}}}$, which is choice b.

LESSON 8-9 (pp. 468-471)
1. -16; 36; -6; 1; $-\frac{1}{6}; -\frac{1}{57}$ 3. 3 5. -4 7. y 9. true 11. defined, real, negative 13. a. -2x b. -2x 15. $x^\frac{1}{3}$ or $\sqrt[3]{x}$ b. undefined when $x < 0$ 17. $-\sqrt[3]{2} + \sqrt[3]{3}$ 19. $10|a|b^2\sqrt[3]{a^2b^2}$ 21. a. 7; 5th b. $x^3 = 7$ c. $16.807$ 23. two times 25. a. 6 feet b. 42 feet c. $= 3.12$ seconds

LESSON 8-10 (pp. 472-475)
1. zero 3. two 5. $w = 64$ 7. $y = 11$ or -11 9. $x = 2$ 13. $x = -8\sqrt{9}$ 15. $s = 175.616$ 17. 12,558 horsepower 19. a. $138$ mph b. $125$ ft 21. $-5x^2$ 23. $20ab^2\sqrt{5}$ 25. $= 74.1$ years

LESSON 8-11 (pp. 476-479)
1. b. Raise both sides to the $n$th power. 3. a. 1000 = $500(1 + r)^7$ b. 10.4% 5. $t = 6\frac{1}{3}$ 7. 8.6% 9. $r = 288$ 11. $r = 1$ 13. a. 5.07% b. yes 15. a. $= 17.45%$ b. $= 17.33%$ 17. 128 19. 25 21. a. $x = -1$ b. $x \geq 0$ 23. $x = 0; 4x + 30 \geq y; 6x \leq y$ 25. $x^{6n}$ 27. 5 years = 14.9%; 10 years = 7.18%; 15 years = 4.73%
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 482–485 or to the Vocabulary (Voc.) on page 480. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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### CHAPTER 8 REVIEW (pp. 482-485)

1. 0.000064  3. 0.034  5. 0.625  7. 10  9. 4.53  11. 1/3  13. 512
15. 18.57  17. False  19. -2  21. 1.41  23. 1.79  25. 3x√2
27. -2a^3√10  29. 7x^2√2  31. No real solution  33. x = 1/3
or -1/3  35. 36  37. b = 81/36  39. c = 117,649  41. x = 15,624
43. r = 2.8  45. x = 6  47. y = 10  49. -4x^3y  51. 3/4  53. 1/3

### LESSON 9-1 (pp. 488-493)

1. 4.800  3. a. 1000 bacteria  b. after about 21/2 hr  5. a. 3.317
b. 3.340  c. 3.322  9. b; x  11. c  13. a. y^10  b. 4/y
15. a. 3%  b. 1.095 billion  17. a. See below.  b. 0
c. ≈.68  d. The domain is all real numbers. The range is all positive real numbers. The graph is an increasing function as x increases.  19. a. geometric  b. a_n = 100(0.9)^n-1  c. a_1 = 100; a_n = a_{n-1} • 0.9 (for n > 1)  21. H^1(x) = √x
23. a. y = 100 = 1/5(x - 100)  b. 72

### LESSON 9-2 (pp. 494-498)

3. $2035  7. 126 g  9. a. 11. a. See below.  b. See below.
c. These graphs are the inverses of each other.  13. a. 2
b. iii  c. all real numbers  d. all positive numbers  15. about
285 million  17. √10 is between 3 and 3.5
19. 5.54% ≤ x < 6.80%

### LESSON 9-3 (pp. 499-503)

3. yes  5. about 40  9. 1000 times  13. a. gastric juice
b. 10^{-2}  15. 10,000 times  17. See below.
19. (m + 5p)^2  21. 15

17.
LESSON 9-4 (pp. 504–508)
3. $10^x = y$ 5. 8 7. .5 9. undefined 11. -3.337 13. b 15. $x \approx 316.23$ 17. 12, 3 19. b 21. 200 23. See below.
25. $10^4 = 27.2$ 29. $x = 0$ 31. a. $x = 0$ and $y = 0$ b. $y = x$ and $y = -x$ c. $y = 0$ and $x = 0$

LESSON 9-5 (pp. 509–512)
1. a. 216, 6  b. 1, 6, 3rd 3. $\log_2 2097,152 = 7$ 5. $b^c = a$
7. 9, 0.5 11. 2 13. 216 15. 729 17. a. See below.
b. True 19. a. 3, $\frac{1}{3}$ b. 2, $\frac{1}{2}$ c. $\log_c b = \frac{1}{\log_b a}$ 21. 5 23. $10^9$
or $1,000,000,000$ 25. $b^{45}$ 27. $x^{2t}$ 29. $\$11,183.32$

LESSON 9-6 (pp. 513–517)
1. 26.8 3. 0 5. 1 7. $\log 25$ 11. 4.9069 13. 4.6438

LESSON 9-7 (pp. 518–523)
5. $\$1.11$ 7. a. initial amount b. rate of continuous growth or decay c. $r$ is negative 9. a. $\$5466.35$ b. $\$350.43$ 11. See below.
13. a. 25% compounded continuously b. $\$25,400$
15. 12 17. -1 19. -0.5 21. $x = 75$ 23. $m = 4$

LESSON 9-8 (pp. 524–528)
5. In $7.39 \approx 2$ 9. 5.991 11. no 13. about 13.7 years
15. at about $(\frac{1}{2}, -0.693)$ 17. $=13.1$ 19. $=8.5\%$ 22. $=22\%$
21. $x = \sqrt{7}$ 23. False 25. $w$ is any real number.
27. 50,000 mm$^3$

LESSON 9-9 (pp. 529–533)
3. True 5. $y = 2.26$ 7. the base of the equation is $e$
9. a little more than 5.1 cm thick 11. $y = 2.36$
13. $r = -0.699$ 15. about 3.5 days 17. 30 19. a, b, c, e
21. $=2.5$ 23. $f = 8$ 25. $[47 45 -26$

CHAPTER 9 PROGRESS SELF-TEST (p. 536)
1. $\log (1,000,000) = \log (10^6) = 6$, since $\log_b b^n = n$.
2. $\log_8 16 = \log_4 4^2 = -2$, since $\log_b b^n = n$.
3. $\ln e^6 = 6$, since $\log_b b^n = n$.
4. $\log 1 = 0$, since $\log_b 1 = 0$
5. Use a calculator; $\ln (42.7) \approx 3.75$.
6. Use a calculator; $\log 25 \approx 1.40$.
7. $e^x = 412$, so $
\ln(e^x) = \ln(412)$, $y = \ln (412) \approx 6.02$.
8. $\log 8 = \frac{1}{3}$, so $x^3 = 8$.
9. $2 \log_6 30 = \log_6 30$, so $m + 1 = 12$.
10. $6^x = 32$, $\log 6^x = \log 32$, $x \log 6 = \log 32$
11. $\log 32 \approx 1.93$.
12. true, by the definition of logarithm.
13. true, by the Quotient Property of Logarithms
14. false, $\log_3 7 + \log_3 13 = \log_3 91$, by the Product Property of Logarithms
15. 8% per hour
16. Substitute into the formula: $x = 8$, $A = 12,000$, so $y = 12,000(0.92)^8 \approx 6159$ bacteria.
17. Substitute $y = 1000$, $x = 2$, and solve for $A$ in the formula: $1000 = A(0.92)^2 \approx 1181$ bacteria.
18. Use the Continuous Compounding Interest Formula: $N = Pe^{rt}$. Substitute $N = 2P$, $r = 0.072$, $2P = Pe^{0.072t}$,
$2 = e^{0.072t}$, $\ln 2 = 0.072t$, $t = 9.90$, so it would take about 10 years.
19. Since $0.0125 - 0.015 = 0.02$, the intensity is $10^{0.02} = 102 = 100$ times.
20. a. See below.
21. d. The inverse of a logarithmic function is an exponential function, so the inverse is $y = e^x$.
22. a. See below.
24. b. domain is positive real numbers and range is all real numbers.
25. c. See below.
26. d. The inverse of a logarithmic function is an exponential function, so the inverse is $y = e^x$.
27. e. See below.
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 537–539 or to the Vocabulary (Voc.) on page 535. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 9 REVIEW** (pp. 537–539)

1. 3 3 9 5. 15 7. -3 9. 4.99 11. 4.47 13. undefined
15. \( x = 3 \) 17. \( n = 14.21 \) 19. \( z = 3.09 \) 21. \( a = 1.78 \)
23. \( x = 11 \) 25. \( z = 100,000 \) 27. \( x = 225 \) 29. \( x = 4 \)
31. \( 6^3 = \frac{1}{16} \) 35. \( \log 0.0631 = -1.2 \) 37. \( \log x = y \)
39. \( \log_b xy = \log_b x + \log_b y \) (product property)
41. \( \log_b (x^n) = n \log_b x \) (powering property)
43. \( \log_b b^n = n \) 45. all positive real numbers 47. True
49. True 51. about \$1.68 53. about 301 days 55. about 31.6 times
57. 100 times 59. See below. 61. 60 represents a decay since the \( y \)-values get smaller as the \( x \)-values increase. 63. a. See below. b. \( y = e^x \)

**LESSON 10-1** (pp. 542–548)

5. d 7. a. hypotenuse  b. \( AC \)  c. \( AB \)  d. \( \angle C \)  e. the tangent of
f. the sine of  g. the cosine of  9. 0.383 11. 0.052
13. a. =88 km  b. =328 km 15. =12.9 ft 17. \( \frac{180(n - 2)}{n} \)
19. \((-x, y)\) 21. False 23. \( \approx 11,460 \) years

**LESSON 10-2** (pp. 549–554)

1. .866 \( \left[ \begin{array}{c} \cos \theta \end{array} \right] \) or \( \cdot .866 \left[ \frac{1}{2} \right] \cos \theta \)
3. 38° 7. 0 9. =113 ft 11. \( \approx 31^\circ \) 13. \( \approx 15^\circ \) 15. \( \approx 400 \) ft 17. \( S = \left[ \begin{array}{c} (2, 0); K = (1, -6); Y = (2, 6) \end{array} \right] \)
19. \( \approx 37^\circ \) 21. SAS—SIDE ANGLE SIDE; SSS—SIDE SIDE SIDE; ASA—ANGLE SIDE
ANGL 23. \( x = \frac{3}{2} \)

**LESSON 10-3** (pp. 555–559)

3. 18° 5. 1 7. \( \frac{1}{2} \) 13. a. \( 3\sqrt{3} \) inches  b. \( 9\sqrt{3} \) sq inches

15. \( \sqrt{3} \) 17. a. \( 10\sqrt{2} \) m  b. 14.1 m 19. =9.5° 21. \( a \approx 4.2 \)
or \( a = 5.2 \) 23. a. (1, 0)  b. (0, -1)  c. (0, -1)

**LESSON 10-4** (pp. 560–564)

3. (1, 0); 1; 0 5. 0 7. -1 9. 0 11. 0 13. \( \approx 0.848 \) 15. B
17. A 19. d 21. d 23. b 25. \( \cos 0^\circ + (\sin 0^\circ) = \)
\( (\cos 270^\circ) + (\sin 270^\circ) = 1; (0)^2 + (-1)^2 = 1 \)
27. \( \frac{1}{2} \) 29. about 930 ft 31. none

**LESSON 10-5** (pp. 565–569)

3. a. See below.  b. negative 5. a. \( \approx -0.469 \)  b. \( \approx 0.883 \)
7. \( \theta = 65^\circ \) 9. -5 11. \( \frac{\sqrt{2}}{2} \) 13. 127° 15. \( -\frac{\sqrt{2}}{2} \) or \( \frac{\sqrt{2}}{2} \)
17. Sample: $45^\circ$, $225^\circ$  
19. \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \)  
21. $6.5^\circ$

LESSON 10-6 (pp. 570-574)
1. False 3. c 5. $=3.3$ miles 7. about $1.59p$ units
9. a. $=49$ mm b. $=29^\circ$ 11. $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

13. $\frac{\sqrt{3}}{2}$ 15. 0.5 17. $20^\circ$ 19. a. 2 b. $\pm 35$ c. See below.
d. parabola 21. $45^\circ$

19. c.
\[
\begin{align*}
(0, -1225) \\
(-35, 0) \\
(35, 0)
\end{align*}
\]

LESSON 10-7 (pp. 575-580)
5. about 22 miles 7. $35.9$ 9. a. $42$ mm b. $55^\circ$ c. $=55^\circ$
d. True 11. a. $m\angle ABD = 142^\circ$ and $m\angle ADB = 13^\circ$
b. $=282$ m c. $=174$ m 13. There is an error somewhere, since the sine of an angle is always between 1 and -1, inclusive. 15. d 17. a. $\cos \theta = \frac{3}{3}$ or $\cos \theta = -\frac{3}{3}$
b. See below. 19. False 21. a. 0 b. no 23. a. $h = -16t^2 + 30t + 12$ b. See below. c. $=26$ feet
d. $=2.2$ seconds after being thrown 17. b.
\[
\begin{align*}
(\cos \theta, \sin \theta) \\
(\cos \theta, \sin \theta)
\end{align*}
\]

LESSON 10-8 (pp. 581-585)
1. a. $0.515$ b. $301^\circ$ c. $0.857$ d. $121^\circ$ 3. a. $=29^\circ$ b. The other possible solution, $151^\circ$, when added to $42^\circ$ is greater than $180^\circ$—the sum of 3 angles of a triangle—and this is not possible. 5. $x = 14.4$ 7. $x = 46.8^\circ$ 9. a. $=66.3^\circ$, $=113.7^\circ$
b. $=22.0$, $=5.9$ 11. $\sin \frac{B}{AC} = \sin \frac{E}{DF}$; $AC = DF$ (given), so $\sin B = \sin E$; thus $\angle B \equiv \angle E$ and $\triangle CAB \equiv \triangle FDE$ by AAS Theorem. 13. i 15. $\frac{1}{2}$ 17. a. b. See below. c. No 19. Yes; the vertical-line test holds

LESSION 10-9 (pp. 586-591)
5. 0; 1 7. decrease 11. a. See below, b. $-315^\circ$, $-135^\circ$, $45^\circ$, $225^\circ$ 15. a 17. No 19. a. Yes b. 3 21. $\cos \theta$
23. $\angle 1 \approx 69.5^\circ$ or $\approx 110.5^\circ$ 25. area: $64\pi$; circumference: $16\pi$

13. a.

LESSON 10-10 (pp. 592-597)
5. $\frac{1}{2}\pi$ 9. $-225^\circ$ 11. $5$ 15. See below. 17. $\frac{\sqrt{2}}{2}$
19. $4\pi$ feet 21. a. all real numbers b. all numbers between -1 and 1, inclusive 23. a. 20 seconds or $\frac{1}{3}$ minute
b. See below. 25. See below.

15.

23. b.
CHAPTER 10 PROGRESS SELF-TEST (p. 600)

1. \( \cos \theta = \frac{adj}{hyp} = \frac{12}{13} \approx 0.923 \) 2. \( \sin \theta = \frac{opp}{hyp} = \frac{5}{13} \approx 0.385 \) 3. \( \theta = 45^\circ \) 4. The coordinates of a point rotated \( \theta \) around the origin are always \((\cos \theta, \sin \theta)\) so (a) represents a correct answer. Furthermore, \( \cos 120^\circ = -\frac{1}{2} = -0.5 \), and \( \sin 120^\circ = \frac{\sqrt{3}}{2} \approx 0.866 \). Thus a., b. and c. are correct.

5. Let \( \theta \) be the angle of elevation, \( \cos \theta = \frac{7}{14} \text{ ft} = \frac{1}{2} \), so \( \theta = 60^\circ \) \( \frac{6}{6} \) height of ladder \( \frac{\text{height}}{\text{length of ladder}} = \frac{14}{2} = \sin 60^\circ \).

So the height = \( \frac{14}{\left(\frac{\sqrt{3}}{2}\right)} = 7\sqrt{3} \approx 12.1 \text{ ft.} \approx 145 \text{ in.} \)

7. cos \( 57^\circ \) is positive. Cosine is also positive for Quadrant IV angles. Reflect the point \((\cos 57^\circ, \sin 57^\circ)\) about the \(x\)-axis to get the image point. So \( x = 360^\circ - 57^\circ = 303^\circ \).

8. cos \( 210^\circ \) = -cos \( 30^\circ \) = -\frac{\sqrt{3}}{2} \( \frac{9}{9} \) If the graph is translated horizontally \( 360^\circ \), the image of the translation coincides with the original graph. Therefore the period is \( 360^\circ \). \( \frac{10}{10} \) to 0, as is seen on the graph (verified with a calculator) \( \frac{11}{11} \). The greatest \( y \)-value on the graph is 1. The lowest \( y \)-value is -1. So the range is \( -1 \leq g(\theta) \leq 1 \). \( \frac{12}{12} \) See below.

13. Since this is an SAS situation, use the Law of Cosines. \((AB)^2 = (110)^2 + (85)^2 - 2(110)(85) \cos 40^\circ \). Then \((AB)^2 \approx 5000 \Rightarrow AB \approx 71 \text{ so the runners are about 71 m apart.} \)

14. Since this is an SSS situation, use the Law of Cosines. \( 8^2 = 11^2 + 5^2 - 2(11)(5) \cos x \). Then \( 0.745 = \cos x \), so \( x \approx 42^\circ \). \( \frac{15}{15} \) Since this is an AAS situation, use the Law of Sines. The angle opposite \( x \) is \( 180^\circ - 40^\circ - 83^\circ = 57^\circ \). So \( \sin 57^\circ \) \( \frac{17}{17} \) to \( \frac{2}{2} \) \( \Rightarrow x = \sin 57^\circ \cdot \frac{2.7}{\sin 83^\circ} \approx 2 \).

16. Draw \( \triangle SLR \): Since this is an SSA situation, use the Law of Sines. First find \( \angle L \). \( \frac{421}{\sin 110^\circ} = \frac{525}{\sin \theta} \Rightarrow \sin L = 0.754 \Rightarrow \angle L = 49^\circ \). Then \( \angle R = 180^\circ - 110^\circ - 49^\circ = 21^\circ \). So \( \frac{r}{\sin 21^\circ} = \frac{525}{\sin 110^\circ} \Rightarrow r \approx 200 \). \( \frac{17}{17} \) Let \( x \) be the vertical distance from the eagle. Then \( \sin 70^\circ = \frac{x}{130} \Rightarrow x = 130(\sin 70^\circ) \approx 122 \text{ feet. Add the 8 feet from the eagle’s beak to the ground, and then the nest is about 130 feet off the ground.} \)

18. \( \frac{7\pi}{6} \cdot \frac{180^\circ}{\pi} = 60^\circ \)

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 601-603 or to the Vocabulary (Voc.) on page 599. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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CHAPTER 10 REVIEW (pp. 601-603)

1. \( 0.29 \) 3. \(-0.77\) 5. \(-0.50\) 7. \(0.923\) 9. \(2.400\) 11. \(\frac{\sqrt{2}}{2}\)

13. \(\frac{\sqrt{3}}{2}\) 15. I 17. \(\frac{\pi}{4}\), \(45^\circ\) and \(135^\circ\), \(\frac{3\pi}{4}\) 19. \(\approx 0.436\) rad

21. \(\approx 42^\circ\) or \(\approx 730\) rad; or \(\approx 138^\circ\) or \(\approx 2.41\) rad 23. \(\frac{7\pi}{12}\)

25. \(3\pi\) 27. \(405^\circ\) 29. \(-225^\circ\) 31. \(139.7^\circ\) 33. \(\approx 25.4\)

35. \(m\angle = 39.3^\circ\) or \(\approx 140.7^\circ\) 37. True 39. True 41. \(-6\)

43. \(41^\circ\) 45. \(\approx 671\) km 47. \(9.5^\circ\) 49. about 2.2 miles

53. \(=169^\circ\) 55. 57. a. See below. b. \(2\pi\) c. \(\frac{\pi}{2}\), \(\frac{3\pi}{2}\)

59. \(T_{90^\circ}, 0\) or \(T_{\frac{\pi}{2}, 0}\)

57. a.
LESSON 11-1 (pp. 606-610)
1. yes, 1, 4 2. no 5. 3. \(a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0\)
7. \(a_n = 7; b_n = 5; c, d, a_n = 4; a_0 = 0\)
8. \(a_0 = 0\) 9. \(P(3) = 196\) (ten thousand) 10. \(b\) differs by 101
11. \(a. 25(1.07)^3 + 50(1.07)^4 + 100(1.07)^5 = 200(1.07)^2 + 400(1.07) + 800\)
12. \(25x^2 + 50x^3 + 100x^3 + 200x^2 + 400x + 800\) 13. 27.54 ft 15. See below.
17. \(105x^2 - 5x - 10\)
19. \(= 45,000\) increase per year

15.

LESSON 11-2 (pp. 611-616)
1. b. degree 9 2. c. degree 3 5. c. degree 2 7. a. \(at + bt + ct + ab + b + bc\) 
9. \(A = (b + c)(a + b + c)\)
11. a. \(x = 3.6\)
12. \((16.8)(12.8)(3.6)^2 = 774\)
13. a. \(V(x) = 4x^3 - 56x^2 + 180x\)
16. \(c. V(2.00) = 168.03\) 15. \(a^2 - b^2 = c^2 + 2bc\)
17. a. \(V = \frac{4}{3} \pi r^2 h\) b. \(r = \sqrt{225 - h^2}\) c. \(V = \frac{4}{3} \pi h^3 + 75 \pi h\)
19. 4000 21. a. 1905 to 1925 b. 1925 23. 16 times

LESSON 11-3 (pp. 617-622)
1. \(3d + e - 2d^2\) 3. \(7x(3x^2 - 4 + 5x^3)\) 5. \(a^2 - b^2\)
9. a. difference of cubes b. \((x - 16)(x + 16)\)
11. a. difference of cubes b. \((4 - 3x)(16 + 12x + 9x^2)\)
13. a. perfect square b. \((7 - 3h)^2\) 17. a. factorable
b. \((5x - 2)(x + 2)\) 19. a. factorable b. \((7z - 8)(z + 1)\)
21. a. \(4x^2 + 9)(4x^2 - 9)\) b. \((4x^2 + 9)(2x^2 + 3)(2x - 3)\)
23. \(8x^2(3x)(25 - 15y - 9y^2)\) 25. d 27. \(S(n) = 6h^2 + 28h + 20\) 29. a. 55 in.\(^2\)
b. \(A(x) = (11 - 2x)(17 - 2x) = 4x^2 - 56x + 187\) 31. a. quadratic; the shape is a parabola
b. \(x = 1\) c. \(x = 5\)

LESSON 11-4 (pp. 623-629)
3. \(k = \frac{4}{7} x^2 + 2\), or \(k = \frac{4}{7} x^2\)
5. \(P(4) = 0\) 7. \(k(x) = x(x - 2)(x - 1)\)
× (3 - 8); zeros are \(x = 0, \frac{1}{2},\) or 8 9. a. See below. b. \(y = (x + 4)(x - 1)(x - 8)\)
11. \(P(x) = k(x + 4)(x - \frac{2}{3})(x - \frac{1}{2})\)
13. \(P(x) = k(x^3 - 4x^2 - 84.8x + 192)\) 15. a. \(g(x) = x(x - 2)(x - 5)(4x^2 + 10x + 25)\)
b. \(x = 0, x = \frac{5}{2}\) are zeros
17. a. \(0 < x < 9\) b. \(x = 9, x = 15, x = 0\)
19. \((3x - 1)(9x + 3x + 1)\) 21. \(3x + y)(x - y)\)
23. \(81, 25, x^3 + 9x^2 + 26x + 24\)

LESSON 11-5 (pp. 630-637)
1. 2 b. \(x = -2.11, x = 1.13\) 3. The zero is the \(x\)-value
where the \(y\)-value changes sign. 5. See below. 7. 5.3
9. \(f(x) = -2x^3 + 4x - 1\) 11. \(x = -8.8\) and \(x = 2.3\)
13. \(x^3 - x^2\) 15. \((2x - 3)^2\) 17. \(100(n^2 + 1)(n + 1)(n - 1)\)
19. slope of \(MT = -1\), slope of \(AH = 1\); \((-1)(1) = -1\)
21. \(\frac{1}{12} + 4\) 6

LESSON 11-6 (pp. 638-643)
5. \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\) 7. a. \(x = 0, x = 5\) 9. \(x = -\frac{1}{2} + i\sqrt{\frac{3}{2}}\)
11. 5 15. \(x = \frac{7}{3} i\) 17. a. \(\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\)
b. 8 19. \((2z + 1)(z + 1)(z - 1) = 0; z = i, -i, 1, -1\)
21. a. \(P(n) = n^3 + n^2 - 1\) b. \(n = .75\) 23. about 275 ft
25. \(x = 1.609\) 27. \(x = 25\)

LESSON 11-7 (pp. 644-650)
5. The second differences are equal. 7. a. yes, \(y = p(x)\)
b. 3 9. a. no, \(y = p(x)\) b. does not apply 11. a. 7, 10, 13, 16, 19, 22, 25 b. yes c. 1 13. a. 6, 6, 6, ... b. 1 c. See below.
d. \(y = 6x + 5\) e. If the 1st differences are equal,
that is the slope of the line that models the data. 15. Consider
the following pattern: a. \(f(5) = 55; f(6) = 91\) b. 3 17, 3, since
the degree is 3 19. See below. 21. a. \(4b = x = 5.0\) or \(x = -4.6\)
23. a. \(y = p(x) = -x^3 + 3x^2 - 2\) b. \(3; 1 - \sqrt{3}\) 25. c

13. c.

LESSON 11-8 (pp. 651-658)
5. a. \(f(6) = \frac{3}{2}(6)^3 + \frac{1}{2}(6)^2 + \frac{3}{4}(6) = 91\)
b. The six layers contain, respectively, 1, 4, 9, 16, 25, and 36 oranges; the
total is 91. 7. a. \(2 b. \ p(x) = 5x^2 - 2x\) 9. \(x, y, z) = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})\)
11. a. \(f(1) = 1^2 - 1 + 2 = 2; f(2) = 2^2 - 2 + 2 = 4; f(3) = 3^2 - 3 + 2 = 8\)
b. \(y = 2^x\)
13. \(t = f(n) = \frac{1}{2n^2 + \frac{1}{2n}}\) 15. a. yes b. 2 17. a. 3
b. \(x = 0, \frac{10}{3}, \frac{10}{3}\) 19. \$334.24 21. \(4x^2y^2(x + 2y)(x - 2y)\)
23. \(z = 5.6\) 25. a. \(S = kwd^2\) b. \(d = \sqrt{4 - w^2}\) c. \(S = -kw^2 + 4kw\) d. 3
CHAPTER 11 PROGRESS SELF-TEST (pp. 660-661)

1. The first money saved earned interest for 5 years. The total is $750x^5 + 600x^4 + 925x^3 + 1075x^2 + 800x. 2. If x = 1.07, then $750x^5 + 600x^4 + 925x^3 + 1075x^2 + 800x = 750(1.07)^5 + 600(1.07)^4 + 925(1.07)^3 + 1075(1.07)^2 + 800(1.07) = 1051.91 + 786.48 + 1133.16 + 1230.77 + 856 = 5058.32. 3. V(x) = x(60 + 2x)(40 - 2x) = x(2400 - 120x - 80x + 4x^2) = x(2400 - 200x + 4x^2) = 4x^3 - 200x^2 + 2400x. See below. 4. The degree is 5.

5. P(-2.5) = (-2.5)^4 + 9(-2.5)^3 - 3 - 8(-2.5)^5 = 39.0625 + 56.25 - 3 - 781.25 = -688.9375

6. (a^2 + 3a - 7)(5a + 2) = a^2(5a + 2) + (3a)(5a + 2) - 7(5a + 2) = 5a^3 + 17a^2 - 29a - 14. 7. If p(x) = 4x^3 - 11(x + \sqrt{7}) = 0, then x = 0, x = 0, x = 0, 5x - 11 so x = \frac{11}{5}, and x + \sqrt{7} = 0 so x = -\sqrt{7}.

8. If f(x) = 3x^4 - 12x^3 + 9x^2 = 0, then 3x(x^2 + 3x + 3) = 0 and 3x^2(x - 3)(x - 1) = 0. There are 4 zeros: x = 0, x = 0, x = 3, and x = 1.

9. See below. 10. Since y = f(x) has degree 3, it has 3 zeros. 11. a. The zeros are between x = -2 and x = 1, x = 1 and x = 2, and x = 2 and x = 3, because those are the intervals for which the polynomial changes signs. b. x = 1.7, x = -1.7. 12. If z^3 - 216 = 0, then z^3 - 6^3 = 0, or (z - 6)(z^2 + 6z + 36) = 0. If z^2 + 6z + 36 = 0, then z = \frac{-6 \pm \sqrt{36 - 144}}{2} = \frac{-6 \pm 6}{2} = 3. The zeros are z = -3 + 3i\sqrt{3}, and z = -3 - 3i\sqrt{3}.

13. c. Never; a polynomial of degree 11 has 11 complex roots. 14. If r is a root or zero of a function, then x - r is a factor. Since we know that f(2) = 0, then 2 is a root and (x - 2) is a factor of the polynomial. That is option d. 15. Since the zeros are -2, 1, 3, and 5, the factors for the function are x - (-2), x - 1, x - 3, and x - 5. Thus the function is f(x) = k(x + 2)(x - 1)(x - 3)(x - 5) = k(x^2 + 2x)(x^2 - 4x + 15) = k(x^2)(x^2 - 8x + 15). 16. 10s^2t^2 + 15s^3t^4 = 5s^3(2s^2 + 3t^2) = 17. 9^2 - 196 = (3s)^2 - (14)^2 = (3s + 14)(3s - 14).

18. 25y^2 + 60y + 36 = (5y + 6)(5y + 6) = (5y + 6)^2. Since the second differences are equal, the data points can be modeled with a polynomial function.

b. The degree of that function is 2. See below. 20. Since the second differences are equal, the general equation is z = f(x) = ax^2 + bx + c. Using these data, f(1) = a(1)^2 + b(1) + c = 0, f(2) = a(2)^2 + b(2) + c = 4, f(3) = a(3)^2 + b(3) + c = 12. Then:

9a + 3b + c = 12
4a + 2b + c = 4 \rightarrow 5a + b = 8 \rightarrow a + b + c = 0 \rightarrow 3a + b = 4 \rightarrow 2a = 4; thus a = 2.

From 3a + b = 4, 3(2) + b = 4, 6 + b = 4, b = -2; and from a + b + c = 0, 2 + (-2) + c = 0, c = 0. The polynomial function is z = f(x) = 2x^2 - 2x. See below.

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 662-665 or to the Vocabulary (Voc.) on page 659. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 11 REVIEW** (pp. 662–665)

1. \(x^3 + 2x - 3\) 3. \(8y^3 + 60y^2 + 150y + 125\)
2. \(6x^3 + 2xy^3 - 3xy - y^2\) 7. \(a^3; 9b^2\) 9. \((x - 7)^2\)
11. \(4x(x^2 - 3x - 7)\) 13. \((r^2s^2 + 9)(rs + 3)(rs - 3)\)
15. \((z - 3)(z^2 + 3z + 9)\) 17. \(x = 0.5, \frac{-1}{3}, 0\) 19. \(x = 0.8\)
21. \(x = 0, -4, -\frac{7}{3}\); no multiple roots 23. \(n = -4, 2 + 2i\sqrt{3}, 2 - 2i\sqrt{3}\); no multiple roots 25. yes; \(a_n = f(n) = -6n + 1\)

**LESSON 12-1** (pp. 668–673)

5. The point on the earth's surface above the point where the earthquake began. 7. b, c 9. a. (0, 0) b. 5 c. See below.
11. a. \(y = \pm 8\) b. See below. 12. \((x + 3)^2 + (y + 2)^2 = 64\)
15. \(30\sqrt{3} \approx 52\) mi 17. a. \(x^2 + y^2 - 300x - 200y + 15,600 = 0\) b. \(A = 1, B = 0, C = 1, D = -300, E = -200, F = 15,600\) 19. ellipse 21. a. See below. b. y-axis
23. a. \(y = \pm 10\) b. \(y = \pm 5\sqrt{3}\) c. \(y = \pm \sqrt{100 - x^2}\)

**LESSON 12-2** (pp. 674–679)

1. a. \(-c\). See below. a. \(x^2 + y^2 = 49\) b. \(y = \sqrt{49 - x^2}\)
2. c. \(y = -\sqrt{49 - x^2}\) 5. \(\{(x, y); 9 < x^2 + y^2 < 36\}\) 7. d
9. See below. 11. a. See below. b. 2 c. \(x^2 + y^2 = 4\)
12. a. \(x^2 + y^2 \leq r^2\)
13. b. \(x^2 + y^2 > r^2\) 15. \(16 < x^2 + y^2 < 36\) 17. A circle is the set of all points in a plane at a given distance (radius) from a fixed point (center).
19. a. \(x^2 + y^2 = 1\) b. the unit circle
21. \(6F + 3F + 2S + P\)

**LESSON 12-3** (pp. 680–686)

1. 12 3. 20 5. vertex 7. true 9. See below. 11. \(\frac{12}{30} = \frac{3}{5}\)
13. Focal constant > distance between foci 15. a. \((-10, 0), (0, -5), (-6, 3), (8, -3), (-8, -3), (-6, 4), (6, -4), (-6, -4)\)
b. See below. c. ellipse d. \(x = 0, y = 0\) 17. a. See below. b. ellipse; \(F_1\) and \(F_2\) 5 c. See below. 19. \(9 < x^2 + y^2 < 49\) 21. a. \((-\frac{1}{2}, \frac{1}{2})\) b. 3 23. \(\sqrt{(x - c)^2 + y^2}\)
25. \(x + 3\) 27. \(4a^2 - 4a\sqrt{p} + p\)
LESSON 12-4 (pp. 687–694)

3. $\overline{AC}$ 5. E 11. See below. 13. $\frac{x^2}{225} + \frac{y^2}{189} \leq 1$ 15. c
17. a. 77.4 million mi  b. 64.2 million mi 19. See below.
21. $(x - 4)^2 + y^2 < 16$ 23. 8π 25. $h = \frac{66}{\pi r} + r$
27. a. $= 20^\circ$ b. $\angle B \approx 60^\circ$; $\angle A \approx 70.3^\circ$

LESSON 12-5 (pp. 695–700)

3. False 5. a. π b. 12π 7. 50π 9. a. False b. See example
11. a. See below. 13. a 15. b 17. f
19. $\frac{x^2}{81} + \frac{y^2}{49} = 1$ 21. a 23. $y + 1 = 2(x - 3)$

LESSON 12-6 (pp. 701–706)

1. 48 mph 3. a. (6, 6), (-6, -6) b. x-axis, y-axis c. 12
5. The equation of the hyperbola is $xy = 8$. Since $8 \cdot 1 = 8$, the point (8, 1) is on the hyperbola. 7. 3600 9. See below.
11. $yx = 64$ 13. a. 7 b. $\frac{x^2}{12.25} + \frac{y^2}{8.25} = 1$ c. $\approx 10\pi$
15. See below. 17. sample: 100i, $\frac{1}{i}$ 19. $x = 0, x = \pm 3, x = \pm 3i$ 21. a. $L = 100 - \frac{1}{2}N$ b. $N = 200$

LESSON 12-7 (pp. 707–712)

5. a. (-5, 0), (5, 0) b. $\frac{y}{\sqrt{11}} = \pm \frac{x}{5}$ 7. $(6, 3.35), (6, -3.35)$ 9. See below. 11. a. $\sqrt{160} - \sqrt{20}$

b. $\frac{x^2}{16.716} - \frac{y^2}{8.284} = 1$ c. See below. 13. a. See below.
15. $x^2 - y^2 = 1$ 17. c 19. $x^2 + (y - 96)^2 = 9216$ 21. (3, -20)

LESSON 12-8 (pp. 713–718)

1. a. no b. it has an $xy^2$ term 3. a. yes b. $1x^2 + 2xy + 3y^2 + 4x + 5y - 6 = 0$ 5. ellipse 7. hyperbola 13. center = (-2, -2) $r = \sqrt{6}$ 17. $3x^2 + 0xy + 0y^2 + 6x - 1y - 5 = 0; A = 3, B = 0, C = 0, D = 6, E = -1, F = -5$
19. a. ellipse b. $y = \pm \sqrt{64 - \frac{16}{3}x^2}$ c. See below.
21. $4\sqrt{74} = 34.4$ 23. a. graphing, substitution or linear combination b. (1.5, 6)
LESSON 12-9 (pp. 719-723)
1. Linear combinations 3. a. See below. b. \((-2 + \sqrt{10}), (6 + 3\sqrt{10}), (-2 - \sqrt{10}), (6 - 3\sqrt{10})\)
5. \((-1, 1), (2, 4)\)
7. a. inconsistent b. Sample: \(y = x - 1\) and \(y = 2x^2\)
9. a. 2 b. \((-3, 0), (4, -7)\) c. Does \(0 = (-3)^2 - 2(3) - 15\), and \(-3 + 0 = -3\)? Yes. Does \(-7 = 4^2 - 2(4) - 15\), \(4 + -7 = -3\)? Yes. 11. \(2x - 7 = x^2 - 8x + 18\); \(x^2 - 10x + 25 = 0; (x - 5)^2 = 0; x = 5, y = 3\) Does \(3 = 5^2 - 8(5) + 18\)? Yes. Does \(3 = 2(5) - 7\)? Yes.
13. \(\left(\frac{4 + \sqrt{41}}{5}, \frac{2 - 2\sqrt{41}}{5}\right) = (2.08, -2.16)\), \(\left(\frac{4 - \sqrt{41}}{5}, \frac{2 + 2\sqrt{41}}{5}\right) = (-2.16, 2.08)\)
15. See below.
17. \(xy = 2\) 19. a. \(5.43 \cdot 10^9\) b. \(1.378 \cdot 10^9\) km c. 75 years
21. \((x - 2)(x^2 + 2xy + 4y^2)\)
23. a. 0°C b. True c. 1

3. a.

19. a.-c.

CHAPTEAR 12 PROGRESS SELF-TEST (p. 734)
1. For \(x^2 + 9x + y^2 - 26y - 163 = 0, A = 1, B = 0, and C = 1\). Thus \(B^2 - 4AC = 0 - 4 = -4 < 0\), so the equation represents an ellipse. Since \(A = C\), the ellipse is a circle.
2. Complete the square for \(x\) and for \(y: x^2 + 9x + \frac{81}{4} + y^2 - 26y + 169 - 163 = 0 + \frac{81}{4} + 169, (x + \frac{9}{2})^2 + (y - 13)^2 = 352.25\).
3. Since the image of \(x^2 + y^2 = 1\) under \(S_{a,b}\) is \(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\), the image under \(S_{3,4}\) is \(\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1\). 4. The equation represents an ellipse, which is choice b.
5. The vertices of \(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\) are \((-a, 0), (a, 0), (0, b), and (0, -b)\), so the vertices of \(\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1\) are \((-3, 0), (3, 0), (0, 4), and (0, -4)\). 6. See below.
Since \(c^2 = 13^2 - 5^2, c = 12\) and the foci are at \((-12, 0)\) and \((12, 0)\). Using a vertex on the minor axis as \(P, F_1P + F_2P = 13 + 13 = 26\). Since \(a = 13\) and \(b = 5\), an equation is \(\frac{x^2}{169} + \frac{y^2}{25} = 1\). 7. The area of the ellipse = \(\pi ab = (13)(5)\pi = 65\pi \approx 204\). 8. See below. From the graph, the intersections are about \((3.5, 1.5)\) and \((-5, -2.5)\). 9. Since \(x - 2 = 4x - x^2\), then \(x^2 - 3x - 2 = 0\) and \(x = \frac{-3 + \sqrt{3^2 - 4(-2)(1)}}{2} = \frac{3 + \sqrt{17}}{2}\) and \(\frac{3 - \sqrt{17}}{2}\). The two points of intersection are \(\left(\frac{3 + \sqrt{17}}{2}, -1 + \sqrt{\frac{17}{2}}\right)\) and \(\left(\frac{3 - \sqrt{17}}{2}, -1 - \sqrt{\frac{17}{2}}\right)\). 10. a. See below. The length of the major axis
is 2.8 + 4.6 = 7.4 billion miles. b. $PO = \frac{1}{2} PQ = 3.7,$
so $SO = 3.7 - 2.8 = 0.9.$ Since $ST = PO = 3.7, TO = \sqrt{3.7^2 - 0.9^2} = \sqrt{12.88} \approx 3.56,$ the length of the minor axis is 7.2 billion miles. 11. See below. 12. See below.

13. a. See below. The two equations are $x^2 + y^2 = 1600$ and $(x + 25)^2 + (y - 60)^2 = 900.$ 14. Solve the system:

\[
\begin{align*}
50x + 625 & - 120y + 3600 = -700 \\
50x & = 120y - 4925 \\
x & = 2.45 - 98.5 \\
\end{align*}
\]
Substitute. $(2.45 - 98.5)^2 + y^2 = 1600$

\[
\begin{align*}
y & = 472.8 \pm \sqrt{472.8^2 - 4(6.76)(8102.25)} \\
& = \frac{472.8 \pm \sqrt{4455}}{2(6.76)} \\
& = \frac{472.8 \pm 66.75}{13.52} \\
\end{align*}
\]

The two $y$-values are 39.9 and 30; the corresponding $x$-values are -2.74 and -26.5. The two points for the epicenter are about (-2.7, 40) and (-27, 30). 15. The $x$-axis and the $y$-axis

6.

8.

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 735-739 or to the Vocabulary (Voc.) on page 732. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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CHAPTER 12 REVIEW (pp. 735-739)

1. $x^2 + 0y + y^2 - 6x + 14y - 42 = 0$ 2. $x^2 + \frac{y^2}{2} = 1$
5. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ 7. $x^2 + y^2 = 8$
9. a. Solve the equation $x^2 + y^2 = 20$ for $y$. $y = \pm \sqrt{20 - x^2}$ b. The graph of $x^2 + y^2 = 20$ is the union of the graphs of $y = \sqrt{20 - x^2}$ and $y = -\sqrt{20 - x^2}$.
11. $\frac{x^2}{144} + \frac{y^2}{169} = 1$ 13. $\frac{x^2}{16} - \frac{y^2}{33} = 1$
15. $33\pi = 104$ 17. The circle has an area of $25\pi$, ellipse has an area of $24\pi$; circle has greater area.
19. (-5, 5.25), (3, 14) 21. (-1, -6) 23. (0, -4), (3, 5)
25. $\left(1 + \sqrt{145}, 1 + \sqrt{145}\right)$, $\left(1 - \sqrt{145}, 1 - \sqrt{145}\right)$
27. center (0, 0), radius $\sqrt{5}$ 29. 26 31. a. (-4, 0), (4, 0)
b. \( \frac{y}{2} = \pm \frac{x}{4} \) 33. ellipse 35. ellipse (circle) 37. hyperbola 39. A: hyperbola; B: parabola; C: ellipse; D: circle 41. True 43. True 45. 19.5\( \pi \) = 61.3 sq m 47. \((x - 200)^2 + (y - 100)^2 < 100\) 49. \((x - 8)^2 + y^2 > 5\) 51. 12 by 18 53. (10.8, -48.8) 55. a. $14 \ b. 420 \ c. 575 \ d. \text{See below.} 59. \text{See below.} 61. \text{See below.} 63. \frac{x^2}{49} + \frac{y^2}{16} = 1 \ e. 65. \ b. \text{See below.} 67. \text{See below.} 69. \text{See below.}

\text{LESSON 13-1 (pp. 742, 747)}

3. a. 20, 18, 16, 14  b. 20 + 18 + 16 + 14 5. 500, 500 7. a. 142  b. 2520 9. The number of terms. 11. a. 27
b. 57  c. 3500 13. a. $24,000; $32,400  b. $225,600
15. a. 15, 52, 466  b. \( T = 10 + 3(n - 1) \); \( S = \frac{3}{2}n^2 + \frac{17}{2}n \)
c. Change lines to following:
20 LET TERM = 2400
40 FOR \( N = 2 \ TO \ 8 \)
50 TERM = TERM + 1200
17. \( \approx 175 \) m 19. \( n(1 - a) \)
21. 514 23. 56 25. a. -13
b. parabola congruent to \( y = 3x^2 \), with vertex at \((4, -13)\)

\text{LESSON 13-2 (pp. 748, 752)}

1. 1, 2 3. 62.496 5. \( \frac{1 - b^{17}}{1 - b} \)
7. \$1267.19 9. a. 6, -4, \( \frac{8}{3} \), \( \frac{10}{9}, \frac{32}{27}, \frac{64}{81}, \frac{128}{243}, \frac{256}{729} \)
b. \( \approx 3.46 \) 11. a. 56, 28, 14, 7
3.5. 1.75, 875, 4375, 21875, 109375  b. \( \approx 111.9 \)
13. 77 15. 165.150 17. \( y = \frac{-3}{2}x + 16 \) 19. \( \approx 20.2 \) sec

\text{LESSON 13-3 (pp. 753, 758)}

3. b 5. a 7. 666 9. 2184 11. 101 13. a. 24  b. 720
c. \( 5.109 	imes 10^{19} \) 15. \( \sum_{i=1}^{5} \frac{7}{100}i^2 \)
19. \( \frac{1}{n} \sum_{i=1}^{n} a_i \) 21. b. 15  b. \( n + 1 \) 23. 5050 25. \text{See below.}
27. a. \( 2.85 \) in.  b. 55.46 in. 29. 100 time as loud

\text{LESSON 13-4 (pp. 759, 764)}

1. a. No.  b. This is an infinite geometric series and the final value cannot be determined. 3. a. no; b. \( |r| > 1 \)

\text{LESSON 13-5 (pp. 765, 770)}

3. a. 28 27. 252 9. 1 11. 15 13. 10 15. true
7. \( x = 11 \), \( y = 6 \) 19. the 2nd elements in each row
21. the 4th elements in each row 23. a. \( 36.365 \)  b. \( 260 \)
25. 60 27. a. one real root  b. No real roots  c. two rational roots 29. \( 6! = 720 \)
3. 1 8 28 56 70 56 28 8 1
1 9 36 84 126 84 36 9 1
I 10 47 120 210 252 210 120 47 10 1

\text{LESSON 13-6 (pp. 771, 775)}

5. \( (a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \)
7. \( a^3 - 3a^2 b + 3ab^2 - b^3 \)
9. \( x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6x y^5 + y^6 \)
11. \( a^4 + 8a^3 b + 24a^2 b^2 + 32ab^3 + 16b^4 \)
13. \( (x + y)^5 \)
15. a. \( a^4 + 4a^2 b^2 + 6a^2 b^2 + 4ab^3 + b^4 \)
b. \( (a + b)^4 \) because \( a^2 + 2ab + b^2 = (a + b)^2 \)
17. \( 1.01004512021252 \)
19. \( 0.2104375 \)
21. The sixth year 23. 7 25. 3825 lb

\text{LESSON 13-7 (pp. 776, 781)}

3. \{ \}, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, \{p, q, r\} 5. 8
9. 120 11. 177, 100 13. \( 2^8 = 256 \)
15. n! 17. a. 52
b. 1326 c. \( \frac{1}{270.725} \)
19. 21. \( a^8 + 8a^7 b + 28a^6 b^2 + \)
56a^5b^3 + 70a^2b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8
23. \frac{45}{25} 25. \frac{165}{27} 27. c

**LESSON 13-8** (pp. 782-787)

1. \frac{1}{30} 3. \frac{1}{5} 5. RRRR WRRR WRWR WRRW
   RRRR WWWR WRRR WRRW
   RRRR WWWR WRRW WRRW
   RRRR WRRW WRRW WRRW
   RRRR WRRW WRRW WRRW
   7. \frac{5}{20} 9. \frac{1}{64} 11. \frac{1}{256} \frac{7}{256} \frac{8}{256} \frac{7}{256} \frac{5}{256} \frac{2}{256} \frac{5}{256}
13. a. b. 0. p^2 + 9p^2q + 36p^2q^2 + 84p^2q^3 + 126p^2q^4 + 126p^2q^5 + 184p^2q^6 + 36p^2q^7 + 9pq^8 + q^9
17. 3 19. 625 21. 6.25\pi \text{ in.}^2 23. 66.69%

**LESSON 13-9** (pp. 788-792)

3. a. mean: -5.14; median: -4; mode: -14  b. median or mean
5. a. mean b. 156.25 7. mean = 30; s.d. = 14.14 9. more spread out 11. a. A few extreme values can affect the mean, but not the median. b. The most common income will not reflect the wealth of the community. 13. mean = 9.14; s.d. = 6.96 15. Sample: \{10, 10, 10, 10\} and \{0, 0, 0, 40\} 17. \frac{11}{140} = .00877

**LESSON 13-10** (pp. 793-797)

1. a. \frac{1}{25} It could represent the probability of getting 3 heads in 5 tosses of a fair coin. b. A probability function 3. \frac{25}{1054} 7. 2.3% 9. 68.2% 11. =66% 13. 15.9% 15. 538,145
17. 6 19. 1 21. False 23. a. \frac{-3 + \sqrt{209}}{10} b. 1.15 or -1.75

**LESSON 13-11** (pp. 798-804)

3. All potential voters; the people who are asked questions 7. 18.6% of all households with TV are tuned into a particular show. 11. 500, 15.8 13. a. Random number program. Student answers will vary. b. The two outputs should be different. 15. The class; the classmate you called 17. =5.16
19. \frac{6}{25} = \frac{3}{12} 21. =5.49 23. y - 5 = (x - 6)^2, y - 5 = -(x - 6)^2 25. x = -|y|

**CHAPTER 13 PROGRESS SELF-TEST** (p. 807)

1. \begin{align*} a_1 &= 12; a_2 &= \frac{1}{2}a_1 = \frac{1}{2}(12) = 6; a_3 &= \frac{1}{2}a_2 = \frac{1}{2}(6) = 3; \\
   g_4 &= \frac{1}{2}a_3 = \frac{1}{2}(3) = \frac{3}{2} \end{align*}
2. The geometric series has first term 12 and constant ratio 1/2. The sum of the first 12 terms is
   \[ S_n = \frac{a(1 - r^n)}{1 - r} \] for \( n = 30, a_1 = 12, \) and \( d = 2, S_{30} = \frac{30}{5} \]
   \[ (2)(12) + (29)(2)) = (15)(82) = 1230. \]
3. \( \sum_{i=2}^{3} 4(n)^i = 4(10)^2 + 4(10)^1 + 4(10)^0 + 4(10)^1 + 4(10)^2 + 4(10)^3 = .04 + .4 + 4 + 40 + 400 + 4000 = 4444.44 \]
4. Use the above method, \( a = 1, b = x \)
5. \( \begin{align*} a = 15, b = 3! \end{align*} \]
6. \( S_c = \frac{8!}{8!} = 17. \)
7. \( \begin{align*} a = \frac{51}{38} \end{align*} \]
20. The expansion of \((x - y)^7\) begins \( \binom{7}{0} x^7 - \binom{7}{1} x^6 y + \binom{7}{2} x^5 y^2 - \ldots \). The second term is \( \binom{7}{1} x^6 y \), so the answer is false.
21. 8! 22. The population is infinite.
16. a. \( \begin{array}{c|c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline P(n) & 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array} \)
18. a. \( \begin{align*} \begin{array}{c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline P(n) & 1 & 1 & 1 & 2 & 1 & 1 & 3 & 3 & 1 & 1 & 4 & 6 & 4 & 1 & 1 & 5 & 10 & 5 & 1 \end{array} \end{align*} \)
The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 808-811 or to the Vocabulary (Voc.) on page 805. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 13 REVIEW** (pp. 808-811)

1. 990 3. 4 5. 1830 7. .354 + .000354 + .000000354...
b. \[ \sum_{n=1}^{72} 2n \]
25. \[ 2^{16} = 67,108,864 \]
27. \[ p^7 - 56p^6 + 1344p^5 - 17,920p^4 + 143,360p^3 - 688,128p^2 + 1,835,008p - 2,097,152 \]
29. \[ \frac{1}{2}a^5 + \frac{1}{8}a^4b + 5a^2b^2 + 20ab^3 + 40b^4 \]
\[ \frac{32}{5}b^5 \]
30. True 31. False 33. 82.125; 83.5; 90 or 68
35. C. 37. \[ |r| < 1 \]
39a. 56.25, 42.1875, 31.640625
41. True 43. 25^2 = 32 45. (r+1)^2 47. 54 mi
49. a. $51,257 b. 31,477.731 51. \frac{n}{2} (1 + n) 53. 6! = 720
55. 316,251 57. 45 59. \[ \frac{10}{32} \]

**LESSON 14-1** (pp. 814-818)

3. False 5. The yz-plane 7. c 9. See below. 11. a, b are negative; c is positive. 13. (6, 8, 10) 15. a. See below.
b. 80 cubic units c. 132 square units 17. a. \( A = (13, 0, 5), B = (0, 0, 5), C = (0, 2, 5), E = (13, 0, 0), G = (0, 2, 0), H = (13, 2, 0) \) b. 130 cubic units c. 202 square units 19. y is divided by 3. 21. y is multiplied by 27 c 23. c 25. c
9.

**LESSON 14-2** (pp. 819-825)

1. True 3. axis 5. The z-value of the point where a plane intersects the z-axis. 7. See below. 9. a. \[ z = \frac{2}{x} \]
\[ x = -3 \]
b. \(-3, 1, 2) 11. y = 5 13. a. 2x + 4y + 5z = 100
b. 0 \leq x \leq 59, 0 \leq y \leq 25, 0 \leq z \leq 20 c. See below.
d. (50, 0, 0), (0, 25, 0), (0, 0, 20); yes

**LESSON 14-3** (pp. 826-832)

3. 60d + 100w + 200p = 21,200
30d + 300w + 400p = 46,850
40d + 80w + 150p = 16,100
\( b. d = \frac{54}{5} \) hr, w = $88/hr, p = $50/hr 5. a. -4; b. 5 or 4; -5
\( b. |4x + 3y + 6z| = 122 \]
\( -12x - 9y - 18z = -399 \)
\( b. |3x + 5y + 4z| = 130 \]
\( 12x + 20y + 16z = 520 \)
So 11y - 2z = 124 c. x = 18, y = 12, z = 4 7. z = -3 c 9. $53 11. y = -6 13. See below.
15. a. 22.0 m b. 39.160192 m c. 40 m 17. a. ellipse b. \((-11, 0), (11, 0), (0, 10), (0, -10)\)
LESSON 14-4 (pp. 833–839)
1. the absolute value of the difference of their x-coordinates
3. Take the absolute value of the difference of the z-coordinates of the endpoints. 5. 6.3 7. 23.2
11. See below. 13. a. \( x^2 + y^2 + z^2 = 25 \)
b. Samples: (3, 4, 0), (4, 3, 0), (4, 0, 3), etc.
15. a. (3, 2, -3) b. 6 c. Sample (9, 0, 0) and (0, 8, 0)
d. \(( x - a)^2 + (y - b)^2 + (z - c)^2 = r^2\) 17. \( \approx 152 \text{ m} \)
(must be longer than \( \sqrt{24,500} \text{ m} \) ) 19. a. \( 12\pi \text{ cm}^2 \)
b. \( 2\sqrt{3} \text{ cm} \) c. \( 3\sqrt{3} \text{ cm} \) d. cylinder

LESSON 14-5 (pp. 840–845)
5. a. a sphere b. \( x^2 + y^2 + z^2 = 50 \) 11. a. sphere
b. \( 288\pi \text{ cubic units} \) c. \( 144\pi \text{ square units} \) 13. See below.
15. See below. 17. a doughnut or torus 19. True 21. It does not intersect the x-axis. 23. False 25. c

CHAPTER 14 PROGRESS SELF-TEST (p. 866)
1. See below. 2. To locate point S, start from R and go 2 units on the x-axis to U, 5 units on the y-axis to I, and 4 units on the z-axis to S. The coordinates are (2, 5, 4). 3. The coordinates of U are (2, 0, 0) and of H are (0, 5, 4). \( UH = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{(2 - 0)^2 + (0 - 5)^2 + (0 - 4)^2} = \sqrt{4 + 25 + 16} = \sqrt{45} = 6.7. \) 4. In the upper left front octant, “upper” means z > 0, “left” means y < 0, and “front” means x > 0.
A sample is (2, -1, 4). 5. The dimensions of the box are 4, 3, and 2; the volume is (4)(3)(2) = 24. 6. a. The set of points 3.5 units from the yz-plane consist of two planes, each parallel to the yz-plane and 3.5 units from it. An algebraic description is \( x = \pm 3.5. \) b. See below. 7. An equation for the sphere is \( x^2 + y^2 + z^2 = 169. \) 8. For the system \( x + y - z = 2 \)
\[ 6x + y - z = 4 \]
\[ 6x + y + 3z = 0 \]
\[ \begin{bmatrix}
1 & 1 & -1 \\
6 & 1 & 4 \\
4 & -1 & 3
\end{bmatrix} \] 9. To solve the system, one method is to find another system without the variable y by adding equations 1 and 3 then 2 and 3: \( 5x + 2z = 2 \) Since these two equations are equivalent, this system (and the original system) has an infinite number of solutions. 10. If a plane is parallel to the z-axis, the coefficient of z in its equation is 0. That is choice d. 11. The choice is b, a cylinder. Make a sketch with a line segment parallel to the x-axis, then sketch its path around the x-axis. 12. Since the equation of the plane \( 4x + y + 2z = 4 \) has non-zero coefficients for all three dimensions, it is false that it is parallel to any coordinate plane. 13. Find the intercepts of \( 4x + y + 2z = 4 \) by letting pairs of coordinate values be zero. See below. 14. The equation for the hypersphere is \( x^2 + y^2 + z^2 + w^2 = 144. \) 15. 0, 1 or infinity, many 16. The intersection of the sphere \( x^2 + y^2 + z^2 = 49 \) with the yz-plane is the circle in the yz-plane \( x^2 + z^2 = 49, \) which has center \((y, z) = (0, 0) \) and radius 7.
17. See below. 18. When the unit is \( \frac{1}{2} \) the size, the boundary
is multiplied by 9. Then \( D = \log 9 \approx 1.37. \) 19. a. For the system the equations are \( 3a + 2c + 1s = 525,20 \) \( 1a + 5c + 1s = 47 \)
b. To solve the system using matrices, start with
\[
\begin{bmatrix}
4 & 4 & 0 & 66 \\
3 & 2 & 1 & 52.50 \\
1 & 5 & 1 & 47 \\
1 & 1 & 0 & 16.50 \\
3 & 2 & 1 & 52.50 \\
1 & 5 & 1 & 47
\end{bmatrix}
\]
Divide the top row by 4;
\[
\begin{bmatrix}
1 & 1 & 0 & 16.50 \\
0 & -1 & 1 & 3 \\
0 & 4 & 1 & 30.50
\end{bmatrix}
\]; multiply the top row by -3 and add to row (2), next multiply the top row by -1 and add to row (3);
\[
\begin{bmatrix}
1 & 0 & 0 & 11.00 \\
0 & -1 & 0 & -5.50 \\
0 & 0 & 1 & 8.50
\end{bmatrix}
\]; multiply row (2) by -4 and add to row (3), next add row (2) to row (1);
\[
\begin{bmatrix}
1 & 0 & 0 & 11.00 \\
0 & -1 & 0 & -5.50 \\
0 & 0 & 1 & 8.50
\end{bmatrix}
\]; subtract row (3) from row (2), next subtract row (3) from row (1);
\[
\begin{bmatrix}
1 & 0 & 0 & 11.00 \\
0 & -1 & 0 & -5.50 \\
0 & 0 & 1 & 8.50
\end{bmatrix}
\]; therefore adult: $11.00; child: $5.50; senior: $8.50.

The chart below keys the Progress Self-Test questions to the objectives in the Chapter Review on pages 867–869 or to the Vocabulary (Voc.) on page 865. This will enable you to locate those Chapter Review questions that correspond to questions you missed on the Progress Self-Test. The lesson where the material is covered is also indicated in the chart.

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**CHAPTER 14 REVIEW** (pp. 867–869)

1. c. \( r = 5, s = 2, t = -4 \)
2. \( (5, 3, -2, -1) \)
3. \( 7, 4, 9, 22 \)
4. \( x^2 + y^2 + z^2 + w^2 = 49 \)
5. center \((0, 0, 0)\); radius = 27
6. \( z = -7 \)
7. \( Ax + By + Cz = D \) where not all of \( A, B, C = 0 \)
8. infinitely many
9. b. The planes for equations 1 and 3 coincide.
10. \( \sqrt{(a - e)^2 + (b - f)^2 + (c - g)^2 + (d - h)^2} \)
11. A fractal is a set of points that is self-similar. 27. See below.
12. 29. m.c. = 2; t.i. = 2; s.a. = 9; c = 8
13. 31. 1st class: $395; 2nd class: $350; 3rd class: $275
14. 33. See below.
15. 35. See below.
16. 37. a. See below, b. a sphere of radius 3
17. 39. a. See below, b. a non-circular ellipse
18. 41. a. a cone b. \( 21\pi \)
19. 43. a. an infinite double cone with the \( x \)-axis as its axis
20. a. a cone b. \( 21\pi \)
21. a. non-circular ellipse
22. a. rectangle with width less than the diameter of the cylinder and height equal to the height of the cylinder
MATHEMATICAL SYMBOLS

\[ A \cap B \] intersection of sets A and B
\[ A \cup B \] union of sets A and B
\[ A' \] image of A
\[ S_k \] size change of magnitude k
\[ S_{a,b} \] scale change with horizontal magnitude a and vertical magnitude b
\[ r_x \] reflection over the x-axis
\[ r_y \] reflection over the y-axis
\[ r_y=x \] reflection over the line \( y = x \)
\[ T_2 \circ T_1 \] composite of transformations \( T_1 \) and \( T_2 \)
\[ R_\theta \] rotation of magnitude \( \theta \) counterclockwise
\[ T_{h,k} \] translation of \( h \) units horizontally and \( k \) units vertically

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \] 2 \times 2 matrix

\[ M^{-1} \] inverse of matrix \( M \)
\[ \det M \] determinant of matrix \( M \)
\[ \sqrt{\ } \] radical sign; square root
\[ \sqrt[\ ]{\ } \] the real \( n \)th root of \( x \)
\[ i \] \( \sqrt{-1} \)
\[ \sqrt{-k} \] a solution of \( x^2 = -k \), \( k > 0 \)
\[ a + bi \] a complex number, where \( a \) and \( b \) are real numbers
\[ g \circ f \] composite of functions \( f \) and \( g \)
\[ |x| \] absolute value of \( x \)
\[ [x] \] greatest integer less than or equal to \( x \)
\[ \lfloor x \rfloor \] smallest integer greater than or equal to \( x \)
\[ f^{-1} \] inverse of a function \( f \)
\[ \log_b m \] logarithm of \( m \) to the base \( b \)
\[ e \] 2.71828 \ldots

\[ \ln x \] natural logarithm of \( x \)
\[ \sin \theta \] sine of \( \theta \)
\[ \cos \theta \] cosine of \( \theta \)
\[ \tan \theta \] tangent of \( \theta \)
\[ \text{rad} \] radian
\[ \theta \] Greek letter theta
\[ a_n \] \( \text{"a sub } n \text{";} \) the \( n \)th term of a sequence
\[ \sum_{i=1}^{n} i \] the sum of the integers from 1 to \( n \).
\[ S_n \] the partial sum of the first \( n \) terms of a sequence
\[ \infty \] infinity
\[ \binom{n}{r} \] the number of ways of choosing \( r \) objects from \( n \) objects
\[ (x, y, z) \] an ordered triple
\[ (x, y, z, w) \] an ordered 4-tuple
\[ \text{INT (X)} \] the BASIC equivalent for \( [x] \)

\[ \sqrt[\ ]{\ }, \sqrt{\ } \] calculator \( n \)th root key
\[ x! \] calculator factorial key
\[ \log \] calculator common logarithm key
\[ e^x \] calculator \( e^x \) key
\[ \ln \] calculator natural logarithm key
\[ \text{DRG} \] calculator degree key
\[ \sin \] calculator sine key
\[ \cos \] calculator cosine key
\[ \tan \] calculator tangent key
absolute value function  A function whose values are the positive distance between \( x \) and 0; \( |x| = x \) if \( x \geq 0 \) and \( |x| = -x \) if \( x < 0 \).

acceleration  The rate at which the velocity of a moving object changes.

addition property of equality  For all real numbers \( a \), \( b \), and \( c \): if \( a = b \), then \( a + c = b + c \).

addition property of inequality  For all real numbers \( a \), \( b \), and \( c \): if \( a < b \), then \( a + c < b + c \).

algebraic expression  A combination of numbers and variables; sometimes called simply an expression.

analytic description  An algebraic description of a situation or object.

angle of depression  The angle between the line of sight and the horizontal when the line of sight points down.

angle of elevation  The angle between the line of sight and the horizontal when the line of sight points up.

arithmetic sequence  A sequence with a constant difference; also called linear sequence.

arithmetic series  An indicated sum of successive terms of an arithmetic sequence.

arrow notation  Function notation used in transformations; also called mapping notation.

asymptote  A line that the graph of a function \( y = f(x) \) approaches as \( x \) increases or decreases without bound.

asymptotes of a hyperbola  Two lines which are approached by the points on the branches of a hyperbola as the points get farther from the foci. The asymptotes of the hyperbola with equation \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) are \( \frac{y}{b} = \pm \frac{x}{a} \).

augmented matrix  A matrix which consists of the coefficients and constants of a system of equations.

automatic graphers  Calculators and computer programs that automatically display graphs.

axis of rotation  The line about which a point, line, or curve is rotated.

axis of symmetry of a parabola  The line perpendicular to the directrix which contains the focus.

base  The variable \( b \) in the expression \( b^n \).

binomial  A polynomial with two terms.

binomial distribution  A probability function in which the values of the function are proportional to binomial coefficients.

binomial expansion  The result of writing the power of a binomial as a polynomial.

binomial square theorem  For all real numbers \( x \) and \( y \): \((x + y)^2 = x^2 + 2xy + y^2 \) and \((x - y)^2 = x^2 - 2xy + y^2 \).

binomial theorem  \((a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \).

boundary  A line or curve separating a plane into two regions.

branches of a hyperbola  The two separate parts of the graph of a hyperbola.

calculator key sequence  A list of keystrokes to be performed on a calculator.

center of a circle  The fixed point from which the set of points of the circle are at a given distance.

center of an ellipse  The intersection of the axes of the ellipse.

center-radius equation for a circle theorem  The circle with center \((h, k)\) and radius \(r\) is the set of points \((x, y)\) that satisfies \((x - h)^2 + (y - k)^2 = r^2\).

central limit theorem  Suppose random samples of size \(n\) are chosen from a population of events in which the probability of an event having certain characteristics is \(p\). Let \(P(x)\) equal the number of elements in that sample with the characteristic. Then \(P\) is approximated by a normal distribution with mean \(np\) and standard deviation \(\sqrt{np(1-p)}\).

circle  The set of all points in a plane at a given distance from a fixed point.

coefficient matrix  A matrix which represents the coefficients of the variables of a system.
coefficients of a polynomial  The numbers
\[a_n, a_{n-1}, a_{n-2}, \ldots, a_0\] in the polynomial
\[a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0\].

combination  Any choice of \(r\) objects from \(n\) objects.

combined variation  A situation in which direct and inverse variations occur together.

common logarithm  A logarithm to the base 10.

complements theorem  For all \(\theta\) between 0° and 90°: \(\sin \theta = \cos(90° - \theta)\) and
\(\cos \theta = \sin(90° - \theta)\).

completing the square  A technique used to transform a quadratic from \(ax^2 + bx + c\) form to \(a(x - h)^2 + k\) form.

complex conjugate  The complex conjugate of \(a + bi\) is \(a - bi\).

complex number  A number of the form \(a + bi\), where \(a, b\) are real numbers and \(i = \sqrt{-1}\).

composite of \(f\) and \(g\)  \(g \circ f\) is the result of first applying function \(f\), then applying function \(g\);
\((g \circ f)(x) = g(f(x))\).

composite of transformations  Suppose transformation \(T_1\) maps figure \(F\) onto figure \(F'\), and
transformation \(T_2\) maps figure \(F'\) onto figure \(F''\). The transformation that maps \(F\) onto \(F''\) is called
the composite of \(T_1\) and \(T_2\), written \(T_2 \circ T_1\).

composition of functions  The operation of first applying one function, then another; denoted by
the symbol \(\circ\).

compounding  The process of earning interest on the interest of an investment.

compound interest formula  \(A = P(1 + r)^t\), where \(P\) is the amount of money invested at an annual
interest rate \(r\) compounded annually and \(A\) is the
total amount after \(t\) years.

compound sentence  A sentence in which two clauses are connected by the word "and" or by
the word "or."

conditional statement  If \(p\) then \(q\); sometimes written \(p \Rightarrow q\).

conic sections  A cross-section of a double cone; also called conic.

conjecture  An educated guess.

consistent system  A system that has solutions.

constant-decrease situation  A situation in which a quantity \(y\) decreases by a constant amount for
every increase in \(x\).

constant-increase situation  A situation in which a quantity \(y\) increases by a constant amount for every
increase in \(x\).

constant matrix  A matrix which represents the constants in a system of equations.

continuous graph  A graph that can be drawn without picking up a pencil from the paper.

continuous compounding  The limit of the process of
earning interest with periods of compounding
approaching zero.

continuously compounded interest formula  \(A = Pe^{rt}\), where an amount \(P\) is invested in an account paying
an annual rate \(r\) compounded continuously, and
\(A\) is the amount in the account after \(t\) years.

converse  The converse of the conditional statement if \(p\) then \(q\) is if \(q\), then \(p\); sometimes written
\(q \Rightarrow p\).

convex regions  A region of the plane in which any
two points of the region can be connected by a line
segment which is itself entirely within the region.

coordinate plane  A plane determined by two axes
in a coordinate system.

cosine function  The correspondence \(\theta \rightarrow \cos \theta\) that associates \(\theta\) with the \(x\)-coordinate of the image
of \((1, 0)\) under \(R_\theta\).

cosine of \(\theta\) (cos \(\theta\))  In a right triangle with acute
angle \(\theta\), \(\cos \theta = \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}}\),
the first coordinate of \(R_\theta(1, 0)\).

counterexample  An instance which proves a
conjecture false.

counting numbers  The set \(\{1, 2, 3, 4, 5, \ldots\}\); also called the natural numbers.

cube root  The cube root \(x\) of \(t\) is a solution to the
equation \(x^3 = t\).

cube root function  The function \(f(x) = \sqrt[3]{x}\).

cubing function  A powering function defined by
\(f(x) = x^3\).
data set  A set in which an element may be listed more than once.

decibel (dB)  A unit of sound intensity; \( \frac{1}{10} \) of a bel.

degenerate form of a conic  The intersection of a double cone and a plane containing the vertex of the cone. The ellipse degenerates to a single point, the parabola to a single line, and the hyperbola to two lines.

degree of a polynomial  The highest exponent of a polynomial in \( x \).

dependent variable  In a formula, a variable whose value always depends on the value of the other variable(s).

depreciation  A situation described by an exponential function where the growth factor is less than one; also called exponential decay.

determinant of a 2×2 matrix  The expression \( ad - bc \), associated with the matrix

\[
M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}
\]

difference of cubes pattern  For all \( a \) and \( b \):

\[
a^3 - b^3 = (a - b)(a^2 + ab + b^2).
\]

difference of squares pattern  For all \( a \) and \( b \):

\[
a^2 - b^2 = (a + b)(a - b).
\]

dimensions \( m \times n \)  A matrix with \( m \) rows and \( n \) columns has dimensions \( m \times n \).

dimensions of a system  A system with \( m \) equations and \( n \) variables has dimensions \( m \times n \).

directly proportional  A situation in which as one variable increases in absolute value, so does the other; \( y \) is directly proportional to \( x^a \) is written as \( y = kx^a \), \( k \neq 0 \), and \( n > 0 \).

directrix  The line whose distance to any point on a parabola is equal to the distance from that point to the focus.

direct variation  A situation in which as one variable increases in absolute value, so does the other; \( y \) varies directly as \( x^a \) or \( y \) is directly proportional to \( x^a \) is written as \( y = kx^a \), \( k \neq 0 \), and \( n > 0 \).

discontinuous graph  A graph that cannot be drawn without picking up a pencil from the paper.

discrete graph  A graph that is made up of unconnected points.

discriminant of a quadratic equation  The value of \( b^2 - 4ac \), which determines whether \( ax^2 + bx + c = 0 \) has real solutions.

discriminant theorem  If \( a, b, \) and \( c \) are real and \( a \neq 0 \), then the equation \( ax^2 + bx + c = 0 \) has:

(a) two real roots, if \( b^2 - 4ac > 0 \); (b) one real root, if \( b^2 - 4ac = 0 \); and (c) no real roots, if \( b^2 - 4ac < 0 \).

discriminant theorem for conics  If \( A, B, C, D, E, \) and \( F \) are real numbers and at least \( A, B, \) or \( C \) is nonzero, then the graph of \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \) is a hyperbola if \( B^2 - 4AC \) is positive, a parabola if \( B^2 - 4AC = 0 \), and an ellipse if \( B^2 - 4AC < 0 \).

disk  The union of a circle and its interior.

distance formula in 3-space  The distance \( d \) between the points \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \) is

\[
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.
\]

distance formula in 4-space  The distance \( d \) between the points \( (x_1, y_1, z_1, w_1) \) and \( (x_2, y_2, z_2, w_2) \) is given by

\[
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2}.
\]

domain of a function  The set of values which are allowable substitutions for the independent variable.

domain of a variable  A set of meaningful numbers or things that can be substituted for a variable.

eccentricity  The ratio of the distance between the foci to the focal constant in an ellipse or hyperbola.

element of a matrix  The object in a particular row and column of a matrix.

ellipse  The ellipse with foci \( F_1 \) and \( F_2 \) and focal constant \( d \) is the set of points \( P \) in a plane which satisfy \( PF_1 + PF_2 = d \), where \( F_1 \) and \( F_2 \) are any two points and \( d \) is a constant with \( d > F_1F_2 \).

equal complex numbers  Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal; \( a + bi = c + di \) if and only if \( a = c \) and \( b = d \).

equal matrices  Two matrices which have the same dimensions and in which corresponding elements are equal.
equation  A sentence stating that two expressions are equal.

equation for a hypersphere  In 4-space, an equation for a hypersphere with center at $(0, 0, 0, 0)$ and radius $r$ is $x^2 + y^2 + z^2 + w^2 = r^2$.

equivalent sentences  Sentences that have the same solutions.

equivalent systems  Systems that have the same solutions.

Euler's $f(x)$ notation  Notation that represents functions by naming the function and enclosing the independent variable in parentheses.

evaluating an expression  Substituting for the variables in an expression and calculating a result.

exact value theorem  (a) $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$;  
(b) $\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$;  
(c) $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$.

expanded form  The result of using the distributive property to rewrite a product of polynomials.

expanded form of an equation of a circle  $x^2 + y^2 + Dx + Ey + F = 0$.

expanded form of an equation of a parabola  An equation of the form $y = ax^2 + bx + c$, where $a \neq 0$.

explicit formula for a geometric sequence  In the geometric sequence with first term $g_1$ and constant ratio $r$, $g_n = g_1r^{n-1}$.

explicit formula for an arithmetic sequence  The $n$th term $a_n$ of an arithmetic sequence with first term $a_1$ and constant difference $d$ is given by the explicit formula $a_n = a_1 + (n - 1)d$.

explicit formula for $n$th term  A formula which describes any term in a sequence according to its position.

exponent  The number $n$ in the expression $b^n$.

$\frac{1}{n}$ exponent theorem  When $x \geq 0$ and $n$ is an integer greater than 1, $x^{1/n}$ is an $n$th root of $x$.

exponential curve  A graph of an exponential equation.

exponential decay  A situation described by an exponential function where the growth factor is less than one; also called depreciation.

exponential function  A function with the independent variable in the exponent; a function with an equation of the form $y = ab^x$.

exponential growth  A situation described by an exponential function where the growth factor is greater than one.

exponential sequence  A sequence with a constant multiplier or constant ratio; also called geometric sequence.

exponentiation  An operation by which a variable is raised to a power; also called powering.

expression  A combination of numbers and variables.

extended distributive property  To multiply two polynomials, multiply each term in the first polynomial by each term in the second.

exterior of a circle  The region outside a circle; given a circle with center $(h, k)$ and radius $r$, the exterior of a circle is described by $(x - h)^2 + (y - k)^2 > r^2$.

extraneous solution  A solution that is gained but does not check in the original equation.

factor theorem  $x - r$ is a factor of a polynomial $P(x)$ if and only if $P(r) = 0$.

factorial function  The function defined by the equation $f(n) = n! = \text{the product of the integers from } n \text{ to } 1$.

fair coin  A coin that has an equal probability of landing on either side; also called an unbiased coin.

feasible region  The set of solutions to a system of linear inequalities; also called feasible set.

Fibonacci sequence  The sequence $1, 1, 2, 3, 5, 8, 13, \ldots$; a recursive definition is $\begin{cases} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} & \text{for } n \geq 3. \end{cases}$

field properties  The assumed properties of addition and subtraction of real numbers.
**local constant** The sum of the distances from a point on an ellipse to the two foci of the ellipse; the absolute value of the difference of the distances from a point on a hyperbola to the two foci of the hyperbola.

**focus (plural foci)** In a parabola, the point along with the directrix from which a point is equidistant; the two points from which the sum (ellipse) or difference (hyperbola) of distances to a point on the conic section is constant.

**formula** A sentence stating that a single variable is equal to an expression with one or more different variables on the other side.

**4-space** The set of ordered 4-tuples \((x, y, z, w)\) of real numbers.

**fractal** An object that is nearly self-similar; an object of fractional dimension.

**function** A relation in which for each ordered pair the first coordinate has exactly one second coordinate.

**fundamental theorem of algebra** Every polynomial equation \(P(x) = 0\) of any degree with complex number coefficients has at least one complex number solution.

**fundamental theorem of variation** If \(y\) varies directly as \(x^n\) and \(x\) is multiplied by a nonzero constant \(c\), then \(y\) is multiplied by \(c^n\); if \(y\) varies inversely as \(x^n\) and \(x\) is multiplied by a nonzero constant \(c\), then \(y\) is divided by \(c^n\).

**general compound interest formula** \(A = P\left(1 + \frac{r}{n}\right)^{nt}\), where \(P\) is the amount invested at an annual interest rate \(r\) compounded \(n\) times per year, and \(A\) is the amount after \(t\) years.

**general form of a quadratic relation** An equation of the form \(Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\), where \(A, B, C, D, E,\) and \(F\) are real numbers and at least one of \(A, B,\) or \(C\) is not zero.

**geometric sequence** A sequence with a constant multiplier or constant ratio; also called exponential sequence.

**geometric series** An indicated sum of successive terms of a geometric sequence.

**graph translation theorem** In a sentence for a graph, replacing \(x\) by \(x - h\) and \(y\) by \(y - k\) causes the graph to undergo the translation \(T_{h,k}\).

**gravitational constant** The acceleration of a moving object due to gravity; near the Earth's surface, it is about 32 ft/sec\(^2\) or 9.8 m/sec\(^2\).

**greatest integer function** The function denoted by \([x]\), whose values are the greatest integer less than or equal to \(x\); also called the rounding down function.

**growth factor** In the exponential function \(y = ab^x\), the amount \(b\) by which \(y\) is multiplied for every unit increase in \(x\).

**half-life** The amount of time required for a quantity to decay to half its original value.

**half-plane** One of the two regions formed by a line dividing a plane.

**horizontal line** A line whose equation is of the form \(y = b\).

**horizontal-line test** The inverse of a function is itself a function if and only if no horizontal line intersects the graph of the function in more than one point.

**horizontal scale change** The stretching or shrinking of a figure in only the horizontal direction; a transformation which maps \((x, y)\) onto \((kx, y)\).

**hyperbola** The set of points \(P\) in a plane which satisfy \(|PF_1 - PF_2| = d\), where \(F_1\) and \(F_2\) are any two points and \(d\) is a constant with \(0 < d < F_1F_2\).

**hypercube** A four-dimensional cube.

**hyperplane** A plane in 4-space with an equation of the form \(Ax + By + Cz + Dw = E\), where not all of \(A, B, C,\) and \(D\) are zero.

**hypersphere** The set of points in 4-space at a given distance from a fixed point.

**identity function** The function defined by \(f(x) = x\).

**identity transformation** A transformation in which each point coincides with its image.
image The object resulting from applying a transformation.

imaginary number A number which is the square root of a negative real number.

imaginary part In a complex number of the form $a + bi$, $b$ is the imaginary part.

inconsistent system A system with no solutions.

independent variable In a formula, a variable upon whose value other variables depend.

index The subscript used for a term in a sequence; the index indicates the position of the term in the sequence.

index variable The variable under the $\Sigma$ sign in summation notation; also called index.

inequality An open sentence containing one of the symbols $<, >, \leq, \geq, \neq$, or $\approx$.

infinite geometric series A geometric series with infinitely many terms.

integers The set $\{0, 1, -1, 2, -2, 3, -3, \ldots \}$.

interior of a circle The region inside a circle; for the circle with center $(h, k)$ and radius $r$: the interior of the circle is described by $(x - h)^2 + (y - k)^2 < r^2$.

intersection of sets The set consisting of those values common to both sets.

interval A solution to an inequality of the form $x \leq a$ or $a \leq x \leq b$, where the $\leq$ can be replaced by $<, >, \text{ or } \geq$.

inverse function theorem f and g are inverse functions if and only if $(f \circ g)(x) = (g \circ f)(x) = x$.

inverse-matrix theorem If $ad - bc \neq 0$, the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

inverse of a function The relation obtained by reversing the order of the coordinates of each ordered pair in the function.

inverse of a matrix Matrices $M$ and $N$ are inverse matrices if and only if their product is the identity matrix.

inverse-square graph The graph of $y = \frac{k}{x^2}$.

inverse-square variation An inverse variation described by the equation $y = \frac{k}{x^2}$, with $k \neq 0$.

inverse variation A situation in which as one variable increases in absolute value, the other variable decreases in absolute value; $y$ varies inversely as $x^n$, or $y$ is inversely proportional to $x^n$ is written as $y = \frac{k}{x^n}$, for $k \neq 0$, $n > 0$.

inversely proportional to In an inverse variation, the same as "varies inversely as."

irrational number A number which cannot be written as a simple fraction; an infinite and nonrepeating decimal.

joint variation A situation in which one quantity varies directly as the product of two or more independent variables, but not inversely as any variable; example: In $y = kxz$, $y$ varies jointly as $x$ and $y$.

lattice point A point with integer coordinates.

lattice point in 3-space A point $(x, y, z)$ in which $x$, $y$, and $z$ are integers.

law of cosines theorem In any triangle $ABC$, $c^2 = a^2 + b^2 - 2ab \cos C$.

law of sines theorem In any triangle $ABC$, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

leading coefficient The coefficient of the variable of highest power in a polynomial in a single variable.

limit A number or figure which the terms of a sequence approach as $n$ gets larger.

linear-combination method A method of solving systems which involves adding multiples of the given equations.

linear-combination situation A situation in which all variables are to the first power and are not multiplied or divided by each other.

linear inequality An inequality in which both sides are linear expressions.
linear polynomial  A polynomial of the first degree, such as \( ax + b \).

linear-programming problem  A problem which leads to systems of linear inequalities whose solution gives a “program” or course of action to follow.

linear-programming theorem  The feasible region of a linear-programming problem is convex, and the maximum or minimum quantity is determined at one of the vertices of the region.

linear sequence  A sequence with a constant difference; also called arithmetic sequence.

line of symmetry  A line through a graph such that if the graph were folded along the line, both sides of the graph would coincide.

logarithm of \( m \) to the base 10  \( n \) is the logarithm of \( m \) to the base 10, written \( n = \log_{10} m \), if and only if \( 10^n = m \).

logarithm of \( m \) to the base \( b \)  Let \( b > 0 \) and \( b \neq 1 \). Then \( n \) is the logarithm of \( m \) to the base \( b \), written \( n = \log_b m \), if and only if \( b^n = m \).

logarithmic curve  The graph of a function of the form \( y = \log_b x \).

logarithmic scale  A scale in which the units are spaced so that the ratio between successive units is the same.

magnitude of a size change  The amount by which distances in a preimage are multiplied; also called scale factor.

major axis of an ellipse  The segment which contains the foci and has two vertices of an ellipse as its endpoints.

mapping notation  The notation \( f: x \to y \); also called arrow notation.

mathematical model  A graph or sentence that describes data or a relation between variables.

matrix  A rectangular arrangement of objects.

matrix addition  If two matrices \( A \) and \( B \) have the same dimensions, their sum \( A + B \) is the matrix in which each element is the sum of the corresponding elements in \( A \) and \( B \).

matrix form of a system  A representation of a system using matrices; the matrix form for \( \begin{cases} ax + by = e \\ cx + dy = f \end{cases} \) is \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \).

matrix multiplication  Suppose \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix. The product \( A \cdot B \) or \( AB \) is the \( m \times p \) matrix whose element in row \( i \) and column \( j \) is the product of row \( i \) of \( A \) and column \( j \) of \( B \).

matrix-solution theorem  A \( 2 \times 2 \) system has exactly one solution if and only if the determinant of the coefficient matrix is not zero.

matrix subtraction  Given two matrices \( A \) and \( B \) having the same dimensions, their difference \( A - B \) is the matrix whose element in each position is the difference of the corresponding elements in \( A \) and \( B \).

mean  The average of all the terms of a data set.

measure of central tendency  A number which in some sense is at the “center” of a data set; the mean, mode, or median of a data set.

measure of dispersion  A number, like standard deviation, which describes the extent to which elements of a data set are dispersed or spread out.

median  The middle term of a data set when the terms are placed in increasing order.

method of finite differences  A method of determining whether a sequence can be described by a polynomial formula, using successive differences of terms of the sequence.

midpoint formula  The midpoint of the segment with endpoints \( (x_1, y_1) \) and \( (x_2, y_2) \) is \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).

minor axis of an ellipse  The segment which does not contain the foci and has two vertices of an ellipse as its endpoints.

mode  The number which occurs most often in a data set.

model for an operation  A pattern that describes many uses of that operation.

monomial  A polynomial with one term.
**multiplication properties of inequality** For all real numbers $a$, $b$, and $c$: if $a < b$ and $c > 0$, then $ac < bc$, and if $a < b$ and $c < 0$, then $ac > bc$.

**multiplication property of equality** For all real numbers $a$, $b$, and $c$: if $a = b$, then $ac = bc$.

**multiplicity of a root** In a polynomial equation, the highest power of $x - r$, where $r$ is a root, that appears as a factor of the polynomial.

**natural logarithm** A logarithm to the base $e$.

**natural numbers** The set $\{1, 2, 3, 4, 5, \ldots\}$; also called the counting numbers.

**negative exponent theorem** If $x > 0$, then $x^{-n} = \frac{1}{x^n}$.

**normal curve** The curve of a normal distribution.

**normal distribution** A function whose graph is the image of the graph of $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ under a composite of translations or scale transformations.

**normalized scores** Scores whose distribution is a normal curve; also called standardized scores.

**nth power function** The function $f(x) = x^n$, where $n$ is a positive integer.

**nth root** Let $n$ be an integer greater than one. Then $b$ is an $n$th root of $x$ if and only if $b^n = x$.

**nth root of nth power theorem** For all real numbers $x$ and integers $n \geq 2$: if $n$ is odd, $\sqrt[n]{x^n} = x$; if $n$ is even, $\sqrt[n]{x^n} = |x|$.

**nth term** The term occupying the $n$th position in the listing of a sequence; the general term of a sequence.

**number of roots of a polynomial equation theorem** Every polynomial equation of degree $n$ has exactly $n$ roots provided that multiple roots are counted as separate roots.

**oblique line** A line that is neither horizontal or vertical.

**octant** One of eight regions determined by the intersection of three coordinate planes in 3-space.

**one-to-one correspondence** A mapping in which each member of one set is mapped to a distinct member of another set, and vice-versa.

**open sentence** A sentence that may be true or false depending on what values are substituted for the variables.

**order of operations** Hierarchy used to evaluate expressions worldwide: (1) Perform operations within grouping symbols from inner to outer; (2) Take powers; (3) Do multiplications or divisions from left to right; (4) Do additions or subtractions from left to right.

**ordered 4-tuple** $(x, y, z, w)$

**ordered triple** $(x, y, z)$

**parabola** The set consisting of every point in the plane of line $l$ and point $F$ not on $l$ whose distance from $F$ equals its distance from $l$.

**paraboloid** A three-dimensional figure created by rotating a parabola in space around its axis of symmetry; the set of points equidistant from a point $F$ (the focus) and a plane $P$.

**partial sum** The sum of the first $n$ terms of a sequence.

**Pascal's triangle** The sequence satisfying

1. \[ \binom{n}{0} = \binom{n}{n} = 1 \] and
2. \[ \binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}, \]

where $n$ and $r$ are any integers with $0 \leq r \leq n$.

The triangular array

\[
\begin{array}{ccccccc}
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

are located next to each other on a row, the element just below and directly between them is $x + y$.

**perfect square trinomial** A trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.

**perfect square trinomial pattern** For all $a$ and $b$: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$.  

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periodic  A relation whose graph can be mapped to itself under a horizontal translation.

permutation  An arrangement of $n$ different objects in order.

piecewise linear graph  A graph made of segments each of which is a piece of a line.

point matrix  A $2 \times 1$ matrix.

point-slope form of a linear equation  An equation of the form $y - y_1 = m(x - x_1)$, where $(x_1, y_1)$ is a point on the line with slope $m$.

difference theorem  $y = f(x)$ is a polynomial function of degree $n$ if and only if, for any set of $x$-values that form an arithmetic sequence, the $n$th differences of the corresponding $y$-values are equal.

polynomial in $x$  An expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, where $n$ is a positive integer and $a_n \neq 0$.

population  The set of all people, events, or items that could be sampled.

postulates  Statements assumed to be true in a mathematical system.

power of a power property  For any nonnegative bases and real exponents or any nonzero base and integer exponents: $(b^m)^n = b^{mn}$.

power of a product property  For any nonnegative bases and real exponents or any nonzero base and integer exponents: $(ab)^m = a^m b^m$.

power of a quotient property  For any nonnegative bases and real exponents or any nonzero base and integer exponents: $(\frac{a}{b})^m = \frac{a^m}{b^m}$.

powering  An operation by which a variable is raised to a power; also called exponentiation.

powering property of logarithms  For any positive real number $x$, $\log_b(x^n) = n \log_b x$.

preimage  The object to which a transformation is applied.

principal  The original amount of money invested.

probability (of an event)  If a situation has a total of $t$ equally likely possibilities and $e$ of these possibilities satisfy conditions for a particular event, then the probability of the event $= \frac{e}{t}$.

probability distribution  A function which maps a set of events onto their probabilities; also called probability function.

product of powers property  For any nonnegative bases and real exponents or any nonzero base and integer exponents: $b^n \cdot b^m = b^{n+m}$.

product property of logarithms  For any base $b$ and for any positive real numbers $x$ and $y$: $\log_b(xy) = \log_b x + \log_b y$.

proof  An argument showing that a statement is true.

properties  Postulates, theorems, or definitions in a mathematical system.

Pythagorean identity  For all $\theta$ between $0°$ and $90°$: $(\cos \theta)^2 + (\sin \theta)^2 = 1$.

quadratic equation  An equation which involves quadratic expressions.

quadratic equation in two variables  An equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A$, $B$, $C$, $D$, $E$, and $F$ are real numbers and at least one of $A$, $B$, or $C$ is not zero.

quadratic expression  An expression which contains one or more terms in $x^2$, $y^2$, or $xy$, but no higher powers of $x$ or $y$.

quadratic form  An expression of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$.

quadratic formula  If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

quadratic-linear system  A system that involves linear and quadratic sentences.

quadratic relation in two variables  The sentence $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ (or the inequality using one of the symbols $>$, $<$, $\geq$, $\leq$) where $A$, $B$, $C$, $D$, $E$, and $F$ are real numbers and at least one of $A$, $B$, or $C$ is not zero.

quadratic-quadratic system  A system that involves two quadratic sentences.

quadratic system  A system that involves at least one quadratic sentence.
quartic equation  A fourth degree polynomial equation.

quintic equation  A fifth degree polynomial equation.

quotient of powers property  For any nonnegative bases and real exponents, or any nonzero bases and integer exponents: \( \frac{b^m}{b^n} = b^{m-n} \).

quotient property of logarithms  For any base \( b \) and for any positive real numbers \( x \) and \( y \):
\[
\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y.
\]

radian  A unit of angle, arc, or rotation measure such that \( \pi \) radians = 180 degrees.

radius  The given distance between a circle and its center.

random numbers  Numbers which have the same probability of being selected.

random sample  A sample in which each element has the same probability as every other element in the population of being selected for the sample.

range of a function  The set of values of the function.

rate of change  Between two points, the quantity \( \frac{y_2 - y_1}{x_2 - x_1} \); for a line, its slope.

rational exponent theorem  For any positive real number \( x \) and positive integers \( m \) and \( n \),
\[
x^{mn} = (x^{1/n})^m,
\]
the \( m \)th power of the positive \( n \)th root of \( x \), and
\[
x^{mn} = (x^m)^{1/n},
\]
the \( n \)th root of the \( m \)th power of \( x \).

rational number  A number which can be written as a simple fraction; a finite or infinitely repeating decimal.

real numbers  Those numbers that can be represented by a decimal.

real part  In a complex number of the form \( a + bi \), \( a \) is a real part.

rectangular hyperbola  A hyperbola with perpendicular asymptotes.

recursive formula  A set of statements that indicates the first term of a sequence and gives a rule for how the \( n \)th term is related to one or more of the previous terms.
row operation  Any of the following operations on rows of a matrix: multiplying a row by any nonzero number; adding two rows and replacing one of the rows with the sum; switching two rows.

sample  The subset of the population actually studied.

scalar  A real number by which a matrix is multiplied.

scalar multiplication  The product of a scalar $k$ and a matrix $A$ is the matrix $kA$ in which each element is $k$ times the corresponding element in $A$.

scale change  The stretching or shrinking of a figure in either a horizontal direction only, in a vertical direction only, or in both directions; a horizontal scale change of magnitude $a$ and a vertical scale change of magnitude $b$ maps $(x, y)$ onto $(ax, by)$, and is denoted by $S_{a,b}$.

scale factor  In a size change, the amount by which distances are multiplied.

self-similar  A property of an object which does not change its appearance significantly when viewed under a microscope of arbitrary magnifying power.

sequence  An ordered list.

shrink  The contraction of a figure in some direction.

simple interest  The amount of money $I$ paid by the bank found by the formula $I = Prt$, where $P$ is the principal, $r$ is the rate, and $t$ is the time.

simulation  A procedure used to ask questions about real-world situations by performing experiments that closely model them.

sine function  The correspondence $\theta \rightarrow \sin \theta$ that associates $\theta$ with the $y$-coordinate of the image of $(1, 0)$ under $R_{\theta}$.

sine of $\theta$ ($\sin \theta$)  In a right triangle with acute angle $\theta$, $\sin \theta = \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}}$; the second coordinate of $R_{\theta}(1, 0)$.

sine wave  A graph which can be mapped onto the graph of $g(\theta) = \sin \theta$ by any composite of reflections, translations, and scale changes.

size change  The stretching or shrinking of a figure by the same amount in both directions; a special kind of scale change; a size change of magnitude $k$, denoted by $S_k$, maps the point $(x, y)$ onto $(kx, ky)$.

slope  The slope of a line through two points $(x_1, y_1), (x_2, y_2)$ equals $\frac{y_2 - y_1}{x_2 - x_1}$; the same as the rate of change between two points on the line.

slope-intercept form of a linear equation  A linear equation of the form $y = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept.

snowflake curve  The limit of a sequence of polygons formed by beginning with $F_1$ as an equilateral triangle, and forming $F_n$ by drawing an equilateral triangle outward on the middle third of each side of $F_{n-1}$ and then deleting the middle third.

solution set for a system  The intersection of the solution sets for individual sentences of a system.

solution to a sentence  A value of a variable or values of variables which make a sentence true.

solving a triangle  The use of trigonometry to find all the missing measures of sides and angles of a triangle.

sphere  The set of points in 3-space at a given distance from a fixed point.

square matrix  A matrix with the same number of rows and columns.

square root  A square root $x$ of $t$ is a solution to $x^2 = t$.

square root function  The function $f(x) = \sqrt{x}$, where $x$ is a nonnegative real number.

squaring function  A powering function defined by $f(x) = x^2$.

standard deviation  Let $S$ be a data set of $n$ numbers $\{x_1, x_2, \ldots, x_n\}$. Let $m$ be the mean of $S$. Then the standard deviation $s.d.$ of $S$ is
s.d. = \sqrt{\frac{\sum_{i=1}^{n} (x_i - m)^2}{n}}.

### standard form of an equation for an ellipse
The ellipse with foci \((c, 0)\) and \((-c, 0)\) and focal constant \(2a\) has equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), where \(b^2 = a^2 - c^2\).

### standard form of an equation for a hyperbola
The hyperbola with foci \((c, 0)\) and \((-c, 0)\) and focal constant \(2a\) has equation \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), where \(b^2 = c^2 - a^2\).

### standard form of an equation for a plane
The equation \(Ax + By + Cz = D\) where not all of \(A, B,\) and \(C\) are zero.

### standard form of a linear equation
An equation of the form \(Ax + By = C\), where \(A\) and \(B\) are not both zero.

### standard form of a quadratic equation
An equation of the form \(ax^2 + bx + c = 0\), where \(a \neq 0\).

### standard position for an ellipse
A location in which the origin of a coordinate system is midway between the foci with the foci on an axis.

### standardized scores
Scores whose distribution is a normal curve; also called normalized scores.

### statistical measure
A single number which is used to describe an entire set of numbers.

### stratified sample
A sample in which the population has first been split into subpopulations and then, from each subpopulation, a sample is selected.

### step function
A graph that looks like a series of steps; the function \(y = [x]\).

### stretch
The expansion of a figure in some direction.

### subscript
A number or variable written below and to the right of a variable.

### subscripted variable
A variable with a subscript.

### subset
A set whose elements are all chosen from another set.

### substitution method
A method of solving a system using the Substitution Property.

### sum and product of roots theorem
\(r_1\) and \(r_2\) are the roots of the equation \(ax^2 + bx + c = 0\), with \(a \neq 0\), if and only if \(r_1 + r_2 = \frac{-b}{a}\) and \(r_1r_2 = \frac{c}{a}\).

### sum of a series
The limit of the partial sums of a series.

### sums of cubes pattern
For all \(a\) and \(b\):
\[a^3 + b^3 = (a + b)(a^2 - ab + b^2)\].

### summation notation
A shorthand notation used to restate a series; also called \(\Sigma\)-notation or sigma notation.

### supplements theorem
For all \(\theta\) in degrees:
\[\sin \theta = \sin(180^\circ - \theta)\].

### surface of revolution
A surface which is generated by rotating a curve in a plane about a line.

### system
A set of conditions joined by the word "and"; a special kind of compound sentence.

### tangent of \(\theta\) (\(\tan \theta\))
In a right triangle with acute angle \(\theta\),
\[\tan \theta = \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta}\].

In general, \(\frac{\sin \theta}{\cos \theta}\) provided \(\cos \theta \neq 0\).

### term of a sequence
An element of a sequence.

### theorems
Statements that can be proved in a mathematical system.

### 3-dimensional coordinate system
A coordinate system used to locate all the points in 3-space.

### transformation
A one-to-one correspondence between sets of points.

### translation
The transformation that maps \((x, y)\) onto \((x + h, y + k)\) is a translation of \(h\) units horizontally and \(k\) units vertically and is denoted by \(T_{h,k}\).

### trial
One occurrence of an experiment.

### triangular matrix
A matrix which has all zeros below the diagonal from top left to bottom right.

### trigonometric ratios
The ratios of the lengths of the sides in a right triangle.

### trinomial
A polynomial with three terms.
unbiased coin A coin that has an equal probability of landing on either side; also called a fair coin.

union of sets The set consisting of those values in either one or both sets.

unit circle The circle with center at the origin and radius 1.

value of a function If \( y = f(x) \), the value of \( y \).

variable A symbol that can be replaced by any one of a set of numbers or other objects.

varies directly as In a direct variation, the same as "directly proportional to."

varies inversely as In an inverse variation, the same as "inversely proportional to."

velocity The rate of change of distance with respect to time.

vertex form of an equation of a parabola An equation of the form \( y - k = a(x - h)^2 \) where \((h, k)\) is the vertex of the parabola.

vertex (vertices) of an ellipse or hyperbola A point (points) of intersection of the ellipse or hyperbola and the line containing its foci.

vertex of a parabola The intersection of a parabola and its axis of symmetry.

vertical line A line with an equation of the form \( x = b \); a line with no slope.

vertical-line test for functions No vertical line intersects the graph of a function in more than one point.

vertical scale change The stretching or shrinking of a figure in only the vertical direction.

whole numbers The set \{0, 1, 2, 3, 4, 5, \ldots \}.

window The part of the coordinate grid shown on the screen of an automatic grapher.

x-intercept The value of \( x \) at a point where a graph crosses the \( x \)-axis.

xy-plane The plane in 3-space determined by the \( x \)-axis and \( y \)-axis.

xz-plane The plane in 3-space determined by the \( x \)-axis and \( z \)-axis.

y-intercept The value of \( y \) at a point where a graph crosses the \( y \)-axis.

yz-plane The plane in 3-space determined by the \( y \)-axis and \( z \)-axis.

z-axis In a 3-dimensional coordinate system, the axis perpendicular to the \( xy \)-plane.

z-coordinate The third coordinate in an ordered triple \((x, y, z)\).

zero exponent theorem If \( b \) is a nonzero real number, \( b^0 = 1 \).

zero of a function An \( x \)-intercept of the graph of a function.

zero product theorem For all \( a \) and \( b \): \( ab = 0 \) if and only if \( a = 0 \) or \( b = 0 \).

zoom A feature on an automatic grapher which enables the window of a graph to be changed without retyping intervals for \( x \) and \( y \).
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1) \( \log_{10} 1 = 0 \)

2) \( \log_{10} 2 + \log_{10} 3 = \log_{10}(2 \cdot 3) = \log_{10} 6 \)  

5) \( \frac{1}{2} \log_{10} 49 = \log_{10} 49^{\frac{1}{2}} = \log_{10} 7 = 1 \)

13) \( \log x = \log 5 + 1 - \log 3 \)  
   \( \log x = \log \left( \frac{5}{3} \right) \)  
   \( \log x = 1.23 \)  
   \( x = 3 \)

\( \log 7 = x \)

\(
\begin{align*}
\log 49 & = 2x \\
\log 7 & = x \\
\log 70 & = 1 + x \\
\log 0.07 & = x - 2 \\
\log 0.07 & = 0.5x
\end{align*}
\)

\( \log 70 = 1.84 \)

\( \log 7 = 0.85 \)

\( \log 70 = 1.85 \)

\( 10^x = 10 \)

\( \log_{10} 10 = 1 \)

\( \log_{10} 100 \)

\( (10^x)^n = 10 \)